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REGULAR RELATION WITH RESPECT TO A HOMOMORPHISM

JUNG OK YU AND YOUNG CHAN LEE

ABSTRACT. In this paper we define a regular relation with respect to a homomorphism in transformation groups and we find the equivalent conditions for the relation to be an equivalence relation.

1. Introduction

In topological dynamics the classification of minimal transformation groups is a central problem. In [2] J. Auslander introduced the regular minimal sets and obtained several characterizations of these notions. P. Shoenfeld [11] extended those concepts by consideration of homomorphisms. Also, the regular relation, as the generalizations of the proximality in transformation groups, has been introduced in [12].

In this paper, we define a generalized regular relation with respect to a homomorphism as the extended notion of regular relation. This relation is obviously reflexive, symmetric and invariant, but is not transitive or closed in general. Some equivalent conditions for this relation to be transitive will be given.

2. Preliminaries

Let (X, T, h) be a transformation group. Then X is called the *phase space* and T is called the *phase group*. If $x \in X, t \in T$, then we shall write xt instead of $h(x,t)$, when there is no danger of ambiguity. Instead of (X, T, h), we shall write (X, T).

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Throughout this paper it will be assumed that the phase spaces of all transformation groups are compact Hausdorff spaces.

A closed nonempty subset A of X is called a *minimal set* if for every $x \in A$ the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called an *almost periodic point*. If X is itself minimal, we say that it is a *minimal transformation group* or *minimal set*.

DEFINITION 2.1. Let (X, T) and (Y, T) be transformation groups. A continuous map $\pi : (X, T) \to (Y, T)$(or simply, $\pi : X \to Y$) is said to be a *homomorphism* if $\pi(xt) = \pi(x)t \quad (x \in X, t \in T)$.

If Y is minimal, the homomorphism π is always onto. Especially, if π is onto, π is called an *epimorphism*. If a homomorphism π is a homeomorphism, π is called an *isomorphism*. A homomorphism π from (X, T) onto itself (not necessarily onto) is called an *endomorphism* of (X, T), and an isomorphism $\pi : (X, T) \to (X, T)$ is called an *automorphism* of (X, T). The set of automorphisms of (X, T) is denoted by $A(X)$.

Let $\{(X_i, T) | i \in I\}$ be a family of transformation groups with the same phase group T. The product transformation group $(\prod_i X_i, T)$ is defined by the condition that $(x_i | i \in I) \in \prod_i X_i$ and $t \in T$ imply $(x_i | i \in I)t = (x_i t | i \in I)$.

We define $E(X)$, the *enveloping semigroup* of (X, T), to be the closure of T in X^X, providing X^X with its product topology. The *minimal right ideal* I is the nonempty subset of $E(X)$ with $IE(X) \subset I$, which contains no proper non-empty subset of the same property.

THEOREM 2.1. ([**6**]) *Let $\pi : (X, T) \to (Y, T)$ be an epimorphism. Then there exists a unique epimorphism $\tilde{\pi} : (E(X), T) \to (E(Y), T)$ such that the diagram*

$$\begin{array}{ccc} E(X) & \xrightarrow{\tilde{\pi}} & E(Y) \\ {\scriptstyle \theta_x}\big\downarrow & & \big\downarrow{\scriptstyle \theta_{\pi(x)}} \\ X & \xrightarrow{\pi} & Y \end{array}$$

commutes $(x \in X)$. *Moreover,* $\check{\pi}$ *is also a semigroup homomorphism.*

Let u and v be two idempotents of $E(X)$. We say that u and v are *equivalent idempotents*, writing $u \sim v$ if $uv = u$ and $vu = v$. If $u \sim v$ and $v \sim w$, then $uw = uvw = uv = u$ and $wu = wvu = wv = w$. Thus $u \sim w$. Hence, the above relation is indeed an equivalence relation.

The compact Hausdorff space X carries a natural uniformity $\mathcal{U}[X]$ whose indices are all the neighborhoods of the diagonal in $X \times X$.

DEFINITION 2.2. Let (X,T) be a transformation group. Two points x and y of X are called *proximal* if for every index $\alpha \in \mathcal{U}[X]$ there exists an element $t \in T$ such that $(xt, yt) \in \alpha$. Two points, which are not proximal are called *distal.* The set of all proximal pairs of points is called the *proximal relation* and is denoted by $P(X,T)$ or, simply $P(X)$. (X,T) is said to be a *proximal(resp. distal) transformation group* if the proximal relation equals to $X \times X$(resp. $\Delta(X)$, the diagonal of X).

LEMMA 2.2. ([**6**]) *Let* (X,T) *be a transformation group, and* $x, y \in X$. *Then the following statements are pairwise equivalent;*

(1) $(x,y) \in P(X,T)$.
(2) *There exists* $p \in E(X)$ *with xp=yp.*
(3) *There exists a minimal right ideal I in E(X) such that* $xq = yq \quad (q \in I)$.

THEOREM 2.3. ([**6**]) *Let* (X,T) *and* (Y,T) *be transformation groups. Let* $\pi : (X,T) \to (Y,T)$ *be an epimorphism and let* $\check{\pi} : X \times X \to Y \times Y$ *be the map defined by* $\check{\pi}(x_1, x_2) = (\pi(x_1), \pi(x_2))$. *Then the following statements hold:*

(1) *If* $P(X,T)$ *is an equivalence relation, so is* $P(Y,T)$.
(2) $\check{\pi}P(X,T) \subset P(Y,T)$.
(3) *If* (Y,T) *is pointwise almost periodic, then* $\check{\pi}P(X,T) = P(Y,T)$.

A minimal transformation group is said to be *regular minimal* if it is isomorphic to a minimal right ideal in the enveloping semigroup.

J. Auslander [2] proved that a minimal transformation group (X, T) is regular minimal if and only if $(x, y) \in X \times X$ implies there is an endomorphism ϕ of (X, T) such that $\phi(x)$ and y are proximal. In regular minimal transformation groups, every endomorphism is in fact an automorphism [2].

DEFINITION 2.3. ([**12**]) Let (X, T) be a transformation group and $x, y \in X$. Then x and y are *regular* if there exists a ϕ in $A(X)$ such that $(\phi(x), y) \in P(X, T)$. The set of all regular pairs in X is called the *regular relation* and is denoted by $R(X, T)$ or, simply $R(X)$.

Note that a minimal transformation group (X, T) is regular minimal if and only if $R(X, T) = X \times X$.

3. Regular relation with respect to a homomorphism

In this section, we define generalized regular relations in transformation groups.

DEFINITION 3.1. Let (X, T) and (Y, T) be transformation groups and let $\pi : (X, T) \to (Y, T)$ be a homomorphism. Two points x_1 and x_2 are said to be *regular with respect to* π if $\pi(x_1)$ and $\pi(x_2)$ are regular in Y, i.e., $(\pi(x_1), \pi(x_2)) \in R(Y, T)$. The set of all regular pair with respect to π is called the *regular relation with respect to* π and is denoted by $R_\pi(X, T)$, or more briefly $R_\pi(X)$, that is,

$$\begin{aligned} R_\pi(X,T) &= \{(x_1, x_2) \in X \times X \,|\, (\pi(x_1), \pi(x_2)) \in R(Y,T)\} \\ &= \{(x_1, x_2) \in X \times X \,|\, (\phi\pi(x_1), \pi(x_2)) \in P(Y,T) \\ &\qquad \text{for some } \phi \in A(Y)\} \end{aligned}$$

For a fixed $\phi \in A(Y)$, we let

$$\begin{aligned} R_\pi^\phi(X,T) &= \{(x_1, x_2) \in X \times X \,|\, (\phi\pi(x_1), \pi(x_2) \in P(Y,T) \\ &\qquad \text{for a fixed } \phi \in A(Y)\} \end{aligned}$$

and the set of all $R_\pi^\phi(X,T)$ is denoted by $G_\pi(X,T)$, i.e.,

$$G_\pi(X,T) = \{R_\pi^\phi | \phi \in A(Y)\}.$$

Similarly, we also define the *proximal relation with respect to* π as follows ;

$$P_\pi(X,T) = \{(x_1,x_2) \in X \times X | (\pi(x_1), \pi(x_2)) \in P(Y,T)\}.$$

We also denote $P_\pi(X,T)$ as $P_\pi(X)$, simply. Clearly $P_\pi(X) \subset R_\pi(X)$.

REMARK 3.1. If we take $X = Y$ and $\pi = 1_X$, the identity map of X, then $R_\pi(X,T)$ coincides with $R(X,T)$.

THEOREM 3.1. *Let (X,T) and (Y,T) be transformation groups and let $\pi : (X,T) \to (Y,T)$ be a homomorphism. Then the followings hold:*

(1) *$R_\pi(X)$ is a reflexive, symmetric and invariant relation.*

(2) *If $E(Y)$ contains just one minimal right ideal, then $R_\pi(X)$ is an equivalence relation.*

Proof. (1) Obvious.

(2) By (1), we only show that $R_\pi(X)$ is transitive. Let I be the only minimal right ideal in $E(Y)$, and let $(x,y) \in R_\pi(X)$ and $(y,z) \in R_\pi(X)$. Then $(h\pi(x), \pi(y)) \in P(Y)$ and $(k\pi(y), \pi(z)) \in P(Y)$ for some automorphisms h and k of Y. Since $E(Y)$ contains just one minimal right ideal I,

$$h\pi(x)p = \pi(y)p \text{ and } k\pi(y)p = \pi(z)p$$

for all $p \in I$. It follows that

$$kh\pi(x)p = k\pi(y)p = \pi(z)p$$

for all $p \in I$. Since $kh \in A(Y)$, we have $(\pi(x), \pi(z)) \in R(Y)$. Therefore $(x,z) \in R_\pi(X)$. □

A minimal transformation group (M,T) is said to be *universal* if every minimal transformation group with phase group T is a homomorphic image of (M,T). That is, if (X,T) is minimal, there is a continuous map $\pi : (M,T) \to (X,T)$ such that $\pi(xt) = \pi(x)t$, for $x \in X$ and $t \in T$. For any group T, a universal minimal set exists and is unique up to isomorphism. This was proved in [2] and [8].

REMARK 3.2. Let (M,T) be a universal minimal transformation group and let $\pi : (M,T) \to (X,T)$ be a homomorphism. If X is minimal, then for a given $\psi \in A(X)$, there exists a $\phi \in A(M)$ such that $\pi\phi = \psi\pi$. Indeed, let $x \in X$. Then $(\psi(x), x)$ is an almost periodic point of $(X \times X, T)$. There exists an almost periodic point (m_1, m_2) of $(M \times M, T)$ such that

$$\check{\pi}(m_1, m_2) = (\pi(m_1), \pi(m_2)) = (\psi(x), x).$$

Define $\phi : M \to M$ by $\phi(m_2) = m_1$. Then

$$\pi\phi(m_2) = \pi(m_1) = \psi(x) = \psi\pi(m_2).$$

This shows that $\pi\phi = \psi\pi$.

REMARK 3.3. Let (M,T) be a universal minimal transformation group, and let $\pi : (M,T) \to (X,T)$ be a homomorphism. If X is regular minimal, then for a given $\phi \in A(M)$, there exists a $\psi \in A(X)$ such that $\pi\phi = \psi\pi$. In fact, let $x \in X$. Then $\pi(m) = x$ for some $m \in M$. Since $(x, \pi\phi(m))$ is an almost periodic point of $(X \times X, T)$ and X is regular minimal, there exists a $\psi \in A(X)$ such that $\psi(x) = \pi\phi(m)$. Therefore $\psi\pi(m) = \pi\phi(m)$. This shows that $\psi\pi = \pi\phi$.

Remark 3.2 and Remark 3.3 lead us to define the following homomorphisms.

DEFINITION 3.2. Let (X,T) and (Y,T) be two transformation groups. Then an epimorphism $\pi : (X,T) \to (Y,T)$ is called a *l-homomorphism* (resp. *r-homomorphism*) if for a given $\psi \in A(Y)$(resp. $\phi \in A(X)$), there exists a $\phi \in A(X)$(resp. $\psi \in A(Y)$) such that $\pi\phi = \psi\pi$.

THEOREM 3.2. *Let $\pi : (X,T) \to (Y,T)$ be a l-homomorphism with (Y,T) minimal and let the proximal relation with respect to π, $P_\pi(X,T)$, be an equivalence relation. If $\phi \neq \psi$ in $A(Y)$, then $R_\pi^\phi \cap R_\pi^\psi = \emptyset$.*

Proof. Assume $R_\pi^\phi \cap R_\pi^\psi \neq \emptyset$. There exists (x,y) such that $(x,y) \in R_\pi^\phi$ and $(x,y) \in R_\pi^\psi$, that is,

$$(\phi\pi(x), \pi(y)) \in P(Y), \ \ (\psi\pi(x), \pi(y)) \in P(Y).$$

Since π is a l-homomorphism, there exist ϕ' and ψ' in $A(X)$ such that

$$\phi\pi = \pi\phi' \text{ and } \psi\pi = \pi\psi'. \tag{1}$$

Therefore, we have

$$(\pi\phi'(x), \pi(y)) \in P(Y), \ \ (\pi\psi'(x), \pi(y)) \in P(Y).$$

From Definition 3.1, we obtain

$$(\phi'(x), y) \in P_\pi(X), \ \ (\psi'(x), y) \in P_\pi(X).$$

Since $P_\pi(X)$ is an equivalence relation, we get

$$(\phi'(x), \psi'(x)) \in P_\pi(X). \tag{2}$$

From (1) and (2), we have

$$(\pi\phi'(x), \pi\psi'(x)) \in P(Y) \text{ and } (\phi\pi(x), \psi\pi(x)) \in P(Y).$$

Put $\pi(x) = z$. Then $(\phi(z), \psi(z)) \in P(Y)$. There exists a minimal right ideal I of $E(Y)$ such that

$$\phi(z)p = \psi(z)p \ (p \in I).$$

This implies

$$\phi(zp) = \psi(zp).$$

Since (Y,T) is minimal, we obtain $\phi = \psi$. This contradicts the fact that $\phi \neq \psi$ in $A(Y)$. □

REMARK 3.4. If $P_\pi(X,T)$ is an equivalence relation, then the followings are verified easily.

(1) $(R_\pi^\phi)^{-1} = R_\pi^{\phi^{-1}}$ for ϕ in $A(Y)$.

(2) $R_\pi^\phi \circ P_\pi = P_\pi \circ R_\pi^\phi = R_\pi^\phi$ for ϕ in $A(Y)$.

THEOREM 3.3. *Let $\pi : (X,T) \to (Y,T)$ be an epimorphism, and let $\breve{\pi} : X \times X \to Y \times Y$ be the map induced by π. Then (1) implies (2):*

(1) *$P(Y,T)$ is closed.*

(2) *$R_\pi^\psi(X,T)$ is closed for all ψ in $A(Y)$.*

Moreover, if π is a l-homomorphism and Y is pointwise almost periodic, then (1),(2) and (3) are all equivalent.

(3) *$R_\pi^\phi(X,T)$ is closed for some ϕ in $A(Y)$.*

Proof. First of all, we will prove that (1) implies (2). Let $\psi \in A(Y)$, and let $((x_n, y_n))$ be a net in $R_\pi^\psi(X)$ such that (x_n, y_n) converges to (x,y). Then $(\psi\pi(x_n), \pi(y_n)) \in P(Y)$ for all n, and $(\psi\pi(x_n), \pi(y_n))$ converges to $(\psi\pi(x), \pi(y))$. Since $P(Y)$ is closed, $(\psi\pi(x), \pi(y)) \in P(Y)$, that is, $(x,y) \in R_\pi^\psi(X)$. This shows that $R_\pi^\psi(X)$ is closed.

Next, assume that π is a l-homomorphism and Y is pointwise almost periodic. It is obvious that (2) implies (3).

Finally, we show that (3) implies (1). Let $\phi \in A(Y)$. Let $((y_n, y_n'))$ be a net in $P(Y)$ such that (y_n, y_n') converges to (y, y'). Since Y is pointwise almost periodic, $\breve{\pi}P(X) = P(Y)$, by Theorem 2.3(3). Therefore, there exists a net $((x_n, x_n'))$ in $P(X)$ such that

$$(\pi(x_n), \pi(x'_n)) = (y_n, y'_n)$$

for all n. By the assumption, for given $\phi \in A(Y)$, there exists a $\psi \in A(X)$ such that $\phi\pi = \pi\psi$. Since $((x_n, x_n')) \in P(X)$ for all n, and

$$\begin{aligned}
&(3) \qquad (\phi\pi(\psi^{-1}(x_n)), \pi(x'_n)) &&= (\pi\psi(\psi^{-1}(x_n)), \pi(x'_n)) \\
&(4) &&= (\pi(x_n), \pi(x'_n)) \\
&(5) &&= (y_n, y'_n) \in P(Y),
\end{aligned}$$

it follows that

$$(\psi^{-1}(x_n), x'_n) \in R_\pi^\phi(X)$$

for all n, and $(\psi^{-1}(x_n), x'_n)$ converges to $(\psi^{-1}(x), x')$. Since $R_\pi^\phi(X)$ is closed, $(\psi^{-1}(x), x') \in R_\pi^\phi(X)$. That is,

$$(\phi\pi\psi^{-1}(x), \pi(x')) = (\pi\psi\psi^{-1}(x), \pi(x')) \tag{6}$$

$$= (\pi(x), \pi(x')) \in P(Y). \tag{7}$$

From (5) and (7), it follows that $(\phi\pi(\psi^{-1}(x_n)), \pi(x'_n)) = (y_n, y'_n)$ converges to $(\phi\pi\psi^{-1}(x), \pi(x')) = (\pi(x), \pi(x')) = (y, y')$. Therefore, we obtain $(y, y') \in P(Y)$ and hence $P(Y)$ is closed. □

THEOREM 3.4. *Let $\pi : (X,T) \to (Y,T)$ be a l-homomorphism. Then the followings hold:*

1. *$P_\pi(X)$ is an equivalence relation if and only if $R_\pi^\psi \circ R_\pi^\phi = R_\pi^{\psi\phi}$ for all ψ,ϕ in $A(Y)$.*
2. *$R_\pi(X)$ is an equivalence relation if and if for ψ, ϕ in $A(Y)$, there exists a θ in $A(Y)$ such that $R_\pi^\psi \circ R_\pi^\phi \subset R_\pi^\theta$.*

Proof. (1) Let $P_\pi(X)$ be an equivalence relation, and let $(x, z) \in R_\pi^\psi \circ R_\pi^\phi$. Then $(x, y) \in R_\pi^\phi$ and $(y, z) \in R_\pi^\psi$ for some $y \in X$, and

$$(\phi\pi(x), \pi(y)) \in P(Y) \text{ and } (\psi\pi(y), \pi(z)) \in P(Y). \tag{8}$$

Since π is a l-homomorphism, for given ϕ and ψ in $A(Y)$, we have

$$\phi\pi = \pi\phi' \text{ and } \psi\pi = \pi\psi' \tag{9}$$

for some ϕ' and ψ' in $A(X)$. From (8) and (9), it follows that

$$(\pi\phi'(x), \pi(y)) \in P(Y) \text{ and } (\pi\psi'(y), \pi(z)) \in P(Y),$$

and hence,

$$(\phi'(x), y) \in P_\pi(X) \text{ and } (\psi'(y), z) \in P_\pi(X).$$

Also, $(\psi'\phi'(x), \psi'(y)) \in P_\pi(X)$. Since $P_\pi(X)$ is an equivalence relation,

$$(\psi'\phi'(x), z) \in P_\pi(X).$$

Therefore,

$$(\pi\psi'\phi'(x), \pi(z)) = (\psi\pi\phi'(x), \pi(z)) = (\psi\phi\pi(x), \pi(z)) \in P(Y).$$

So, we obtain $(x, z) \in R_\pi^{\psi\phi}$. This show that $R_\pi^\psi \circ R_\pi^\phi \subset R_\pi^{\psi\phi}$.

Next, let $(x, z) \in R_\pi^{\psi\phi}$. Then

$$(\psi\phi\pi(x), \pi(z)) = (\psi\pi\phi'(x), \pi(z)) \in P(Y)$$

for some ϕ' in $A(X)$. Put $y = \phi'(x)$. Then $(y, z) \in R_\pi^\psi$ and

$$(\phi\pi(x), \pi(y)) = (\phi\pi(x), \pi\phi'(x)) = (\phi\pi(x), \phi\pi(x)) \in P(Y).$$

It follows that, $(x, y) \in R_\pi^\phi$ and $(y, z) \in R_\pi^\psi$ implies $(x, z) \in R_\pi^\psi \circ R_\pi^\phi$, and therefore $R_\pi^{\psi\phi} \subset R_\pi^\psi \circ R_\pi^\phi$.

Conversely, suppose that $R_\pi^\psi \circ R_\pi^\phi = R_\pi^{\psi\phi}$ for all ψ,ϕ in $A(Y)$. We show that $P_\pi(X)$ is transitive. Let $(x, y) \in P_\pi(X)$ and $(y, z) \in P_\pi(X)$. Then $(x, y) \in R_\pi^{1_Y}$ and $(y, z) \in R_\pi^{1_Y}$ imply $(x, z) \in R_\pi^{1_Y} \circ R_\pi^{1_Y} = R_\pi^{1_Y}$, which shows that $(x, z) \in P_\pi(X)$.

(2) Suppose that $R_\pi(X)$ is an equivalence relation. Let $(x, z) \in R_\pi^\psi \circ R_\pi^\phi$. There exists a $y \in X$ such that $(x, y) \in R_\pi^\phi \subset R_\pi(X)$ and $(y, z) \in R_\pi^\psi \subset R_\pi(X)$. Since $R_\pi(X)$ is an equivalence relation, $(x, z) \in R_\pi(X)$ and hence $(\theta\pi(x), \pi(z)) \in P(Y)$ for some θ in $A(Y)$, which shows that $(x, z) \in R_\pi^\theta$.

Conversely, let $(x, y) \in R_\pi(X)$ and $(y, z) \in R_\pi(X)$. Then $(x, y) \in R_\pi^\phi$ and $(y, z) \in R_\pi^\psi$ for some ϕ and ψ in $A(Y)$. Therefore $(x, z) \in R_\pi^\psi \circ R_\pi^\phi \subset R_\pi^\theta \subset R_\pi(X)$ for some θ in $A(Y)$. This shows that $R_\pi(X)$ is transitive. □

From Theorem 3.4 and Remark 3.4, we have the following theorem.

THEOREM 3.5. *If $P_\pi(X,T)$ is transitive, then $G_\pi(X,T)$ is a group under the composition.*

THEOREM 3.6. *Let $\pi : (X,T) \to (Y,T)$ be an epimorphism. The followings are mutually equivalent:*

(1) *$P_\pi(X,T)$ is an equivalence relation.*
(2) *Let $u^2 = u \in E(Y)$. Then $(\pi(x)u, \pi(y)u) \in P(Y)$ for all $(x,y) \in P_\pi(X)$.*
(3) *Let $x \in X$ and let u_1, u_2 be any equivalent idempotents of $E(Y)$. Then $(\pi(x)u_1, \pi(x)u_2) \in P(Y)$.*

Proof. (1) implies (2): Let $(x,y) \in P_\pi(X)$ and $u^2 = u \in E(Y)$. There exists an idempotent $u' \in E(X)$ such that $\tilde{\pi}(u') = u$. Since the pairs $(\pi(x), \pi(y))$, $(\pi(x), \pi(x)u) = (\pi(x), \pi(xu'))$, and $(\pi(y), \pi(y)u) = (\pi(y), \pi(yu'))$ are all in $P(Y)$, (x,y), (x, xu'), and (y, yu') are all in $P_\pi(X)$.

Since $P_\pi(X)$ is an equivalence relation, we obtain

$$(xu', yu') \in P_\pi(X),$$

and therefore

$$\begin{aligned}(\pi(xu'), \pi(yu')) &= (\pi(x)\tilde{\pi}(u'), \pi(y)\tilde{\pi}(u')) \\ &= (\pi(x)u, \pi(y)u) \in P(Y).\end{aligned}$$

(2) implies (3): Let $x \in X$ and let u_1 and u_2 be any equivalent idempotents of $E(Y)$. Then

$$(\pi(x)u_1, \pi(x)) \in P(Y)$$

for every $x \in X$ and $u_1^2 = u_1 \in E(Y)$.

Therefore (2) implies

$$(\pi(x)u_1u_2, \pi(x)u_2) = (\pi(x)u_1, \pi(x)u_2) \in P(Y).$$

(3) implies (1): It is sufficient to show that $P_\pi(X)$ is transitive. Let $(x,y) \in P_\pi(X)$ and $(y,z) \in P_\pi(X)$. There exist minimal right ideals I

and I' in $E(Y)$ such that

$$\pi(x)p = \pi(y)p \text{ and } \pi(y)q = \pi(z)q \tag{10}$$

for all $p \in I$ and $q \in I'$.

Let u_1 and u_2 be any equivalent idempotents in I and I', respectively. Then from (10),

$$\pi(x)u_1 = \pi(y)u_1 \text{ and } \pi(y)u_2 = \pi(z)u_2.$$

By the hypothesis, $(\pi(y)u_1, \pi(y)u_2) \in P(Y)$. Since

$$\begin{aligned}(\pi(y)u_1, \pi(y)u_2) &= (\pi(y)u_1u_2, \pi(y)u_2^2) \\ &= (\pi(y)u_1, \pi(y)u_2)u_2,\end{aligned}$$

$(\pi(y)u_1, \pi(y)u_2)$ is an almost periodic point of $(Y \times Y, T)$, and hence $\pi(y)u_1 = \pi(y)u_2$. Similarly, we obtain $\pi(z)u_1 = \pi(z)u_2$.

Therefore

$$\pi(x)u_1 = \pi(y)u_1 = \pi(y)u_2 = \pi(z)u_2 = \pi(z)u_1,$$

which shows that

$$(\pi(x), \pi(z)) \in P(Y),$$

and hence

$$(x, z) \in P_\pi(X).$$

This completes the proof. □

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JUNG OK YU
DEPARTMENT OF MATHEMATICS
HANNAM UNIVERSITY
TAEJON 300-791, KOREA
E-mail: joy@math.hannam.ac.kr

YOUNG CHAN LEE
DEPARTMENT OF MATHEMATICS
HANNAM UNIVERSITY
TAEJON 300-791, KOREA