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TWISTING COPRODUCTS ON HOPF ALGEBRAS

KANG JU MIN AND JUN SEOK PARK

ABSTRACT. Let (H, K) be a paired Hopf algebras and let A be arbitrary left H-module coalgebra. We construct twisting coproduct on $A \otimes K$. We show that the well known construction of the smash coproduct can be viewed as a particular case of the construction above.

Throughout the paper we let k be a field. Tensor products are assumed to be over k. Let H be a Hopf algebra over k ; that is, H is an algebra with 1 and a coalgebra over k with:

(1) comultiplication $\Delta\colon H \to H \otimes H$
(2) counit $\epsilon\colon H \to k$
(3) antipode $S\colon H \to H$
(4) multiplication $\mu\colon H \otimes H \to H$
(5) unit $u\colon k \to H$,

where Δ and ϵ are algebra homomorphisms and S is an algebra antihomomorphism. In the case that H is finite dimensional H^* is also a Hopf algebra and its structure is given by maps denoted as $\Delta^*, \epsilon^*, \mu^*, u^*$and S^*.

The following notations are used in this paper:

1. Given an arbitrary bialgebra H, the *cooposite coalgebra* H^{cop} is given as follows : $H^{cop} = H$ as a vector space, with new comultiplication Δ' given by $\Delta' = \tau\Delta$.

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2. The symbol $s\colon A\otimes B \to B\otimes A$ denotes the map that switches the tensor factors. More generally if τ is a permutation of n elements, and $V_i, i=1,...,n$ are arbitrary k-spaces, we call

$$s_\tau\colon V_1\otimes V_2\otimes\cdots\otimes V_n \to V_{\tau(1)}\otimes V_{\tau(2)}\otimes\cdots\otimes V_{\tau(n)}$$

the map that permutes the tensor factors in the same fashion as τ. In particular $s=s_{(1,2)}$.

3. If H is a bialgebra and M and N are right H-modules with structures ϕ_M and ϕ_N respectively, we denote as $M\boxtimes N$ the vector space $M\otimes N$ equipped with the tensor product right H-module structure ([3, P.14]). :

$$\phi_M\boxtimes\phi_N=(\phi_M\otimes\phi_N)(id\otimes s\otimes id)(\Delta\otimes id\otimes id).$$

4. If A and K are coalgebras with structures Δ_A and Δ_K, the map $\Delta_A\boxtimes\Delta_K\colon A\otimes K\to A\otimes K\otimes A\otimes K$ given by

$$\Delta_A\boxtimes\Delta_K=(id\otimes s\otimes id)(\Delta_A\otimes\Delta_K)$$

defines a coalgebra structure on $A\otimes K$. If μ_A and μ_K are multiplications in A and K the tensor product multiplication $\mu_A\boxtimes\mu_K$ is given by

$$\mu_A\boxtimes\mu_K=(\mu_A\otimes\mu_K)(id\otimes s\otimes id).$$

DEFINITION 1. Suppose that H and K are k-bialgebras. We say that H and K are *paired* if there exists a k-linear map $q\colon k\to H\otimes K$ (called the *pairing*) such that the diagrams below commute:

$$\text{(1-1)}\qquad \begin{array}{ccccc} k & \xrightarrow{q} & H\otimes K & \xrightarrow{id\otimes\Delta_k} & H\otimes K\otimes K \\ {\scriptstyle\Delta_k}\big\downarrow & & & & \big\uparrow{\scriptstyle\mu_H\otimes id\otimes id} \\ k\otimes k & \xrightarrow{q\otimes q} & H\otimes K\otimes H\otimes K & \xrightarrow{id\otimes s\otimes id} & H\otimes H\otimes K\otimes K \end{array}$$

$$
\text{(1-2)}\qquad
\begin{array}{ccccc}
k & \xrightarrow{q} & H\otimes K & \xrightarrow{\Delta_H\otimes id} & H\otimes H\otimes K \\
{\scriptstyle\Delta_k}\downarrow & & & & \uparrow{\scriptstyle id\otimes id\otimes\mu_K} \\
k\otimes k & \xrightarrow{q\otimes q} & H\otimes K\otimes H\otimes K & \xrightarrow{id\otimes s\otimes id} & H\otimes H\otimes K\otimes K
\end{array}
$$

$$
\text{(1-3)}\qquad
\begin{array}{ccccc}
k & \xrightarrow{id} & k & \xrightarrow{id} & k \\
\downarrow{\scriptstyle 1_H\otimes} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \otimes 1_K} \\
H\otimes k & \xleftarrow[id\otimes\epsilon_K]{} & H\otimes K & \xrightarrow{\epsilon_H\otimes id} & k\otimes K
\end{array}
$$

The commutativity of the diagrams above can be expressed equationally in the following way:

$$(1-1')\qquad \sum_{i,(b_i)} a_i\otimes(b_i)_1\otimes(b_i)_2=\sum_{i,j} a_ia_j\otimes b_i\otimes b_j$$

$$(1-2')\qquad \sum_{i,(a_i)} (a_i)_1\otimes(a_i)_2\otimes b_i=\sum_{i,j} a_i\otimes a_j\otimes b_ib_j$$

$$(1-3')\qquad 1_H\cong 1_H\otimes 1_k=\sum_i a_i\otimes\epsilon_K(b_i)\cong\sum_i\epsilon_K(b_i)a_i$$

$$1_k\cong 1_k\otimes 1_K=\sum_i\epsilon_H(a_i)\otimes b_i\cong\sum_i\epsilon_H(a_i)b_i$$

where $q(1)=\sum a_i\otimes b_i$, $\Delta_K(b_i)=\sum(b_i)_1\otimes(b_i)_2$ and $\Delta_H(a_i)=\sum(a_i)_1\otimes(a_i)_2$. The dual concept was defined in [2].

Let V be a finite dimensional vector space with basis $\{v_i\}$. The dual vector space V^* has the dual basis $\{v^i\}$. Let us express the isomorphism

$$\lambda_{U,V}\colon V\otimes U^*\to Hom(U,V)$$

by

$$v \otimes \alpha \mapsto \lambda_{U,V}(v \otimes \alpha)$$

where $\lambda_{U,V}(v \otimes \alpha)(u) = \alpha(u)v, u \in U$.

Let $f: U \to V$ be a linear map. Using bases for U and V, we have

$$f(u_j) = \sum_i f^i_j v_i$$

for some family $(f^i_j)_{ij}$ of scalars. It is easily checked that

$$f = \lambda_{U,V}(\sum_{ij} f^i_j v_i \otimes u^j).$$

In particular, taking for f the identity of V, we get

$$id_V = \lambda_{V,V}(\sum_i v_i \otimes v^i).$$

This allows us to define the *coevaluation map* of any finite dimensional vector space V as the linear map $\delta_V: k \to V \otimes V^*$ defined by ([1, P.29])

$$\delta_V(1) = \lambda^{-1}_{V,V}(id_V) = \sum_i v_i \otimes v^i.$$

LEMMA 1. *If H is a finite dimensional bialgebra then the coevaluation map δ is a pairing between H and H^*.*

Proof. Define $\delta: k \to H \otimes H^*$ as

$$\delta(1) = \sum_l h_l \otimes h^l$$

where $h_1, h_2, ..., h_n$ is a basis for H.

Let $\mu_H(h_i \otimes h_j) = \sum \alpha_{l,i,j} h_l$ then we have

$$\begin{aligned}
&(\mu_H \otimes id \otimes id)(id \otimes s \otimes id)(\delta \otimes \delta)\Delta_k(1) \\
&\qquad = (\mu_H \otimes id \otimes id)(id \otimes s \otimes id)(\delta \otimes \delta)(1 \otimes 1) \\
&\qquad = (\mu_H \otimes id \otimes id)(id \otimes s \otimes id)(\sum_i h_i \otimes h^i \otimes h_j \otimes h^j) \\
&\qquad = (\mu_H \otimes id \otimes id)(\sum h_i \otimes h_j \otimes h^i \otimes h^j) \\
&\qquad = \sum \alpha_{l,i,j} h_l \otimes h^i \otimes h^j \\
&\qquad = \sum h_l \otimes (\sum \alpha_{l,i,j} h^i \otimes h^j) \\
&\qquad = (id \otimes \Delta_{H^*})(\sum h_l \otimes h^l) \\
&\qquad = (id \otimes \Delta_{H^*})\delta(1)
\end{aligned}$$

where sixth equality follows from the fact: if λ: $H^* \otimes H^* \to (H \otimes H)^*, \lambda(f \otimes g)(u \otimes v) = f(u)g(v), u, v \in H$ is an isomorphism then

$$\begin{aligned}
\Delta_{H^*}(h^i) &= (s^{-1}_{H^*,H^*} \circ \lambda^{-1})\mu^*(h^i) \\
&= (s^{-1}_{H^*,H^*} \circ \lambda^{-1})(h^i \circ \mu) \\
&= \sum \alpha_{l,i,j} h^i \otimes h^j.
\end{aligned}$$

Similarly

$$(id \otimes id \otimes \mu_{H^*})(id \otimes s \otimes id)(\delta \otimes \delta)\Delta_k = (\Delta_H \otimes id)\delta.$$

And

$$\begin{aligned}
((\epsilon_H \otimes id)\delta)(1) &= (\epsilon_H \otimes id)(\sum h_i \otimes h^i) \\
&= \sum \epsilon_H(h_i)h^i = \epsilon_H = 1_{H^*}, \\
((id \otimes \epsilon_{H^*})\delta)(1) &= \sum h_i \otimes \epsilon_{H^*}(h^i) = \sum u^*(h^i)h_i \\
&= \sum h^i(1_H)h_i = 1_H.
\end{aligned}$$

□

LEMMA 2. *If H_1, K_1 are paired and H_2, K_2 are paired then $H_2 \boxtimes H_1, K_1 \boxtimes K_2 (H_1 \boxtimes H_2, K_2 \boxtimes K_1)$ are paired.*

Proof. Let $q_1 \colon k \to H_1 \otimes K_1, q_1(1) = \sum_i h_{1i} \otimes k_{1i}$ and $q_2 \colon k \to H_2 \otimes K_2, q_2(1) = \sum_j h_{2j} \otimes k_{2j}$ be pairing. We can show that $q = (s \otimes s)(id \otimes s \otimes id)(q_1 \otimes q_2) \colon k \to (H_2 \boxtimes H_1) \otimes (K_1 \boxtimes K_2), q(1) = \sum h_{2j} \otimes h_{1i} \otimes k_{2j} \otimes k_{1i}$ is a pairing. □

If H is a finite dimensional algebra and V is a left H-module then V is a right H^*-module. In the same way, if H and K are paired and V is a left H(right K)-module then V has an associated right K(left H)-comodule structure:

(1) Let $\psi \colon H \otimes V \to V$ be a left H-module structure map of V then $\hat{\psi} = (\psi \otimes id)(s \otimes id)(id \otimes q) \colon V \cong V \otimes k \to V \otimes K$ is a right K-comodule structure map on V.
(2) Let $\phi \colon V \otimes K \to V$ be a right K-module structure map of V then $\hat{\phi} = (id \otimes \phi)(id \otimes s)(q \otimes id) \colon V \cong k \otimes V \to H \otimes V$ is a left H-comodule structure map on V.

DEFINITION 2. Assume that H and K are paired bialgebras. Let $\psi \colon H \otimes V \to V, \psi(h \otimes v) = h \cdot v$ be a left H-module structure map of V and $\phi \colon V \otimes K \to V, \phi(v \otimes k) = v \odot k$ be a right K-module structure map of V. Define the map $\psi \vee \phi \colon V \otimes k \otimes V \cong V \otimes V \to V \otimes V$ by the following diagram :

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\psi \vee \phi} & V \otimes V \\ {\scriptstyle id \otimes q \otimes id} \downarrow & & \uparrow {\scriptstyle \psi \otimes \phi} \\ V \otimes H \otimes K \otimes V & \xrightarrow{s \otimes s} & H \otimes V \otimes V \otimes K. \end{array}$$

We show that

$$(\psi \vee \phi)(u \otimes v) = \sum (a_i \cdot u) \otimes (v \odot b_i)$$

where $q(1) = \sum a_i \otimes b_i$. And

$$\psi \vee \phi = (id \otimes \phi)(id \otimes s)(\bar{\psi} \otimes id) = (\psi \otimes id)(s \otimes id)(id \otimes \bar{\phi}).$$

DEFINITION 3. Let H and K be bialgebras. Let $\psi\colon H \otimes V \to V$ be left H-module structure map and $\phi\colon V \otimes K \to V$ be right K-module structure map. We say that ψ and ϕ are *compatible* (or ψ and ϕ define an H, K-bimodule structure on V) if the following diagram commutes.

$$\begin{array}{ccc} H \otimes V \otimes K & \xrightarrow{id \otimes \phi} & H \otimes V \\ {\scriptstyle \psi \otimes id}\downarrow & & \downarrow{\scriptstyle \psi} \\ V \otimes K & \xrightarrow{\phi} & V \end{array}$$

The definition above is equivalent to the fact that the map

$$\phi\colon {}_H(V \otimes K) \to_H V$$

is a left H-module homomorphism where ψ and $\psi \otimes id$ are the H-module structure maps of V and $V \otimes K$ respectively: for $h \in H$,

$$\begin{aligned} \phi(h \cdot (v \otimes k)) &= \phi(\psi(h \otimes v) \otimes k) \\ &= \phi(\psi \otimes id)(h \otimes v \otimes k) \\ &= \psi(id \otimes \phi)(h \otimes v \otimes k) \\ &= \psi(h \otimes \phi(v \otimes) \\ &= h \cdot \phi(v \otimes k). \end{aligned}$$

We note that if H and K are paired bialgebras and $\psi\colon H \otimes V \to V$ and $\phi\colon V \otimes K \to V$ are compatible, then $\bar{\psi}\colon V_K \to (V \otimes K)_K$ is a

K-module homomorphism where we endow $V \otimes K$ with the structure $(\phi \otimes id)(id \otimes s)$.

DEFINITION 4. Let K and K' be bialgebras. Let $\phi\colon V \otimes K \to V$ and $\phi'\colon V \otimes K' \to V$ be right K and K'-module structures. We say that ϕ and ϕ' are *compatible* if the following diagram commutes.

$$\begin{array}{ccccc} V \otimes K' \otimes K & \xrightarrow{\phi' \otimes id} & V \otimes K & \xrightarrow{\phi} & V \\ \uparrow{\scriptstyle id \otimes s} & & & & \uparrow{\scriptstyle id} \\ V \otimes K \otimes K' & \xrightarrow{\phi \otimes id} & V \otimes K' & \xrightarrow{\phi'} & V \end{array} \tag{4-1}$$

$$\begin{array}{ccccc} V \otimes K' \otimes K & \xrightarrow{\phi' \otimes id} & V \otimes K & \xrightarrow{\phi} & V \\ {\scriptstyle id \otimes s}\downarrow & & & & {\scriptstyle id}\downarrow \\ V \otimes K \otimes K' & \xrightarrow{\phi \otimes id} & V \otimes K' & \xrightarrow{\phi'} & V \end{array} \tag{4-2}$$

DEFINITION 5. Let $\phi\colon V \otimes K \to V$ and $\phi'\colon V \otimes K' \to V$ be right module structures on V. We define the map $\phi \boxtimes \phi'\colon V \otimes K \otimes K' \to V$ as

$$\phi \boxtimes \phi' = \phi'(\phi \otimes id).$$

ϕ is compatible with ϕ' if and only if $\phi \boxtimes \phi' = (\phi' \boxtimes \phi)(id \otimes s)$.

THEOREM 3. *In the situation above, if ϕ is compatible with ϕ' then the map $\phi \boxtimes \phi'\colon V \otimes K \otimes K' \to V$ is a structure of $K \boxtimes K'$-module on V. Conversely, any structure $\theta\colon V \otimes K \otimes K' \to V$ of $K \boxtimes K'$-module on V is of the form $\theta = \phi \boxtimes \phi'$ for some pair $\phi\colon V \otimes K \to V$ and $\phi'\colon V \otimes K' \to V$ of compatible right module structures on V.*

Proof. Suppose that $\phi\colon V\otimes K\to V$ and $\phi'\colon V\otimes K'\to V$ are compatible. Then we have

$$\begin{aligned}
&(\phi\boxtimes\phi')(id_V\otimes\mu_K\boxtimes\mu_{K'})\\
&=(\phi'\boxtimes\phi)(id_V\otimes s)(id_V\otimes\mu_K\otimes\mu_{K'})\\
&=(\phi'\boxtimes\phi)((\phi'\boxtimes\phi)\otimes id_{K'}\otimes id_K)s_{45}\\
&=\phi(\phi'\otimes id_K)(\phi'\otimes id_{K'}\otimes id_K)(\phi\otimes id_{K'}\otimes id_{K'}\otimes id_K)s_{45}\\
&=\phi(\phi'\otimes id_K)(id_V\otimes\mu_{K'}\otimes id_K)(\phi\otimes id_{K'}\otimes id_{K'}\otimes id_K)s_{45}\\
&=\phi'(\phi\otimes id_{K'})(id_V\otimes s)(id_V\otimes\mu_{K'}\otimes id_K)(\phi\otimes id_{K'}\otimes id_{K'}\otimes id_K)s_{45}\\
&=\phi'(\phi\otimes id_{K'})(\phi\otimes id_K\otimes id_{K'})(id_V\otimes id_K\otimes id_K\otimes\mu_{K'})s_{34}\\
&=\phi'(\phi\otimes id_{K'})(id_V\otimes\mu_{K'}\otimes id_K)(id_V\otimes id_K\otimes id_K\otimes\mu_{K'})s_{34}\\
&=(\phi\boxtimes\phi')((\phi\boxtimes\phi')\otimes id_K\otimes id_{K'}).
\end{aligned}$$

The first equality follows from the compatibility of ϕ and ϕ' and the third equality follows from the definition of $\phi'\boxtimes\phi$. Since V is a right K'-module, fourth equality holds and sixth equality follows from the compatibility of K and K'. Therefore $\phi\boxtimes\phi'\colon V\otimes K\otimes K'\to V$ verifies the associativity condition.

Let $\theta\colon V\otimes K\otimes K'\to V$ be a $K\boxtimes K'$-module structure. Define $\phi\colon V\otimes K\cong V\otimes K\otimes 1\to V$ as

$$\phi=\theta(id_v\otimes id_K\otimes\mu_{K'})$$

and $\phi'\colon V\otimes K'\cong V\otimes 1\otimes K'\to V$ as

$$\phi'=\theta(id_V\otimes\mu_K\otimes id_{K'}).$$

Since $id_K\otimes\mu_{K'}\colon K\cong K\otimes 1\to K\boxtimes K'$ is a bialgebra morphism, $\phi\colon V\otimes K\to V$ is a K-module structure map on V. Similarity for ϕ

and $\mu_K \otimes id_{K'}$.

$$\begin{aligned}
&(\phi \boxtimes \phi')(v \otimes a \otimes b) \\
&\quad = (\theta(id_V \otimes \mu_K \otimes id_{K'}))(\theta(id_V \otimes id_K \otimes \mu_{K'}) \otimes id_{K'})(v \otimes a \otimes b) \\
&\quad = \theta(\theta(v \otimes a \otimes \mu_{K'}(1)) \otimes \mu_K(1) \otimes b) \\
&\quad = \theta(\theta \otimes id_K \otimes id_{K'})((v \otimes a \otimes \mu_{K'}(1)) \otimes \mu_K(1) \otimes b) \\
&\quad = \theta(id_V \otimes \mu_K \boxtimes \mu_{K'})((v \otimes a \otimes \mu_{K'}(1)) \otimes \mu_K(1) \otimes b) \\
&\quad = \theta(v \otimes \mu_K(a \otimes \mu_K(1)) \otimes \mu_{K'}(\mu_{K'}(1) \otimes b)) \\
&\quad = \theta(v \otimes a \otimes b).
\end{aligned}$$

Therefore $\phi \boxtimes \phi' = \theta$. The compatibility of ϕ and ϕ' can be verified in a similar way. This completes the proof. □

DEFINITION 6. Let K be a bialgebra and let B be a right K-module with structure map $\phi\colon B \otimes K \to B$. A k-linear map $\Delta\colon B \to B \otimes B$ is said to be *compatible with* ϕ , or that *ϕ-comultiplication* if the map $\Delta\colon B \to B \otimes B$ is a right K-module morphism. If Δ is a structure of coassociative k-coalgebra on B, we say that B is *a right K-module coalgebra.*

We consider the following situation. Let H and K be paired bialgebras and let B be k-space equipped with an coassociative comultiplication. Let $\Delta_B\colon B \to B \otimes B$ and with maps verifying the following conditions:

(1) $\psi\colon H \otimes B \to B, h \otimes b \mapsto h \cdot b$ is a left H-module structure and $\phi\colon B \otimes K \to B, b \otimes k \mapsto b \odot k$ is a right K-module structure.
(2) $\Delta_B\colon B \to B \otimes B$ is a ψ-comultiplication
(3) $\Delta_B\colon B \to B \otimes B$ is a ϕ-comultiplication.
(4) ψ, ϕ are compatible.

DEFINITION 7. $(\psi \vee \phi)\Delta_B \colon B \to B \otimes B$ is a k-linear map from B to $B \otimes B$ and is called the *twist of the comultiplication* Δ_B with the actions ψ and ϕ. It is denoted as $\Delta_{B,\psi,\phi}$.

LEMMA 4. *In the situation above, the twist coproduct $\Delta_{B,\psi,\phi}$ is coassociative.*

Proof. The map $\Delta_{B,\psi,\phi} \colon B \to B \otimes B$ was defined by $\Delta_{B,\psi,\phi}(b) = \sum h_i \cdot b_1 \otimes b_2 \odot k_i$ where the pairing $q(1) = \sum h_i \otimes k_i$ and $\Delta_B(b) = \sum b_1 \otimes b_2$.

$$\begin{aligned}
&((id \otimes \Delta_{B,\psi,\phi})\Delta_{B,\psi,\phi})(b) \\
&\quad = \sum h_i \cdot (h_j \cdot (b_1)_1) \otimes h_i \cdot ((b_1)_2 \odot k_j) \otimes b_2 \odot k_i \\
&\quad = \sum h_i \cdot b_1 \otimes (h_j \cdot (b_2)_1) \odot (k_i)_1 \otimes ((b_2)_2 \odot k_j) \odot (k_i)_2 \\
&\quad = ((\Delta_{B,\psi,\phi} \otimes id)\Delta_{B,\psi,\phi})(b).
\end{aligned}$$

The first equality follows from the ψ-comultiplicity of Δ_B and the second equality follows from the compatibility of ψ and ϕ and from the pairing of H and K. The third equality follows from the ϕ-comultiplicity of Δ_B. □

Let H and K be paired bialgebras with multiplication $\mu_K \colon K \otimes K \to K$ and let A be an arbitrary left H-module coalgebra with comultiplication $\Delta_A \colon A \to A \otimes A, a \mapsto \sum a_1 \otimes a_2$ and with module structure $\chi_A \colon H \otimes A \to A, h \otimes a \mapsto h \cdot a$. Then $B = A \otimes K$ is a right K-module with structure map

$$\phi \colon (A \otimes K) \otimes K \to A \otimes K,$$

$$(a \otimes k) \otimes k' \mapsto (a \otimes k) \odot k' = a \otimes \mu_K(k \otimes k') = a \otimes kk'$$

and left H-module with structure map

$$\psi \colon H \otimes (A \otimes K) \to A \otimes K,$$

$$h \otimes (a \otimes k) \mapsto h \bullet (a \otimes K) = h \cdot a \otimes k.$$

Define $\Delta_B \colon (A \otimes K) \to (A \otimes K) \otimes (A \otimes K)$ as

$$\Delta_B = (id \otimes s \otimes id)(\Delta_A \otimes \Delta_K).$$

We notice that B satisfies (1),(2) and (3) in above of the Lemma 4. So we can define

$$\Delta_{B,\psi,\phi} \colon B \to B \otimes B,$$

$$\begin{aligned} a \otimes k \mapsto & \sum h_i \bullet (a \otimes k)_1 \otimes (a \otimes k)_2 \odot k_i \\ & = \sum h_i \bullet (a_1 \otimes k_1) \otimes (a_2 \otimes k_2) \odot k_i \\ & = \sum h_i \cdot a_1 \otimes k_1 \otimes a_2 \otimes k_2 k_i. \end{aligned}$$

Let H be a finite dimensional Hopf algebra and let H^* be the dual Hopf algebra of H. Then H and H^* are paired Hopf algebras by the Lemma 1. The multiplication of H^* is

$$\mu_{H^*} \colon H^* \otimes H^* \to H^*,$$

$$\mu_{H^*}(f \otimes g)(h) = \sum f(h_1) g(h_2)$$

where $\Delta_H(h) = \sum h_1 \otimes h_2$ and comultiplication of H^* is

$$\Delta_{H^*} \colon H^* \to H^* \otimes H^*,$$

$$f \mapsto \sum g_i \otimes t_i$$

where $\{g_i\}$ is basis for $H \rightharpoonup f$. Therefore

(7-1) $$\Delta_{H^*}(f)(a \otimes b) = f(ab) = \sum g_i(a) t_i(b).$$

Let A be an arbitrary left H-module coalgebra with comultiplication $\Delta_A \colon A \to A \otimes A, a \mapsto \sum a^1 \otimes a^2$ and structure map

$$\chi_A \colon H \otimes A \to A,$$

$$h \otimes a^1 \mapsto h \cdot a^1 \equiv h^{**}(a^1)_1(a^1)_0 =< h^{**}, (a^1)_1 > (a^2)_0$$

where the H^*-comodule structure map is $A \to A \otimes H^*, a^2 \mapsto \sum (a^2)_0 \otimes (a^2)_1 \cong \sum a_i \otimes f_i$ where $a^2 \in A, \{a_1, ..., a_n\}$ is a basis for $H \cdot a^2 \leq A$. Then

$$h \cdot a^2 = \sum f_i(h) a_i, h \in H \tag{7-2}$$

for some $f_i \in H^*$. The coalgebra $B = A \otimes H^*$ is a right H^*-module with structure map

$$\phi\colon (A \otimes H^*) \otimes H^* \to A \otimes H^*,$$

$$(a \otimes f) \otimes g \mapsto (a \otimes f) \odot g = a \otimes \mu_{H^*}(f \otimes g),$$

and left H-module with structure

$$\psi\colon H \otimes (A \otimes H^*) \to A \otimes H^*,$$

$$h \otimes (a \otimes f) \mapsto h \bullet (a \otimes f) = h \cdot a \otimes f \equiv h^{**}(a_1) a_0 \otimes f$$

where the H^*-comodule structure map $A \to A \otimes H^*, a \mapsto \sum a_0 \otimes a_1$. Therefore $\Delta_{B,\psi,\phi}\colon A \otimes H^* \to (A \otimes H^*) \otimes (A \otimes H^*)$ is defined by

$$\begin{aligned} a \otimes f \to & \sum h_i \cdot a^1 \otimes g_j \otimes a^2 \otimes t_j h^i \\ & \equiv \sum h_i^*(a^1)_1(a^1)_0 \otimes g_j \otimes a^2 \otimes \mu_{H^*}(t_j \otimes h^i) \end{aligned}$$

where the pairing $\delta(1) = \sum h_i \otimes h^i$.

DEFINITION 8. Let H be a bialgebra and A be a left H-module coalgebra. The *smash coproduct* $A \sharp H^{cop}$ is defined to be $A \otimes H^{cop}$ as a vector space, with comultiplication given by

$$\Delta(a \sharp h) = \sum a^1 \sharp (a^2)_1 h_2 \otimes (a^2)_0 \sharp h_1$$

and counit

$$\epsilon(a \sharp h) = \epsilon_A(a) \epsilon_H(h),$$

for all $a \in A, h \in H^{cop}$.

THEOREM 5. *Let H be a finite dimensional Hopf algebra and let A be a left H-module coalgebra. The smash coproduct is a special case of twisting coproduct.*

Proof. Consider the twisting coproduct of $B = A \otimes H^*$. We will show that smash coproduct $A\sharp(H^*)^{cop}$ is same as the twisting coproduct of B under the isomorphism. Let $\Delta_A \colon A \to A \otimes A, a \mapsto \sum a^1 \otimes a^2$ and $\Delta_H \colon H \to H \otimes H, h_s \mapsto \sum (h_s)_1 \otimes (h_s)_2$ be comultiplication of A and H and let $\{h_1, ..., h_n\}$ be a basis for H. Then for any $u, v \in H^*$ and $h_r, h_s \in \{h_1, ..., h_n\}$,

$$
\begin{aligned}
&(\Delta_{B,\psi,\phi}(a \otimes f)(u \otimes h_r \otimes v \otimes h_s) \\
&\quad = (\sum h_i \cdot a^1 \otimes g_j \otimes a^2 \otimes t_j h^i)(u \otimes h_r \otimes v \otimes h_s) \\
&\quad = \sum u(h_i \cdot a^1) \otimes g_j(h_r) \otimes v(a^2) \otimes t_j((h_s)_1) h^i((h_s)_2) \\
&\quad = \sum u(h_i \cdot a^1) \otimes 1 \otimes v(a^2) \otimes (\sum g_j(h_r) t_j((h_s)_1)) h^i((h_s)_2) \\
&\quad = \sum u(h_i \cdot a^1) \otimes 1 \otimes v(a^2) \otimes f(h_r(h_s)_1) h^i((h_s)_2) \\
&\quad = \sum u((\sum h^i((h_s)_2) h_i) \cdot a^1) \otimes 1 \otimes v(a^2) \otimes f(h_r(h_s)_1) \\
&\quad = \sum u((h_s)_2 \cdot a^1) \otimes 1 \otimes v(a^2) \otimes f(h_r(h_s)_1).
\end{aligned}
$$

The fourth equality follows from the (7-1). And

$$
\begin{aligned}
&\Delta_{A\sharp(H^*)^{cop}}(a \otimes f)(u \otimes h_r \otimes v \otimes h_s) \\
&\quad = (\sum a^1 \otimes f_i t_i \otimes a_i \otimes g_i)(u \otimes h_r \otimes v \otimes h_s) \\
&\quad = \sum u(a^1) \otimes \mu_{H^*}(f_i \otimes t_i)(h_r) \otimes v(a_i) \otimes g_i(h_s) \\
&\quad = \sum u(a^1) \otimes f_i((h_r)_1) t_i((h_r)_2) \otimes v(a_i) \otimes g_i(h_s) \\
&\quad = \sum u(a^1) \otimes t_i((h_r)_2) \otimes v(\sum f_i((h_r)_1) a_i) \otimes g_i(h_s) \\
&\quad = \sum u(a^1) \otimes 1 \otimes v((h_r)_1 \cdot a^2) \otimes \sum g_i(h_s) t_i((h_r)_2) \\
&\quad = \sum u(a^1) \otimes 1 \otimes v((h_r)_1 \cdot a^2) \otimes f(h_s(h_r)_2) \\
&\quad = \sum u(a^1) \otimes 1 \otimes v((h_r)_1 \cdot a^2) \otimes f(h_s(h_r)_2).
\end{aligned}
$$

The fifth equality follows from the (7-2) and the last equality follows from the (7-1). It is easy to show that

$$\Delta_{A\sharp(H^*)^{cop}} = (\tau\Delta_H)s_{13}(\tau\Delta_A)(\Delta_{B,\psi,\phi}(a \otimes f)s_{24}s_{13}.$$

This completes the proof. □

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Kang Ju Min
Department of Mathematics
Chungnam National University
Taejon 305-764,Korea

Jun Seok Park
Department of Mathematics
Hoseo University
Asan 336-795,Korea

E-mail: junspk@dogsuri.hoseo.ac.kr