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ON δ-FRAMES AND STRONG δ-FRAMES

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ABSTRACT. We introduce δ-frames, strong δ-frames and completely distributive lattices, and investigate some relationships among those frames.

1. Introduction

It is well known [6,7,11] that for any topological space X, its topology $\Omega(X)$ is a frame. In $\Omega(X)$, there are no points of X but open subsets of X, so we call the frame $\Omega(X)$ a pointfree topology or a pointless topology.

The study of topological properties from a lattice-theoretic viewpoint was initiated by H. Wallman [17] and further developed by J. C. C. Mckinsey and A. Tarski [14], G. Nöbeling [15], and L. Lesieur [13]. In particular, C. Ehresmann [5] and J. Bénabou [2] took the decisive step of regarding local lattices as generalized topological spaces in their own right. Such a local lattice is called a frame, a term introduced by C. H. Dowker and studied by D. Papert [3,4], J. R. Isbell [10], B. Banaschewski [1], P. T. Johnstone [11], G. Gierz et al. [6], Jorge Picado [12], A. Schauerte [16], and J. Wick Pelletier [18].

In a complete lattice, there are various conditions of distributivity. The strongest one is the completely distributive law which arises very rarely. Indeed, complete Boolean algebra is completely distributive iff L is isomorphic with the power set lattice of some set X. We also

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note that continuous lattices and frames are characterized by certain distributive laws. We note that a frame L is a complete lattice but in the theory of frames, we use only finite meets. Considering countable meets, we will get more properties of frames.

We introduce the concepts of δ-frames, strong δ-frames and completely distributive lattices, and study some relationships among those concepts.

DEFINITION 1.1. Let L be a poset. We say that L is :

(1) a *lattice* if every finite subset of L has the least upper bound and the greatest lower bound.
(2) *complete* if every subset A of L has the least upper bound and the greatest lower bound.

DEFINITION 1.2 ([8,9]). Let L be a lattice.

(1) L is said to be *distributive* if for any $x,\ y,\ z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

or equivalently,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

(2) For any $x,\ y \in L$, y is said to be a *complement* of x if $x \vee y = e$ and $x \wedge y = 0$.

If L is a distributive lattice, then every element x of L has at most one complement. If x has the unique complement, then the complement of x is denoted by x'.

DEFINITION 1.3. A distributive lattice L is called a *Boolean algebra* if every element x in L has the complement x'.

DEFINITION 1.4. A complete lattice L is called a *frame* (or *complete Heyting algebra*) if for any $a \in L$ and $S \subseteq L$,

$$a \wedge (\bigvee S) = \bigvee\{\ a \wedge s\ :\ s \in S\ \}.$$

EXAMPLE 1.5.

(1) Let X be a set and $\Omega(X)$ a topology on X. Then $(\Omega(X), \subseteq)$ is a frame, where $\subseteq$ is the inclusion relation.

(2) Every complete chain is a frame.

(3) Every complete Boolean algebra is a frame.

2. δ-Frames and Strong δ-Frames

DEFINITION 2.1. A frame L is called a *δ-frame* if for any $a \in L$ and countable subset K of L,

$$a \vee (\bigwedge K) = \bigwedge\{a \vee k : k \in K\}.$$

REMARK 2.2.

(1) In a complete lattice L, $a \vee (\bigwedge K) \leq \bigwedge\{a \vee k : k \in K\}$ holds for any $K \subseteq L$ and $a \in L$, because $a \leq a \vee k$ for all $k \in K$ imply $a \leq \bigwedge\{a \vee k : k \in K\}$ and $k \leq a \vee k$ for all $k \in K$ imply $\bigwedge K \leq \bigwedge\{a \vee k : k \in K\}$; hence $a \vee (\bigwedge K) \leq \bigwedge\{a \vee k : k \in K\}$.

(2) Every complete chain L is a δ-frame. Because for any $a \in L$ and $K \subseteq L$, we have :

i) If $a \leq k$ for all $k \in K$, then $a \leq \bigwedge K$; hence

$$a \vee (\bigwedge K) = \bigwedge K = \bigwedge\{a \vee k : k \in K\}.$$

ii) If there is $k_0 \in K$ with $k_0 \leq a$, then $\bigwedge K \leq k_0 \leq a$; hence

$$a \vee (\bigwedge K) = a = a \vee k_0 \geq \bigwedge\{a \vee k : k \in K\}.$$

Thus by i), ii) and (1), L is a δ-frame.

(3) Every complete Boolean algebra L is a δ-frame. Thus the frame of regular open subsets of $\mathbb{R}$ is a δ-frame. To show this, let L be a complete Boolean algebra. Then for $a \in L$ and for any $T \subseteq L$ and $x \in T$,

$$\begin{aligned} x &= 0 \vee x \\ &= (a \wedge a') \vee x \\ &= (a \vee x) \wedge (a' \vee x)\ ; \end{aligned}$$

hence

$$\begin{aligned} \bigwedge T &= \bigwedge\{(a \vee x) \wedge (a' \vee x) : x \in T\} \\ &= (\bigwedge\{a \vee x : x \in T\}) \wedge (\bigwedge\{a' \vee x : x \in T\}). \end{aligned}$$

Thus

$$\begin{aligned} a \vee (\bigwedge T) &= \langle a \vee (\bigwedge\{a \vee x : x \in T\})\rangle \wedge \langle a \vee (\bigwedge\{a' \vee x : x \in T\})\rangle \\ &= \langle a \vee (\bigwedge\{a \vee x : x \in T\})\rangle \wedge e \\ &= \bigwedge\{a \vee x : x \in T\}. \end{aligned}$$

Therefore, L is a δ-frame by (1).

PROPOSITION 2.3. *Every δ-frame is a frame.*

EXAMPLE 2.4. A frame need not be a δ-frame. In fact, the open set lattice $C_f(\mathbb{N})$ is not a δ-frame but a frame, where $C_f(\mathbb{N})$ is the cofinite topology on the set $\mathbb{N}$ of natural numbers. To show this, let

$$K = \{\mathbb{N} - \{m\} : m \text{ is a positive odd integer}\},\ a = \mathbb{N} - \{2\}.$$

Then $a = a \vee (\bigwedge K) \neq \bigwedge\{a \vee k : k \in K\} = e$, where $\bigwedge K = int(\bigcap K)$ and $\bigvee K = \bigcup K$.

DEFINITION 2.5. A frame L is called a *strong δ-frame* if for any countable family $(A_k)_{k\in\mathbb{N}}$ of subsets of L,

$$\bigwedge_{k\in\mathbb{N}} \left(\bigvee A_k\right) = \bigvee_{f\in\prod_{k\in\mathbb{N}} A_k} \left(\bigwedge_{n\in\mathbb{N}} f(n)\right),$$

where $f = (f(n))_{n\in\mathbb{N}}$.

EXAMPLE 2.6. Let X be an infinite set endowed with the cocountable topology $C_c(X)$. Then $L = C_c(X)$ is a strong δ-frame. We note that $C_c(X)$ is closed under countable intersections. Indeed, take any countable family $(\mathcal{A}_k)_{k\in\mathbb{N}}$ of subsets of L,

$$\begin{aligned}\bigwedge_{k\in\mathbb{N}} \left(\bigvee \mathcal{A}_k\right) &= \bigcap_{k\in\mathbb{N}} \left(\bigcup \mathcal{A}_k\right)\\ &= \bigcup_{f\in\prod_{k\in\mathbb{N}} \mathcal{A}_k} \left(\bigcap_{n\in\mathbb{N}} f(n)\right)\\ &= \bigvee_{f\in\prod_{k\in\mathbb{N}} \mathcal{A}_k} \left(\bigwedge_{n\in\mathbb{N}} f(n)\right).\end{aligned}$$

PROPOSITION 2.7. *Let L be a strong δ-frame. If for each $n \in \mathbb{N}$, A_n is cover of L, then $\{\bigwedge_{n\in\mathbb{N}} f(n) : f \in \prod_{k\in\mathbb{N}} A_k\}$ is the meet of $(A_n)_{n\in\mathbb{N}}$ in $(Cov(L), \leq)$.*

Proof. Let $B = \{\bigwedge_{n\in\mathbb{N}} f(n) : f \in \prod_{k\in\mathbb{N}} A_k\}$, then B is a cover of L, because

$$\begin{aligned}\bigvee B &= \bigvee\{\bigwedge_{n\in\mathbb{N}} f(n) : f \in \prod_{k\in\mathbb{N}} A_k\}\\ &= \bigwedge_{k\in\mathbb{N}} \left(\bigvee A_k\right)\\ &= e.\end{aligned}$$

Clearly $B \leq A_n$ for any $n \in \mathbb{N}$. Suppose there is C with $C \leq A_n$ for any $n \in \mathbb{N}$. For any $c \in C$, there is a $f \in \prod_{k\in\mathbb{N}} A_k$ with $f(n) \in A_n$ and $c \leq f(n)$ $(n \in \mathbb{N})$; hence $c \leq \bigwedge_{n\in\mathbb{N}} f(n) \in B$. Thus one has $C \leq B$. □

PROPOSITION 2.8. *Every strong δ-frame L is a δ-frame.*

Proof. For any countable $K \subseteq L$ and $a \in L$, put $A_k = \{a, k\}$ $(k \in K)$, then the equation in Definition 2.5 is precisely one in Definition 2.1. □

EXAMPLE 2.9. Let $L = \{G : G$ is a regular open subset of $\mathbb{R}\}$. Then L is a δ-frame but not a strong δ-frame. Because, let $A_k = \{(p-1/k,\ p+1/k) : p \in \mathbb{Q}\}$ $(k \in \mathbb{N})$, then since $\mathbb{Q} \subseteq \bigvee A_k$, $\bigvee A_k = \mathbb{R}$ $(k \in \mathbb{N})$; hence $\bigwedge_{k\in\mathbb{N}} (\bigvee A_k) = \mathbb{R}$. Take any $f \in \prod_{k\in\mathbb{N}} A_k$, then $\bigwedge_{n\in\mathbb{N}} f(n) = \emptyset$. Thus $\bigvee_{f\in\prod_{k\in\mathbb{N}} A_k} (\bigwedge_{n\in\mathbb{N}} f(n)) = \emptyset$. Hence $\bigwedge_{k\in\mathbb{N}} (\bigvee A_k) \neq \bigvee_{f\in\prod_{k\in\mathbb{N}} A_k} (\bigwedge_{n\in\mathbb{N}} f(n))$.

DEFINITION 2.10. Let L be a complete lattice. L is said to be *completely distributive* if for any family $(A_i)_{i\in I}$ of subsets of L,

$$\bigwedge_{i\in I} (\bigvee A_i) = \bigvee_{f\in\prod_{i\in I} A_i} (\bigwedge_{j\in I} f(j)).$$

PROPOSITION 2.11. *A complete chain is completely distributive.*

Proof. Since $\bigvee A_i$ is an upper bound for $\{\bigwedge_{j\in I} f(j) : f \in \prod_{i\in I} A_i\}$,

$$\bigvee A_i \geq \bigwedge_{j\in I} f(j) \text{ for all } f \in \prod_{i\in I} A_i \text{ ; hence}$$
$$\bigvee A_i \geq \bigvee_{f\in\prod_{i\in I} A_i} (\bigwedge_{j\in I} f(j)) \text{ ; and hence}$$
$$\bigwedge_{i\in I} (\bigvee A_i) \geq \bigvee_{f\in\prod_{i\in I} A_i} (\bigwedge_{j\in I} f(j)).$$

Put $x = \bigwedge_{i\in I} (\bigvee A_i)$ and $y = \bigvee_{f\in\prod_{i\in I} A_i} (\bigwedge_{j\in I} f(j))$. If $x > y$, then we have the following cases.

Case 1. If there is no element of L strictly between x and y, then since $x \leq \bigvee A_i$ $(i \in I)$, there is $a_j \in A_j$ with $x \leq a_j$ $(j \in I)$; hence there is a choice function $f \in \prod_{i \in I} A_i$ with $f(j) \geq x$ $(j \in I)$. Thus

$$x \leq \bigwedge_{j \in I} f(j) \leq \bigvee_{f \in \prod_{i \in I} A_i} (\bigwedge_{j \in I} f(j)) = y,$$

which contradicts to the fact that $x > y$.

Case 2. If there is $z \in L$ with $x > z > y$, then since $z < \bigwedge_{i \in I} (\bigvee A_i)$, $\bigvee A_i > z$ for all $i \in I$ and there is $a_j \in A_j$ with $a_j > z$. Then there is a choice function $f \in \prod_{i \in I} A_i$ with $f(j) > z$ $(j \in I)$. Thus

$$z \leq \bigwedge_{j \in I} f(j) \leq \bigvee_{f \in \prod_{i \in I} A_i} (\bigwedge_{j \in I} f(j)) = y,$$

which contradicts to the fact that $y < z$. Therefore $x = y$. □

PROPOSITION 2.12. *If a complete lattice L is completely distributive, then L is a strong δ-frame.*

The converse of Proposition 2.12 is not true.

EXAMPLE 2.13. Let $L = C_c(\mathbb{R})$ and $A_\alpha = \{\mathbb{R} - \{\alpha\}, \mathbb{R} - \{-\alpha\}\}$ $(\alpha \in \mathbb{R}^+)$. Then

$$\bigwedge_{\alpha \in \mathbb{R}^+} (\bigvee A_\alpha) = \mathbb{R} \neq \emptyset = \bigvee_{f \in \prod_{\alpha \in \mathbb{R}^+}} (\bigwedge_{\beta \in \mathbb{R}^+} f(\beta)).$$

Thus $L = C_c(\mathbb{R})$ is not completely distributive.

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