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BCK–ALGEBRAS INDUCED BY EXTENDED POGROUPOIDS

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ABSTRACT. In this paper we study (positive) implicativeness of $BCK^*(X^*)$, and investigate some properties of ideals in $BCK^*(X)$.

1. Introduction

BCK-algebras and BCI-algebras were introduced by K. Iséki and Y. Imai in 1966 ([IT1, IT2, Is, MJ]), and then many authors have investigated various properties of these algebras. On the while, J. Neggers ([Ne]) introduced the notion of pogroupoid, and J. Neggers and H. S. Kim ([NK]) obtained a necessary and sufficient condition that a pogroupoid is to be a semigroup. Recently, C. K. Hur and H. S. Kim ([HK]) constructed a BCK-algebra $(X^*; *, w)$ from the extended pogroupoid $(X^*, \cdot)$, and obtain a necessary and sufficient condition for the algebraic system $(X^*; *, \cdot, w)$ to have a property $(x \cdot y) * z = (x * z) \cdot (y * z)$ for all $x, y, z \in X^*$. In this paper we study (positive) implicativeness of $BCK^*(X^*)$, and investigate some properties of ideals in $BCK^*(X)$.

2. Preliminaries

A groupoid $(X, \cdot)$ is called a *pogroupoid*([Ne]) if

(i) $x \cdot y \in \{x, y\}$,

(ii) $x \cdot (y \cdot x) = y \cdot x$,

(iii) $(x \cdot y) \cdot (y \cdot z) = (x \cdot y) \cdot z$

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for all $x, y, z \in X$. J. Neggers ([Ne]) defined an associated partially ordered set $(X, \leq)$ by $x \leq y$ iff $y \cdot x = y$. On the one hand, for a given poset $(X, \leq)$ he also defined a binary operation on X by $y \cdot x = y$ if $x \leq y$, $y \cdot x = x$ otherwise, and proved that $(X, \cdot)$ is a pogroupoid. Let $(X, \cdot)$ be a pogroupoid and let $w \notin X$. Define $w \cdot a = a \cdot w = w$ for all $a \in X^\star := X \cup \{w\}$. Then $(X^\star, \cdot)$ is a pogroupoid, called the *extended pogroupoid* of $(X, \cdot)$. Define a partial order $\leq$ on $X^\star$ by $x \leq y$ iff $y \cdot x = y$. Then $(X^\star, \leq)$ is a poset, called the *associated poset with respect to* $(X^\star, \cdot)$.

PROPOSITION 2.1. ([HK]) *Let $(X, \cdot)$ be a pogroupoid and let $(X^\star := X \cup \{w\}, \cdot)$ be the extended pogroupoid of X. Then w is the greatest element of the associated poset $(X^\star, \leq)$.*

Let X be a set with a binary operation "$*$" and a constant 0. Then $(X; *, 0)$ is called a *BCK-algebra* if it satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(V) $0 * x = 0$,

for all $x, y, z \in X$. We construct a BCK-algebra $(X^\star; *, w)$ from the extended pogroupoid $(X^\star, \cdot)$ motivated from S. Tanaka ([Ta]).

THEOREM 2.2. ([HK]) *Let $(X^\star := X \cup \{w\}, \cdot)$ be the extended pogroupoid of a pogroupoid $(X, \cdot)$. Define*

$$x * y := \begin{cases} w & \text{if } x \cdot y = x, \\ x & \text{otherwise.} \end{cases}$$

Then $(X^\star; *, w)$ is a BCK-algebra.

We denote such a BCK-algebra by $BCK^\star(X^\star)$.

3. (Positive) implicativeness in $BCK^\star(X^\star)$

The notion of positive implicative BCK-algebra was introduced by K. Iséki and S. Tanaka ([IT2]). A BCK-algebra X is said to be *positive implicative* if it satisfies $x * y = (x * y) * y$ for all x, y in X.

THEOREM 3.1. *If $(X^\star := X \cup \{w\}, \cdot)$ is an extended pogroupoid of a pogroupoid X, then the BCK-algebra $BCK^\star(X^\star)$ is positive implicative.*

Proof. For $x, y \in X$ if $y \cdot x = x$, then $x * y = x$ and $(x*y)*y = x*y = x$ in $BCK^\star(X^\star)$. If $y \cdot x = x$ does not hold, then $x * y = (x * y) * y$. □

A BCK-algebra X is said to be *commutative* if it satisfies $x * (x * y) = y * (y * x)$ for all $x, y \in X$.

THEOREM 3.2. *Let $(X^\star := X \cup \{w\}, \cdot)$ be an extended pogroupoid of a pogroupoid X. Then the BCK-algebra $BCK^\star(X^\star)$ is commutative if and only if $x \cdot y = x$ for any $x, y \in X$.*

Proof. Suppose that the BCK-algebra $BCK^\star(X^\star)$ is commutative. If there are x, y in X with $x \cdot y = x$, then $x * y = w$ in $BCK^\star(X^\star)$ and hence $x * (x * y) = x * w = x$. Since $BCK^\star(X^\star)$ is commutative, $y * (y * x) = y * y = w$, a contradiction.

Conversely, assume that $x \cdot y = x$ for any $x, y \in X$. If $x < y$ in $BCK^\star(X^\star)$, then $y = w$ and so $x * (x * y) = x * x = w$ and $y * (y * x) = y * w = w * w = w$. Otherwise, $x * (x * y) = x * x = w = y * (y * x)$. Hence $BCK^\star(X^\star)$ is commutative. □

A BCK-algebra X is said to be *implicative* if $x = x * (y * x)$ for all $x, y \in X$. With this concept, K. Iséki and S. Tanaka proved the following theorem:

THEOREM 3.3. ([IT2]) *A BCK-algebra is implicative if and only if it is both commutative and positive implicative.*

Combining with Theorem 3.1 we obtain :

COROLLARY 3.4. *Let* $(X^\star := X\cup\{w\}, \cdot)$ *be an extended pogroupoid of a pogroupoid* X. *If* $x \cdot y = x$ *for any* $x, y \in X$, *then* $BCK^\star(X^\star)$ *is implicative.*

4. $Z(x)$ and $Z(x,y)$ in $BCK^\star(X^\star)$

For a pogroupoid $(X, \cdot)$, we define $Z(x) := \{y \in X | y \cdot x = y\}$ and it is called a *terminal section* of $x \in X$. For any x and y in a BCK-algebra X, define $Z(x,y) := \{v \in X | x \leq v * y\}$. In this section we investigate the relation $Z(x)$ and $Z(x,y)$ in $BCK^\star(X^\star)$.

THEOREM 4.1. *If* $(X^\star := X \cup \{w\}, \cdot)$ *is an extended pogroupoid of a pogroupoid* X, *then* $Z(x,y) = Z(x) \cup Z(y)$ *in* $BCK^\star(X^\star)$.

Proof. Let $u \in Z(x) \cup Z(y)$. Then $u \in Z(x)$ or $u \in Z(y)$. If $u \in Z(x)$, then $u \cdot x = u$. Hence $u * x = w$ and $u * y = u$ for any $y(\neq x) \in X$ and so $x \leq u = u * y$. Therefore $u \in Z(x,y)$. Thus $Z(x) \subseteq Z(x,y)$. If $u \in Z(y)$, then $u \cdot y = u$ and so $u * y = w$. Since w is the greatest element in X, $x \leq w = u * y$, i.e., $x \leq u * y$. Therefore $u \in Z(x,y)$ and so $Z(y) \subseteq Z(x,y)$. Thus $Z(x) \cup Z(y) \subseteq Z(x,y)$.

Assume that $Z(x) \cup Z(y) \subsetneq Z(x,y)$. Then there is an element $u \in X$ such that $x \leq u * y$, $u \cdot x = x$ and $u \cdot y = y$. This means that $w = (u * y) * x = u * x = u$, a contradiction. □

Let $(X^\star := X\cup\{w\}, \cdot)$ be an extended pogroupoid of a pogroupoid X. A non-empty subset I of the X is called an *ideal* of $BCK^\star(X^\star)$ if

(i) $w \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$

for all $x, y \in X$.

THEOREM 4.2. *Let* $(X^\star := X\cup\{w\}, \cdot)$ *be an extended pogroupoid of a pogroupoid* X *and* $\emptyset \neq I \subseteq X^\star$. *Then* I *is an ideal of* $BCK^\star(X^\star)$ *if and only if, for any* x, y *in* I, $Z(x,y) \subseteq I$.

Proof. Assume that I is an ideal of $BCK^\star(X^\star)$. Let $x, y \in I$. If $u \in Z(x,y)$, then $x \leq u * y$ and so $(u * y) * x = w \in I$. Since I is an ideal of $BCK^\star(X^\star)$, $u * y \in I$ and $u \in I$. Thus $Z(x,y) \subseteq I$.

Conversely, suppose $Z(x,y) \subseteq I$ for all $x, y \in I$. Note that $w \in Z(x,y) \subseteq I$. Let $a * b \in I$ and $b \in I$. It is enough to show that $a \in I$. Since $w = (a*b)*(a*b)$, $a * b \leq a * b$ and so $a \in Z(a*b, b) \subseteq I$. Hence $a \in I$. Thus I is an ideal of $BCK^\star(X^\star)$. □

We can easily prove that the following lemma in $BCK^\star(X^\star)$.

LEMMA 4.3. *If $y \in Z(x)$ in $BCK^\star(X^\star)$, then $Z(y) \subseteq Z(x)$.*

THEOREM 4.4. *If $(X^\star := X \cup \{w\}, \cdot)$ is an extended pogroupoid of a pogroupoid X and $x_i \in X (i = 1, 2, \cdots)$, then $\cup_{i=1} Z(x_i)$ is an ideal of $BCK^\star(X^\star)$.*

Proof. Clearly, $w \in \cup_{i=1} Z(x_i)$. Let $x, y \in \cup_{i=1} Z(x_i)$. Then $x \in Z(x_j)$ and $y \in Z(x_k)$ for some j, k. By applying Theorem 4.1 and Lemma 4.3, we obtain

$$Z(x_j, x_k) = Z(x_j) \cup Z(x_k) \subseteq \cup_{i=1} Z(x_i).$$

If follows from Theorem 4.2 that $\cup_{i=1} Z(x_i)$ is an ideal of $BCK^\star(X^\star)$. □

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