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FIBREWISE H–COMPLETE EXTENSION OF A QUASI–UNIFORM SPACE OVER A BASE

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ABSTRACT. The purpose of this paper is to show that there exists a fibrewise H-complete extension of a quasi-uniform space over a base.

0. Introduction

The fibrewise view point is standard in the theory of the fibre bundles. It has been recognized only recently that the same view point is also of great value in other theories, such as general topology. I. M. James [5-9] has been promoting the fibrewise view point systematically in topology and uniformity and developed the theory of uniform space over a base as an extension from the category of uniform spaces.

After A. Császár had introduced quasi-uniform spaces, many researchers have been concerned with the concept of completeness of quasi-uniform structures and studied on the completions of a quasi-uniform space. Since a quasi-uniformity need not have symmetry, various type of completions can be constructed.

In this paper, we develop a theory of fibrewise H-completeness of quasi-uniform spaces over a base and construct fibrewise H-complete extension of a quasi-uniform space over a base.

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1. Preliminaries

For what we are going to do a modified form of the ordinary theory of filters is required. We begin with an outline of this, following the terminology, etc. of Bourbaki [1] as far as possible. Proofs which are simply routine modifications of the proofs of the corresponding results in the ordinary theory will be indicated briefly or omitted altogether.

We work over a quasi-uniform space B. A set over B, we recall, is a set X together with a function $p : X \to B$, called the *projection*. Usually X alone is sufficient notation. For each point $b \in B$ the fibre $X_b = p^{-1}(b)$ is defined, and for each neighborhood V of b the neighborhood $X_V = p^{-1}(V)$ of the fibre. If A is a subset of X then A may be regarded as a set over B by restriction of the projection.

Let X be a set over B. By a *tied filter* on X we mean a pair $(b, \mathcal{F})$, where b is a point of B and $\mathcal{F}$ is a filter on X of which b is a cluster point 'along p'. This means that each member M of $\mathcal{F}$ has a non-empty intersection with X_V for each neighborhood V of b. Instead of saying that $(b, \mathcal{F})$ is a tied filter we shall often use the phrase '$\mathcal{F}$ is a *b-filter*' to mean precisely the same thing. By a refinement of a tied filter $(b, \mathcal{F})$ we mean a tied filter $(b, \mathcal{F}')$, where $\mathcal{F}'$ is a refinement of $\mathcal{F}$. This relation of refinement imposes a partial order on the set of b-filters, for each b, and a b-filter which is maximal with respect to the partial order will be called a *b-ultrafilter*. In that case b is a limit point of $\mathcal{F}$ along p and so it follows that $\mathcal{F}$ is an ultrafilter in the ordinary sense. Since the union of a collection of b-filters is again a b-filter the usual argument depending on Zorn's lemma shows that each b-filter can be refined be a b-ultrafilter.

Let X and Y be sets over B. If $f : X \to Y$ is a function over B, then f preserves fibres, since for each $b \in B$, $f(X_b) \subseteq Y_b$. Hence a function over B is also called a *fibre-preserving function*.

To conclude these preliminaries consider the situation where X, as well as B, has a quasi-uniform structure and p is (quasi-)uniformly continuous. In that case we call X a *quasi-uniform space over* B.

DEFINITION 1.1. Let $(b, \mathcal{F})$ be a tied filter on X. By a *cluster point* of $(b, \mathcal{F})$ we mean a point of the fibre X_b which is a cluster point of $\mathcal{F}$ in the usual sense. Similarly, by a *limit point* of $(b, \mathcal{F})$ we mean a point of the fibre X_b which is a limit point of $\mathcal{F}$ in the usual sense.

DEFINITION 1.2. A topological space $Y$is said to be *T_2-reduced with respect to a subspace* $X \subseteq Y$ if $a \in Y, b \in Y - X$ and $a \neq b$ imply that a and b have disjoint neighborhoods.

PROPOSITION 1.3. *Let $\mathcal{B}$ be a base for a quasi-uniform space $(X, \mathcal{U})$ and for each $x \in X$, let $\mathcal{B}[x] = \{V[x] \mid V \in \mathcal{B}\}$ where $V[x] = \{y \in X \mid (x, y) \in V\}$. Then there is a unique topology on X such that, for each $x \in X$, $\mathcal{B}[x]$ is a base for the neighborhood filter of x in this topology.*

If $(X, \mathcal{U})$ is a quasi-uniform space, *the topology induced by $\mathcal{U}$* (or simply *the topology $\mathcal{T}(\mathcal{U})$ of $\mathcal{U}$*) means the topology defined in proposition 1.3.

A topological space $(X, \mathcal{T})$ is said to be *quasi-uniformizable* if there exists a quasi-uniformity $\mathcal{U}$ on X which induces $\mathcal{T}$, i.e., $\mathcal{T}(\mathcal{U}) = \mathcal{T}$. Then $\mathcal{U}$ is said to be *compatible* with $\mathcal{T}$ and $(X, \mathcal{T})$ is said to *admit* $\mathcal{U}$.

DEFINITION 1.4. Let $(X, \mathcal{U})$ be a quasi-uniform space. A filter $\mathcal{F}$ is said to be *round* in $(X, \mathcal{U})$ if, for $F \in \mathcal{F}$, there are $U_0 \in \mathcal{U}$ and $F_0 \in \mathcal{F}$ such that $U_0[F_0] \subseteq F$, where $U_0[F_0] = \cup\{U_0[x] \mid x \in F_0\}$.

The following propositions can be found in [2,3], but we restate them for the further purpose.

PROPOSITION 1.5. [3, Lemma 1.2] *Every round filter in a quasi-uniform space is an open filter.*

PROPOSITION 1.6. [2, Lemma 2.1] *Let $(X,\mathcal{U})$ be a quasi-uniform space, $X \subseteq Y$ and assign, to every element $y \in Y - X$, a round filter $\gamma(y)$ in $(X,\mathcal{U})$. Let Σ denote the collection of all maps $S : Y - X \to 2^X$ such that $S(y) \in \gamma(y)$ for $y \in Y - X$. Then, define, for $U \in \mathcal{U}$ and $S \in \Sigma$, the set $B(U,S) \subseteq Y \times Y$ in the following manner ;*

$(x_1, x_2) \in B(U,S)$ iff $(x_1, x_2) \in U$ for $x_1, x_2 \in X$

$(x, y) \notin B(U,S)$ for $x \in X$ and $y \in Y - X$

$(y, x) \in B(U,S)$ iff $x \in U[S(y)]$ for $x \in X$ and $y \in Y - X$

$(y_1, y_2) \in B(U,S)$ iff $y_1 = y_2$ for $y_1, y_2 \in Y - X$.

Now$\{B(U,S) \mid U \in \mathcal{U}, S \in \Sigma\}$ is a base for some quasi-uniform structure $\mathcal{V}$ on Y such that $(X,\mathcal{U})$ is a dense subspace of $(Y,\mathcal{V})$ and $\gamma(y)$ is the trace in X of the neighborhood filter of $y \in Y - X$. Moreover, for the topology of $(Y,\mathcal{V})$, $Y - X$ is discrete and closed.

LEMMA 1.7. *Let Y be a topological space, $X \subseteq Y$ dense, $Y - X$ discrete and closed, and $f : X \to Z$ a continuous map into a topological space Z. If, for any $y \in Y - X$, the filter $\mathcal{F}_y$ generated by the image $f(\mathcal{N}_y|_X)$ of the trace filter $\mathcal{N}_y|_X$ of the neighborhood filter $\mathcal{N}_y$ of y converges in Z, then f has a continuous extension $g : Y \to Z$.*

Proof. It suffices to define $g|_X = f$ and for $y \in Y - X$, $g(y) = z$ such that $\mathcal{F}_y$ converges to z. □

PROPOSITION 1.8. [3, Lemma 2.3] *Let $(X,\mathcal{U})$ be a quasi-uniform space, $(Y,\mathcal{V})$ one of its extensions constructed by the method of proposition 1.7 and g a map from Y into a quasi-uniform space $(Z,\mathcal{W})$ that is continuous and its restriction to X is quasi-uniformly continuous. Then g is quasi-uniformly continuous as well.*

2. Fibrewise H-complete quasi-uniform spaces

DEFINITION 2.1. Let $(X,\mathcal{U})$ be a quasi-uniform space over B with projection $p : X \to B$ and $b \in B$. A tied filter $(b,\mathcal{F})$ on X is said to be a *round b-filter* (or *round tied filter*) in $(X,\mathcal{U})$ if

(1) b is a cluster point of $p(\mathcal{F})$

(2) $\mathcal{F}$ is round in $(X,\mathcal{U})$.

PROPOSITION 2.2. *Let $(X,\mathcal{U})$ be a quasi-uniform space over B and $b \in B$. If $\mathcal{N}_x$ is the neighborhood filter in X of a point $x \in X_b$, then $(b,\mathcal{N}_x)$ is a round b-filter.*

Proof. Recall that $\mathcal{N}_x = \langle\{U[x] \mid U \in \mathcal{U}\}\rangle$. Since $x \in X_b$, $(b,\mathcal{N}_x)$ is a tied filter. Let $N \in \mathcal{N}_x$, then there exists an entourage $U \in \mathcal{U}$ such that $U[x] \subseteq N$. Since $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ with $V \circ V \subseteq U$. Then $V[V[x]] \subseteq N$ and hence $\mathcal{N}_x$ is round in $(X,\mathcal{U})$. □

PROPOSITION 2.3. *Let $(X,\mathcal{U})$ be a quasi-uniform space over B. If $(b,\mathcal{F})$ and $(b,\mathcal{G})$ are round b-filter, then $(b,\{F \cap G \mid F \in \mathcal{F}, G \in \mathcal{G}\})$ is a round b-filter finer than both $(b,\mathcal{F})$ and $(b,\mathcal{G})$ provided its elements are non-empty.*

Proof. Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Since $\mathcal{F}$ and $\mathcal{G}$ are round in $(X,\mathcal{U})$, there are $U_1, U_2 \in \mathcal{U}, F_0 \in \mathcal{F}$ and $G_0 \in \mathcal{G}$ such that $U_1[F_0] \subseteq F$ and $U_2[G_0] \subseteq G$. Then $(U_1 \cap U_2)[F_0 \cap G_0] \subseteq F \cap G$. Hence $(b,\{F \cap G \mid F \in \mathcal{F}, G \in \mathcal{G}\})$ is a round b-filter. Clearly, it is finer than $(b,\mathcal{F})$ and $(b,\mathcal{G})$. □

PROPOSITION 2.4. *Every round b-filter in $(X,\mathcal{U})$ is contained in a maximal round b-filter.*

Proof. By the above proposition 2.3 and Zorn's lemma. □

PROPOSITION 2.5. *If $(b,\mathcal{F})$ and $(b,\mathcal{G})$ are distinct maximal round b-filters, then there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \emptyset$.*

Proof. Suppose not. Then we have a round b-filter $(b, \{F \cap G \mid F \in \mathcal{F}, G \in \mathcal{G}\})$ finer that $(b,\mathcal{F})$ and $(b,\mathcal{G})$. This is a contradiction. □

PROPOSITION 2.6. *If $x \in X_b$ is a cluster point of the maximal round b-filter $\mathcal{F}$, then $\mathcal{F}$ converges to x.*

Proof. Consider the neighborhood filter $\mathcal{N}_x$. Then $(b,\mathcal{N}_x)$ is also a round b-filter. Since x is a cluster point of $(b,\mathcal{F})$, for each $F \in \mathcal{F}$ and $N \in \mathcal{N}_x$, $F \cap N \neq \emptyset$. Then $(b, \{F \cap N \mid F \in \mathcal{F}, N \in \mathcal{N}_x\})$ is also a round b-filter finer than $(b,\mathcal{F})$. Since $(b,\mathcal{F})$ is maximal, $\mathcal{N}_x \subseteq \mathcal{F}$. Hence $\mathcal{F}$ converges to x. □

PROPOSITION 2.7. *Let $(Y,\mathcal{V})$ be a quasi-uniform space over B and X be a subspace of Y.. If $(b,\mathcal{F})$ is a round b-filter in $(Y,\mathcal{U})$ and $F \cap X \neq \emptyset$ for each $F \in \mathcal{F}$, then $(b,\mathcal{F}|_X)$ is a round b-filter in $(X,\mathcal{U})$, where $\mathcal{F}|_X = \{F \cap X \mid F \in \mathcal{F}\}$ is a trace filter of $\mathcal{F}$ in X and $\mathcal{U} = \{V \cap (X \times X) | V \in \mathcal{V}\}$.*

Proof. Choose $F \in \mathcal{F}$, then there are $V_0 \in \mathcal{V}$ and $F_0 \in \mathcal{F}$ such that $V_0(F_0) \subseteq F$. Let $U = V_0 \cap (X \times X) \in \mathcal{U}$, then $U(F_0 \cap X) \subseteq F \cap X$. Hence $\mathcal{F}|_X$ is round in $(X,\mathcal{U})$. □

THEOREM 2.8. *For a quasi-uniform space $(X,\mathcal{U})$ over B, the following statements are equivalent:*

(a) Every maximal round b-filter in X is convergent in X_b.

(b) Every round b-filter has a cluster point in X_b.

(c) $(X,\mathcal{U})$ is a closed subspace of each larger space $(Y,\mathcal{V})$ T_2-reduced with respect to X where $(Y,\mathcal{V})$ is a quasi-uniform space over B with projection $q : Y \to B$ and $q|_X = p$.

Proof. (a)⇒(b) Let $(b,\mathcal{F})$ be a round b-filter. Then by proposition 2.4, there is a maximal round b-filter $(b,\mathcal{G})$ such that $\mathcal{F}\subseteq\mathcal{G}$. By the hypothesis, $(b,\mathcal{G})$ converges to x for some $x\in X_b$, i.e., $\mathcal{N}_x\subseteq G$. Then $F\cap N\neq\emptyset$ for any $F\in\mathcal{F}$ and $N\in\mathcal{N}_x$. Thus x is a cluster point of $(b,\mathcal{F})$.

(b)⇒(c) Let $y\in\overline{X}$ and $q(y)=b$, i.e., $y\in Y_b$. By the proposition 2.2 and 2.7, the trace $\mathcal{N}_y|_X$ in X of the neighborhood filter $\mathcal{N}_y$ is a round b-filter. By (b), $\mathcal{N}_y|_X$ has a cluster point x in X_b. If $y\in Y-X$, then since Y is T_2-reduced with respect to X and $x\in X$, there are neighborhoods V_1 and V_2, respectively, such that $V_1\cap V_2=\emptyset$. This is a contradiction. Hence $y\in X$. Thus X is closed in Y.

(c)⇒(a) Suppose there is a non-convergent maximal round b-filter in X. Let $X\subsetneq Y$ be chosen such that there is a bijective map γ from $Y-X$ to the set of these filters. With the help of this γ, construct the quasi-uniformity $\mathcal{V}$ described in proposition 1.6. And define a projection $q:Y\to B$ on Y by $q|_X=p$ and, for $y\in Y-X$, $q(y)=b$ such that $\gamma(y)=(b,\mathcal{F}_y)$. Then by lemma 1.7 and 1.8, q is quasi-uniformly continuous and $q|_X=p$. Hence $(Y,\mathcal{V})$ is a quasi-uniform space over B with projection $q:Y\to B$ and X is dense in Y. Let $y_1,y_2\in Y-X$ with $y_1\neq y_2$. Then there are non-convergent maximal round tied filters $\gamma(y_1)$ and $\gamma(y_2)$ in $(X,\mathcal{U})$ with $\gamma(y_1)\neq\gamma(y_2)$. By the proposition 2.5, there are $F\in\gamma(y_1)$ and $G\in\gamma(y_2)$ with $F\cap G=\emptyset$. Since $\gamma(y_1)$ and $\gamma(y_2)$ are round b-filters, there exist $U_1,U_2\in\mathcal{U}$ and $F_1\in\gamma(y_1),G_1\in\gamma(y_2)$ such that $U_1[F_1]\subseteq F$ and $U_2[G_2]\subseteq G$, and hence $U_1[F_1]\cap U_2[G_1]=\emptyset$. Choose $S_1,S_2\in\Sigma$ such that $S_1(y_1)=F_1$ and $S_2(y_2)=G_1$. Then $y_1\in B(U_1,S_1)[y_1]=U_1[S_1(y_1)]\cup\{y_1\}=U_1[F_1]\cup\{y_1\}$ and $y_2\in B(U_2,S_2)[y_2]=U_2[S_2(y_2)]\cup\{y_2\}=U_2[G_1]\cup\{y_2\}$. And, for $x\in X$ and $y\in Y-X$, there exists a non-convergent maximal round b-filter $\gamma(y)$ in $(X,\mathcal{U})$. Since $\gamma(y)$ is not convergent to x, x is not a cluster point of $\gamma(y)$ by proposition 2.6. Then there exists a neighborhood N of x and $F \in \gamma(y)$ such that $N \cap F = \emptyset$. Since $\gamma(y)$ is round in $(X, \mathcal{U})$, there are $U \in \mathcal{U}$ and $F_0 \in \gamma(y)$ such that $U[F_0] \subseteq F$. Choose $S \in \Sigma$ with $S(y) = F_0$. Then $B(U,S)[y] = U[S(y)] \cup \{y\} = U[F_0] \cup \{y\} \subseteq F \cup \{y\}$ and $N \cap B(U,S)[y] = \emptyset$. In all, Y is T_2-reduced with respect to X. Since $X \neq Y$ and $\overline{X} = Y$, $X \neq \overline{X}$, hence X is not closed in $(Y, \mathcal{V})$. □

DEFINITION 2.9. Let $(X, \mathcal{U})$ be a quasi-uniform space over B. X is said to be *fibrewise H-complete* if it satisfies the condition (a) (or (b)) in theorem 2.8.

THEOREM 2.10. *A quasi-uniform space over B is fibrewise H-complete if and only if it is closed in every larger T_2-reduced quasi-uniform space over B.*

Proof. By theorem 2.8, it is proved. □

PROPOSITION 2.11. *Let $(X, \mathcal{U})$ be a quasi-uniform space over B with projection $p : X \to B$. Let $\mathcal{G}$ be a b-filter in $(X, \mathcal{U})$. Then $\mathcal{F} = \{U[G] \mid U \in \mathcal{U}, G \in \mathcal{G}\}$ is a round b-filter coarser than $\mathcal{G}$.*

Proof. Since $\mathcal{G}$ is a b-filter, $b \in \bigcap_{G \in \mathcal{G}} \overline{p(G)}$. Since for any $G \in \mathcal{G}$ and any $U \in \mathcal{U}$, $G \subseteq U[G]$, so $p(G) \subseteq p(U[G])$. Hence $x \in \bigcap_{G \in \mathcal{G}} \overline{p(G)} \subseteq \cap\{\overline{p(U[G])} \mid U \in \mathcal{U}, G \in \mathcal{G}\}$. Thus $\mathcal{F}$ is also a b-filter. Since $G \subseteq U[G]$ for any $G \in \mathcal{G}$, $\mathcal{F} \subseteq \mathcal{G}$, i.e., $\mathcal{F}$ is coarser than $\mathcal{G}$.

Let $U[G] \subseteq \mathcal{F}$. Then there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Take any $y \in V[V[S]]$. Then $(x,y) \in V$ for some $x \in V[S]$ and so $(x,y) \in V, (a,x) \in V$ for $a \in S$. Thus $(a,y) \in V \circ V \subseteq U$, i.e., $y \in U[S]$. Hence $V[V[S]] \subseteq U[S]$, this shows that $\mathcal{F}$ is round in $(X, \mathcal{U})$. □

PROPOSITION 2.12. *Let $(X,\mathcal{U})$ be a uniform space over B, $\mathcal{G}$ an b-ultrafilter in X and $\mathcal{F}$ defined by $\{U[G] \mid U \in \mathcal{U}, G \in \mathcal{G}\}$. Then $\mathcal{F}$ is a maximal round b-filter.*

Proof. By the proposition 2.11, $\mathcal{F}$ is a round b-filter. If $\mathcal{W} \supseteq \mathcal{F}$ is a round b-filter and $R \in \mathcal{W}$, let $R_0 \in \mathcal{W}$ and $U_0 \in \mathcal{U}$ such that $U_0[R_0] \subseteq R$.Then choose a symmetric entourage $U_1 \in \mathcal{U}$ with $U_1 \circ U_1 \subseteq U_0$. Now, $X - U_1[R_0] \notin \mathcal{G}$, since $\mathcal{G}$ is a b-ultrafilter, $U_1[R_0] \in \mathcal{G}$ and $R \supseteq U_0[R_0] \supseteq U_1[U_1[R_0]] \in \mathcal{F}$, so $R \in \mathcal{F}$. Hence f is a maximal round b-filter. □

THEOREM 2.13. *A uniform space over B id fibrewise H-complete if and only if every b-ultrafilter converges in X_b.*

Proof. Let $(X,\mathcal{U})$ be a fibrewise H-complete uniform space over B and $\mathcal{G}$ be any b-ultrafilter in $(X,\mathcal{U})$. Then $\mathcal{F} = \{U[G] \mid U \in \mathcal{U}, G \in \mathcal{G}\}$ is a maximal round b-filter coarser than $\mathcal{G}$ in $(X,\mathcal{U})$ by proposition 2.12. Since $(X,\mathcal{U})$ is fibrewise H-complete, $\mathcal{F}$ converges in X_b and hence $\mathcal{G}$ also converges in X_b. Clearly the converse is true. □

3. Fibrewise H-complete extension

THEOREM 3.1. *Let $(X,\mathcal{U})$ be a quasi-uniform space over B that is not fibrewise H-complete. Then $(Y,\mathcal{V})$ constructed in the proof of theorem 2.8 is a T_2-reduced fibrewise H-complete extension of $(X,\mathcal{U})$.*

Proof. It suffices to show that $(Y,\mathcal{V})$ is fibrewise H-complete. Let $(b,\mathcal{F})$ be a round b-filter in $(Y,\mathcal{V})$. Since every round is an open filter and X is dense in Y, $F \cap X \neq \emptyset$ for any $F \in \mathcal{F}$. Hence $\mathcal{F}_1 = \mathcal{F}|_X = \{F \cap X \mid F \in \mathcal{F}\}$ is a round b-filter in $(X,\mathcal{U})$ by proposition 2.7. Let $(b,\mathcal{F}_2)$ be the maximal round b-filter containing $(b,\mathcal{F}_1)$. Then either $\mathcal{F}_2$ converges in X or $(b,\mathcal{F}_2) = \gamma(y)$ for some $y \in Y - X$. In both cases,$(b,\mathcal{F}_2)$ converges in Y_b, since $\gamma(y)$ is the trace of the neighborhood filter of $y \in Y - X$ and $q(y) = b$. Hence $(b, \mathcal{F}_1)$ and $(b, \mathcal{F})$ has a cluster point in Y_b. $\square$

There are several essentially different fibrewise H-complete extensions of a quasi-uniform space over B, in general.

EXAMPLE Let $X = (c, d)$ be an open interval with the Euclidean uniformity and $B = \{b\}$ a singleton set with trivial quasi-uniformity. Consider X as a quasi-uniform space over B with constant projection map $p : X \to B$. Then X has the T_2-reduced fibrewise H-complete extension $[c, d]$ which is a uniform space over B while the quasi-uniformity $\mathcal{V}$ in 3.1 never is a uniformity over B, since $B(U_1, S_1) \subseteq B^{-1}(U, S)$ is impossible.

However, the extension $(Y, \mathcal{V})$ in theorem 3.1, is, in the natural sense, the finest of all fibrewise H-complete extensions.

PROPOSITION 3.2. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be quasi-uniform spaces over B with projections $p : X \to B$ and $q : Y \to B$ of X and Y, respectively. Suppose $f : (X.\mathcal{U}) \to (Y, \mathcal{V})$ is a quasi-uniformly continuous map over B. Let $(b, \mathcal{F})$ be a round b-filter in $(Y, \mathcal{V})$ whose members all intersect $f(X)$. Then $\{f^{-1}(F) | F \in \mathcal{F}\}$ generates a round b-filter in $(X, \mathcal{U})$.*

Proof. Let $\mathcal{G} = \langle \{f^{-1}(F) \,|\, F \in \mathcal{F}\} \rangle$. Since $q \circ f = p$, $\mathcal{G}$ is a b-filter. For $F \in \mathcal{F}$, choose $V_0 \in \mathcal{V}$ and $F_0 \in \mathcal{F}$ such that $V_0[F_0] \subseteq F$. Since f is quasi-uniformly continuous on X, there is $U \in \mathcal{U}$ such that $U[f^{-1}(F_0)] \subseteq f^{-1}(F)$. Hence $\mathcal{G}$ is round in X. In all, $\mathcal{G}$ is a round b-filter in $(X, \mathcal{U})$. $\square$

COROLLARY 3.3. *Let X and Y be quasi-uniform space over B and $f : X \to Y$ a quasi-uniformly continuous onto function over B. If X is fibrewise H-complete, then so is Y.*

Proof. Let $(b, \mathcal{F})$ be a round b-filter in Y. Then by proposition 3.2, $\mathcal{G} = \langle \{f^{-1}(F) \,|\, F \in \mathcal{F}\} \rangle$ is a round b-filter in X. Since X is fibrewise H-complete, $\mathcal{G}$ has a cluster point x in X_b. Let $f(x) = y \in Y_b$. Since $x \in \bigcap_{F \in \mathcal{F}} \overline{f^{-1}(F)}$ and f is continuous over B, for any neighborhood N of y, there is a neighborhood W of x such that $f(W) \subseteq N$ and $W \cap f^{-1}(F) \neq \emptyset$ for all $F \in \mathcal{F}$, and so $\emptyset \neq f(W \cap f^{-1}(F)) \subseteq f(W) \cap f(f^{-1}(F)) \subseteq N \cap F$ for any $F \in \mathcal{F}$. Hence $y = f(x) \in \bigcap_{F \in \mathcal{F}} \overline{F}$. Thus $f(x)$ is a cluster point of $\mathcal{F}$. Therefore Y is fibrewise H-complete. □

LEMMA 3.4. *Let $(X, \mathcal{U})$ be a quasi-uniform space over B, $(Y, \mathcal{V})$ one of its extensions constructed by the method of theorem 3.1, and g a fibrewise map from Y into a quasi-uniform space $(Z, \mathcal{W})$ over B that is continuous and its restriction to X is quasi-uniformly continuous. Then g is also quasi-uniformly continuous over B.*

Proof. Given $W \in \mathcal{W}$, choose $W_1 \in \mathcal{W}$ with $W_1 \circ W_1 \subseteq W$, then there is $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $(g(x), g(y)) \in W_1$ and, for $y \in Y - X$, $S(y) \in \gamma(y)$ such that $g(S(y)) \subseteq W_1(g(y))$, since g is continuous on Y. Then $S \in \Sigma$. Then $(y, x) \in B(U, S)$ if and only if $x \in U(S(y))$ for $x \in X$ and $y \in Y - X$. So $(z, x) \in U$, $z \in S(y)$ and hence $(g(z), g(x)) \in W_1$ and $g(z) \in W_1(g(y))$. Therefore $(g(y), g(z)) \in W_1$ and hence $(g(y), g(x)) \in W_1 \circ W_1 \subseteq W$. □

LEMMA 3.5. *Let $(X, \mathcal{U})$ be a quasi-uniform space over B and $(Z, \mathcal{W})$ an arbitrary fibrewise H-complete extension of $(X, \mathcal{U})$. Then every maximal round b-filter in $(X, \mathcal{U})$ converges in Z.*

Proof. Let $\mathcal{G}$ be a maximal round b-filter in $(X, \mathcal{U})$, $\mathcal{H}$ the b-filter in Z generated by $\mathcal{G}$ and $\mathcal{F} = \langle \{U[S] \,|\, U \in \mathcal{G}, S \in \mathcal{H}\} \rangle$ the round b-filter in $(Z, \mathcal{W})$ constructed from $\mathcal{H}$. If $z \in Z_b$ is a cluster point of $\mathcal{F}$, then it is a cluster point of $\mathcal{G}$. In fact, every open neighborhood V of z

and every $F \in \mathcal{F}$, $V \cap F \neq \emptyset$, and hence $V \cap (F \cap X) \neq \emptyset$, since $\overline{X} = Z$. Now, for $R \in \mathcal{G}$, choose $R_0 \in \mathcal{U}$ such that $U_0[R_0] \subseteq R$. Then there is $W \in \mathcal{G}$ such that $W \cap (X \times X) = U_0$. Since $W[R_0] \in \mathcal{F}$, $\emptyset \neq V \cap W[R_0] \cap X = V \cap U_0[R_0] \subseteq V \cap R$. Since the trace in X of the neighborhood filter of z is round in $(X, \mathcal{U})$, by proposition 2.3, and maximality of $\mathcal{G}$, $\mathcal{N}_y \subseteq \mathcal{G}$, i.e., $\mathcal{G}$ converges to $z \in Z_b$. □

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