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SOME PROPERTIES OF SEQUENCES IN THE FUZZY REAL LINE

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ABSTRACT. In this paper, we shall define the usual fuzzy distance between two real fuzzy points, using the usual distance between two points in $\mathbb{R}$. We introduce the fuzzy sequence in the fuzzy real line and the notion of limit of fuzzy sequence in $F_p(\mathbb{R})$, and obtain the fuzzy increasing(decreasing) sequence and fuzzy Cauchy sequence of real fuzzy points.

1. Introduction

Throughout this paper, we denote the closed interval $[0,1]$ by I, while $I_0 = (0,1]$ and $\mathbb{R}^+ = [0,\infty)$. A fuzzy set A in the set X is characterized by a membership function μ_A from X to I. The set $\tilde{P}(X)$ is the set of all fuzzy sets in X. Two fuzzy sets A and B are said to be equal iff $\mu_A(x) = \mu_B(x)$ for all $x \in X$. The support of $A \in \tilde{P}(X)$, denoted by $S(A)$, is the ordinary subset of X defined by $S(A) = \{x \in X | \mu_A(x) > 0\}$. A is said to be included in B, denoted by $A \subseteq B$, iff $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.

A fuzzy point in $\mathbb{R}$ is a fuzzy set in $\mathbb{R}$ which is zero everywhere except at the one point, say x, where it takes a value , say α , in I_0. This fuzzy point is denoted by x_α, which is called a fuzzy point with support x and the value α. The collection of all fuzzy points in X will be denoted by $F_p(X)$. A fuzzy point x_α is called an element of A, denoted by $x_\alpha \in A$, iff $\alpha < \mu_A(x)$.

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If $x_\alpha \in F_p(\mathbb{R})$, the *fuzzy absolute value* of x_α, denoted by $|x_\alpha|$, is defined by

$$|x_\alpha| = \begin{cases} x_\alpha & \text{if} \quad x \geq 0, \\ (-x)_\alpha & \text{if} \quad x < 0. \end{cases}$$

Define a relation $\leq$ in $F_p(\mathbb{R})$ by for any $x_\alpha, y_\beta \in F_p(\mathbb{R})$, $x_\alpha \leq y_\beta$ if $x < y$ or $x = y$ and $\alpha \leq \beta$.

The relation $\leq$ in $F_p(\mathbb{R})$ satisfies that for any $x_\alpha, y_\beta, z_\gamma \in F_p(\mathbb{R})$,

(1) $x_\alpha \leq x_\alpha$,

(2) $x_\alpha \leq y_\beta$ and $y_\beta \leq x_\alpha$ imply $x_\alpha = y_\beta$,

(3) $x_\alpha \leq y_\beta$ and $y_\beta \leq z_\gamma$ imply $x_\alpha \leq z_\gamma$.

We call that the relation $\leq$ in $F_p(\mathbb{R})$ is the *usual fuzzy order.* In particular, we write $x_\alpha \ll y_\beta$ if $x < y$ and $\alpha < \beta$.

For any x_α and y_β in $F_p(\mathbb{R})$, we have

$$|x_\alpha + y_\beta| \leq |x_\alpha| + |y_\beta|$$

because $x_\alpha + y_\beta = (x + y)_{\alpha\wedge\beta}$.

It is said to be *fuzzy triangle inequality* in real fuzzy points.

For a metric space (X, D), a fuzzy distance $\tilde{D}$ between fuzzy sets A and B in X is defined using D as

$$\mu_{\tilde{D}(A,B)}(\delta) = \bigvee_{\delta=D(A,B)} (\mu_A(u) \wedge \mu_B(v)) \quad \text{for all} \quad \delta \in \mathbb{R}^+$$

where $\tilde{D}$ is a mapping from $[\tilde{P}(X)]^2$ to $\tilde{P}(\mathbb{R}^+)$.

We define a distance function $d : F_p(X) \times F_p(X) \to \tilde{P}(\mathbb{R}^+)$ by the restriction of $\tilde{D}$ to $[F_p(X)]^2$. Note that each pair (x_α, y_β) in $[F_p(X)]^2$ corresponds to the fuzzy point $D(x, y)_{\alpha\wedge\beta}$ with support $D(x, y)$ and the value $\alpha \wedge \beta$.

DEFINITION 1.1. ([3]) The *usual fuzzy metric* $d : F_p(\mathbb{R}) \times F_p(\mathbb{R}) \to \tilde{P}(\mathbb{R}^+)$ is defined by

$$d(x_\alpha, y_\beta) = |x - y|_{\alpha\wedge\beta} \quad \text{for every } (x_\alpha, y_\beta) \in F_p(\mathbb{R}) \times F_p(\mathbb{R}).$$

We call the pair $(\mathbb{R}, d)$ the usual fuzzy metric space.

REMARK. ([6]) Let (X, d) be an induced fuzzy metric space. Then for all $x_\alpha, y_\beta, z_\gamma \in F_p(\mathbb{R})$ the followings hold;

(1) $d(x_\alpha, y_\beta) \in F_p(\mathbb{R}^+)$.
(2) $d(x_\alpha, y_\beta) = 0_{\alpha\wedge\beta}$ iff $x = y$.
(3) $d(x_\alpha, y_\beta) = d(y_\beta, x_\alpha)$.
(4) Except z_γ with $x \leq z \leq y$(or $y \leq z \leq x$) and $0 < \gamma < \alpha \wedge \beta$, we have $d(x_\alpha, y_\beta) \leq d(x_\alpha, z_\gamma) + d(z_\gamma, y_\beta)$.

DEFINITION 1.2. ([3]) The open fuzzy ball $B(x_\alpha; r_\alpha)$ with center x_α and radius r_α is the fuzzy set

$$B(x_\alpha; r_\alpha) = \cup\{y_\beta \in F_p(\mathbb{R}) : d(x_\alpha, y_\beta) \ll r_\alpha\}.$$

We see that $S[B(x_\alpha; r_\alpha)] = (x - r, x + r)$ and $\mu_{B(x_\alpha;r_\alpha)}(y) = \alpha$ for all $y \in (x - r, x + r)$. In this case we denote $B(x_\alpha; r_\alpha)$ by $(x - r, x + r)_\alpha$ and call it the *open fuzzy interval with the value* α.

A fuzzy set A in $\mathbb{R}$ is called an *open fuzzy set* if and only if for every $x \in S(A)$ and for every $0 \leq \lambda < \mu_A(x)$ there exits an $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon)_\lambda \subset A.$$

A fuzzy set A in $\mathbb{R}$ is *bounded* iff $S(A)$ is bounded in ordinary set $\mathbb{R}$.

DEFINITION 1.3. ([4]) A function s from the set $\mathbb{N}$ into the set $F_p(\mathbb{R})$ is called a *usual fuzzy real sequence*, denoted by $< x^{(n)}_{\alpha_n} >$, where the n-th term is the fuzzy point $x^{(n)}_{\alpha_n}$ with the support $x^{(n)}$, and the value $\alpha_n \in I_0$.

DEFINITION 1.4. ([4]) A usual real fuzzy sequence $< x^{(n)}_{\alpha_n} >$ *converges* to a fuzzy point x_α if $x^{(n)} \to x$ and $\alpha_n \to \alpha$, that is $d(x^{(n)}_{\alpha_n}, x_\alpha) \to 0_\alpha$. Thus, we have that $x^{(n)}_{\alpha_n} \to x_\alpha$ if for any given

$\epsilon > 0$ there exists a natural number N such that $n > N$ implies $|x^{(n)} - x| < \epsilon$ and $|\alpha_n - \alpha| < \epsilon$. In this case, x_α is called the *limit* of $< x^{(n)}_{\alpha_n} >$.

PROPOSITION 1.1. *A fuzzy sequence can converge to at most one fuzzy point of* $\mathbb{R}$.

PROPOSITION 1.2. ([5]) *Let* $< x^{(n)}_{\alpha_n} > \to x_\alpha$ *and* $< y^{(n)}_{\beta_n} > \to y_\beta$. *Then, we have the followings;*

(1) $< x^{(n)}_{\alpha_n} \pm y^{(n)}_{\beta_n} >$ *converges to* $x_\alpha \pm y_\beta$.
(2) *For any given* $k_\gamma \in F_p(\mathbb{R})$, $< k_\gamma x^{(n)}_{\alpha_n} >$ *converges to* $(kx)_{\gamma\wedge\alpha}$.
(3) $< x^{(n)}_{\alpha_n} y^{(n)}_{\beta_n} >$ *converges to* $(xy)_{\alpha\wedge\beta}$.
(4) *Let* $y^{(n)} \neq 0$ *for all* $n \in \mathbb{N}$ *and let* $y \neq 0$. *Then,* $< x^{(n)}_{\alpha_n} / y^{(n)}_{\beta_n} >$ *converges to* $(x/y)_{\alpha\wedge\beta}$.

Here, $x^{(n)}_{\alpha_n} \pm y^{(n)}_{\beta_n} = (x^{(n)} \pm y^{(n)})_{\alpha_n\wedge\beta_n}, x^{(n)}_{\alpha_n} y^{(n)}_{\beta_n} = (x^{(n)} y^{(n)})_{\alpha_n\wedge\beta_n}$ *and* $x^{(n)}_{\alpha_n} / y^{(n)}_{\beta_n} = (x^{(n)}/y^{(n)})_{\alpha_n\wedge\beta_n}$.

DEFINITION 1.5. ([3]) A usual real fuzzy sequence $< x^{(n)}_{\alpha_n} >$ is said to be *bounded* if $< x^{(n)} >$ is bounded, i.e, there exists a real number $M > 0$ such that $|x^{(n)}| \leq M$ for all $n \in \mathbb{N}$.

REMARK. A sequence $< x^{(n)}_{\alpha_n} >$ is bounded if and only if the set $\{x^{(n)}_{\alpha_n} | n \in \mathbb{N}\}$ is bounded in $F_p(\mathbb{R})$.

PROPOSITION 1.3. *Let the fuzzy sequence* $< x^{(n)}_{\alpha_n} >$ *converge to* x_α. *Then the fuzzy sequence* $< |x^{(n)}_{\alpha_n}| >$ *of absolute fuzzy points converges to* $|x_\alpha|$.

THEOREM 1.4. *Suppose that* $< x^{(n)}_{\alpha_n} >, < y^{(n)}_{\beta_n} >$, *and* $< z^{(n)}_{\gamma_n} >$ *are fuzzy sequences such that* $x^{(n)}_{\alpha_n} \ll y^{(n)}_{\beta_n} \ll z^{(n)}_{\gamma_n}$ *for all* $n \in \mathbb{N}$ *and* $\lim x^{(n)}_{\alpha_n} = \lim z^{(n)}_{\gamma_n}$. *Then* $< y^{(n)}_{\beta_n} >$ *is convergent and* $\lim x^{(n)}_{\alpha_n} = \lim y^{(n)}_{\beta_n} = \lim z^{(n)}_{\gamma_n}$.

Proof. Let $w_\alpha = \lim x_{\alpha_n}^{(n)} = \lim z_{\gamma_n}^{(n)}$. Then, $x^{(n)} \to w$, $z^{(n)} \to w$, $\alpha_n \to \alpha$ and $\gamma_n \to \alpha$. Since the hypothesis implies that

$$x^{(n)} < y^{(n)} < z^{(n)}$$

squeeze theorem implies that $y^{(n)} \to w$ and $\beta_n \to \alpha$. Thus $y_{\beta_n}^{(n)} \to w_\alpha$. □

2. Fuzzy increasing and decreasing sequences and Cauchy sequences in $F_p(\mathbb{R})$

DEFINITION 2.1. Let $< x_{\alpha_n}^{(n)} >$ be a sequence of real fuzzy points. We say that $< x_{\alpha_n}^{(n)} >$ is *fuzzy increasing* (*decreasing*) if it satisfies that

$$x_{\alpha_1}^{(1)} \ll x_{\alpha_2}^{(2)} \ll \cdots \ll x_{\alpha_n}^{(n)} \ll x_{\alpha_{n+1}}^{(n+1)} \ll \cdots$$
$$(x_{\alpha_1}^{(1)} \gg x_{\alpha_2}^{(2)} \gg \cdots \gg x_{\alpha_n}^{(n)} \gg x_{\alpha_{n+1}}^{(n+1)} \gg \cdots)$$

where $\inf\{\alpha_n | n \in \mathbb{N}\} \neq 0$.

PROPOSITION 2.1. *A convergent sequence of real fuzzy points is bounded.*

THEOREM 2.2.

(1) *If* $< x_{\alpha_n}^{(n)} >$ *is bounded increasing sequence, then* $\lim x_{\alpha_n}^{(n)} = x_\alpha$, *where* $x = \sup\{x^{(n)} | n \in \mathbb{N}\}$ *and* $\alpha = \sup\{\alpha_n | n \in \mathbb{N}\}$.

(2) *If* $< y_{\beta_n}^{(n)} >$ *is bounded decreasing sequence, then* $\lim y_{\beta_n}^{(n)} = y_\beta$, *where* $y = \inf\{y^{(n)} | n \in \mathbb{N}\}$ *and* $\beta = \inf\{\beta_n | n \in \mathbb{N}\}$.

Proof. (1) Since $< x^{(n)} >$ and $< \alpha_n >$ is increasing and bounded, we have $\lim\limits_{n\to\infty} x^{(n)} = \sup\{x^{(n)} | n \in \mathbb{N}\}$ and $\lim\limits_{n\to\infty} \alpha_n = \sup\{\alpha_n | n \in \mathbb{N}\}$. Similarly, we can have (2). □

DEFINITION 2.2. Let $< x_{\alpha_n}^{(n)} >$ be a sequence of real fuzzy points and let $r_1 < r_2 < \cdots < r_n < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $< x_{\alpha_{r_n}}^{(r_n)} >$ in $F_p(\mathbb{R})$ is called a *subsequence* of $< x_{\alpha_n}^{(n)} >$.

PROPOSITION 2.3. *If a sequence* $< x_{\alpha_n}^{(n)} >$ *of real fuzzy points converges to* x_α*, then every subsequence of* $< x_{\alpha_n}^{(n)} >$ *also converges to* x_α.

DEFINITION 2.3. A fuzzy sequence $< x_{\alpha_n}^{(n)} >$ of real fuzzy points is said to be a *fuzzy Cauchy sequence* if $< x^{(n)} >$ and $< \alpha_n >$ are Cauchy sequences, that is, for any given $\epsilon > 0$ there is a natural number N such that for all natural number $n, m \geq N$, we have $|x^{(n)} - x^{(m)}| < \epsilon$ and $|\alpha_n - \alpha_m| < \epsilon$.

LEMMA 2.4. *If* $< x_{\alpha_n}^{(n)} >$ *is a convergent sequence of real fuzzy points, then* $< x_{\alpha_n}^{(n)} >$ *is a fuzzy Cauchy sequence.*

LEMMA 2.5. *A fuzzy Cauchy sequence of real fuzzy points is bounded.*

THEOREM 2.6. *A fuzzy sequence* $< x_{\alpha_n}^{(n)} >$*of real fuzzy points is convergent if and only if it is a fuzzy Cauchy sequence and* $< \alpha_n >$ *does not converge to* 0.

Proof. By the Cauchy convergence criterion, it is obvious. □

DEFINITION 2.4. A fuzzy sequence $< x_{\alpha_n}^{(n)} >$ is *contractive* if there exist $C, 0 < C < 1$,

$$d(x_{\alpha_{n+2}}^{(n+2)}, x_{\alpha_{n+1}}^{(n+1)}) \leq C d(x_{\alpha_{n+1}}^{(n+1)}, x_{\alpha_n}^{(n)}) \quad \text{for all} \quad n \in \mathbb{N}$$

The number C is called the constant of the contractive fuzzy sequence.

THEOREM 2.7. *If* $< x_{\alpha_n}^{(n)} >$ *is contractive, fuzzy increasing or decreasing and* $< \alpha_n >$ *does not converge to* 0, *then it is convergent.*

COROLLARY 2.8. *If* $< x_{\alpha_n}^{(n)} >$ *is contractive and fuzzy increasing or decreasing, then it is a fuzzy Cauchy sequence.*

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