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THE RELATIVE LUSTERNIK-SCHNIRELMANN CATEGORY OF A SUBSET IN A SPACE WITH RESPECT TO A MAP

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ABSTRACT. In this paper we shall define a relative Lusternik-Schnirelmann category of a subset in a space with respect to a map which generalizes the category of a space, the category of a map and the relative category of a subset in a space. We shall study some properties of the relative Lusternik-Schnirelmann category of a subset in a space with respect to a map and generalize many results of the above categories.

1. Introduction

The notion of category of a space was proposed by Lusternik and Schnirelmann[7] in 1934, and proved that, when X is a smooth manifold, cat X gives a lower bound for the number of critical points of a smooth function on X. The definition adopted here is due to R. H. Fox[3]. He altered the origin definition by replacing closed by open coverings as follows; *The category, cat X, of a topological space X* is the least integer n such that X can be covered by the n open subsets each of which is contractible in X; if there is no such integer, cat $X = \infty$. The notion of category of a space can be generalized in a number of ways. One of these is the notion of the *category of a map*, due to Berstein and Ganea[1]. For a map $f : X \to Y$, the *category, cat f, of a map f* is the least integer $n \geq 1$ with the property that X may be covered by n open subsets on each of which f is homotopic to a constant map; if

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no such integer exists, we put $cat\ f = \infty$. Of course $cat\ f$ reduces to $cat\ X$ when $X = Y$ and f is the identity. For a subset A of X and an inclusion $i : A \to X$, $cat\ i$ is generally written as $cat_X A$ and called *the relative category, $cat_X A$, of a subset A in X*. In this paper we shall define a Lusternik-Schnirelmann category of a subset with respect to a map which generalizes the above several Lusternik-Schnirelmann categories. We shall study some properties of the Lusternik-Schnirelmann category of a subset with respect to a map and generalize many results of the above categories.

2. The relative Lusternik-Schnirelmann Category of a subset in a space with respect to a map

DEFINITION 2.1. Let A be a subset of X and $f : X \to Y$ a map. Then the *relative Lusternik-Schnirelmann category, $cat_Y\ f_A$, of A in X with respect to f* is the least integer n with the property that A can be covered by the n open subsets in A each of which f is homotopic to a constant map. If no such covering exists we say that the relative category of A in X with respect to f is infinite.

REMARK 2.2. (1) If $f = 1_X$,then $cat_X\ (1_X)_A = cat_X\ A$.

(2) In fact, $cat_Y\ f_A = cat\ (fi)$, where $i : A \to X$ is the inclusion.

(3) If $f \sim g : X \to Y$,then $cat_Y\ f_A = cat_Y\ g_A$.

(4) $cat_Y\ f_A = 1$ if and only if $f\,|_A \sim * : A \to Y$.

(5) If $A = X$, then $cat_Y\ f_A = cat\ f$.

The following Theorem 2.3 says that $cat_Y\ f_A$ is subadditive.

THEOREM 2.3. *If $f : X \to Y$ and $A \subset X$ and $A = A_1 \cup A_2$ and A_1, A_2 be open subsets of A, then $cat_Y\ f_A \leq cat_Y\ f_{A_1} + cat_Y\ f_{A_2}$.*

Proof. Let $\mathrm{cat}_Y\ f_{A_1} = m$ and $cat_Y\ f_{A_2} = n$. Then there exist coverings $\{U_i \,|\, U_i$: open in A_1, $f\,|_{U_i} \sim * : U_i \to Y, i = 1, \cdots, m\}$ of A_1 and $\{V_j \,|\, V_j$: open in A_2, $f\,|_{V_j} \sim *' : V_j \to Y, i = 1, \cdots, n\}$ of A_2. Now we show that $\{U_i,\ V_j \,|\, U_i,\ V_j$ are open in A_1, A_2 respectively,

$f\,|_{U_i} \sim * : U_i \to Y,\ f\,|_{V_j} \sim * : V_j \to Y,\ i = 1, \cdots, m,\ j = 1, \cdots, n\}$ is an open covering of A. Since U_i is open in A_1 and A_1 is open in A, U_i is open in A. Similarly, V_j is open in A. Since $A_1 = \bigcup_i U_i$, $A_2 = \bigcup_j V_j$ and $A = A_1 \cup A_2$, $\{U_i,\ V_j\}$ is an open covering of A. Hence $cat_Y\ f_A \leq cat_Y\ f_{A_1} + cat_Y\ f_{A_2}$. □

COROLLARY 2.4. [1] *If* $X = A \cup B$ *and* A, B *are open in* X, *then* $cat\ f \leq cat\ f\,|_A + cat\ f\,|_B$.

THEOREM 2.5. *If* $A \subset B \subset X$, *then* $cat_Y\ f_A \leq cat_Y\ f_B$.

Proof. Let $\{V_\alpha\}$ be a covering of B such that each V_α is open in B and on each V_α f is null homotopic. Since $B \subset \bigcup_\alpha V_\alpha$ and A is subset of B, $A \subset \bigcup_\alpha (V_\alpha \cap A)$. Thus $\{V_\alpha \cap A\}$ is a covering of A such that each $V_\alpha \cap A$ is open in A and on each $V_\alpha \cap A$ f is null homotopic. This proves the theorem. □

Taking $B = X$ in Theorem 2.5, we have the following corollary.

COROLLARY 2.6. $cat_Y\ f_A \leq cat\ f$ *for any subset* A *of* X.

THEOREM 2.7. *For any two maps* $f : X \to Y$, $g : Y \to Z$ *and a subset* A *of* X, $cat_Z\ (g \circ f)_A \leq \min\ \{cat_Y\ f_A,\ cat_Z\ g_{f(A)}\}$.

Proof. (1) We show that $cat_Z\ (g \circ f)_A \leq cat_Y\ f_A$. Let $\{U_\alpha\}$ be a covering of A such that for each α, U_α is open in A and $f \circ i_\alpha \sim * : U_\alpha \to Y$. Then $g \circ f \circ i_\alpha \sim g \circ * = c_{g(*)} : U_\alpha \to Z$. Thus $cat_Z\ (g \circ f)_A \leq cat_Y\ f_A$. (2) We show that $cat_Z\ (g \circ f)_A \leq cat_Z\ g_{f(A)}$. Let $\{V_\alpha\}$ be a covering of $f(A)$ such that for each α, V_α is open in $f(A)$ and $g \circ i'_\alpha \sim * : V_\alpha \to Z$. Then for each α, there exists an open U_α in Y such that $V_\alpha = f(A) \cap U_\alpha$. Since f is continuous, $f^{-1}(U_\alpha)$ is open in X and $f^{-1}(U_\alpha) \cap A$ is open in A. Moreover $A \subset f^{-1} \circ f(A) = f^{-1}(\bigcup_\alpha V_\alpha) = \bigcup_\alpha (f^{-1}(U_\alpha) \cap A)$.

Consider the following commutative diagram

$$\begin{array}{ccc} f^{-1}(U_\alpha)\cap A & \xrightarrow{i_\alpha} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ V_\alpha & \xrightarrow{i'_\alpha} & Y. \end{array}$$

Then $g\circ f\circ i_\alpha = g\circ i'_\alpha\circ f \sim *\circ f = *' : f^{-1}(V_\alpha)\to Z$. Thus $cat_Z\ (g\circ f)_A \le cat_Z\ g_{f(A)}$. From (1) and (2), we know that $cat_Z\ (g\circ f)_A \le \min\ \{cat_Y\ f_A,\ cat_Z\ g_{f(A)}\}$. □

From Theorem 2.7 and Corollary 2.6, we have the following corollary.

COROLLARY 2.8. [4] *For any two maps $f: X\to Y$ and $g: Y\to Z$ we have cat $(g\circ f)\le$ min $\{cat\ f,\ cat\ g\}$. In particular cat $f\le$ cat X and cat $f\le$ cat Y.*

COROLLARY 2.9. *For any map $f: X\to Y$ and any subset A of X, $cat_Y\ f_A \le \min\ \{cat_Y\ f(A),\ cat_X\ A\}$.*

Proof. In Theorem 2.7, take $g = i_{f(A)} : f(A)\to Y$. Since $cat\ i_{f(A)} = cat_Y\ f(A)$ and $cat_Y\ f_A \le cat\ i_A = cat_X\ A$, we know, from Theorem 2.7, that $cat_Y\ f_A \le \min\ \{cat_Y\ f(A),\ cat_X\ A\}$. □

COROLLARY 2.10. *Let $h: X'\to X$ has a left homotopy inverse $k: X\to X'$. Then for a subset A' of X', $cat_X\ h_{A'} = cat_{X'}\ A'$.*

Proof. Since $cat_X\ h_{A'} = cat\ hi'$ and $cat_{X'}\ A' = cat\ i'$, we know, from Corollary 2.8, that $cat_X\ h_{A'} \le cat_{X'}\ A'$. Thus we show that $cat_{X'}\ A' \le cat_X\ h_{A'}$. Let $\{V_\alpha\}$ be a covering of A' such that for each α, V_α is open in A' and $hi_\alpha \sim * : V_\alpha \to X$. Thus $i_\alpha \sim khi_\alpha \sim c_{k(*)} : V_\alpha \to X'$. Therefore $cat_{X'}\ A' \le cat_X\ h_{A'}$. This proves the corollary. □

THEOREM 2.11. *For a map $f: X\to Y$, let $h: X'\to X$ be a homotopy equivalence with homotopy inverse $k: X\to X'$. Let A' be a subset of X' such that $k\circ h(A')\subset A'$. Then $cat_Y\ (f\circ h)_{A'} = cat_Y\ f_{h(A')}$.*

Proof. From Theorem 2.7, we have that $cat_Y\ (f\circ h)_{A'} \leq cat_Y\ f_{h(A')}$. Thus we show that $cat_Y\ f_{h(A')} \leq cat_Y\ (f\circ h)_{A'}$. Let $\{V_\alpha\}$ be a covering of A' such that for each α, V_α is open in A' and $f\circ h\circ i'_\alpha \sim * : V_\alpha \to Y$. Then for each α, there exists an open set U_α in X' such that $V_\alpha = A'\cap U_\alpha$. Since k is continuous, $k^{-1}(U_\alpha)$ is open in X and $h(A')\cap k^{-1}(U_\alpha)$ is open in $h(A')$. Since $\{V_\alpha\}$ is a covering of A' and $k\circ h(A')\subset A'$, $h(A')\subset k^{-1}(A')\subset k^{-1}(\bigcup_\alpha V_\alpha)\subset \bigcup_\alpha[k^{-1}(U_\alpha)]$. Thus $h(A')\subset\bigcup_\alpha(h(A')\cap k^{-1}(U_\alpha))$. Since $f\circ h\circ i'_\alpha \sim * : V_\alpha\to Y$, there exists a continuous function $K: V_\alpha\times I\to Y$ such that $K(\ ,0) = f\circ h\circ i'_\alpha$ and $K(\ ,1) = c_*$. Define $H: h(A')\cap k^{-1}(U_\alpha)\times I\to Y$ by $H(x,t) = K(k(x),t)$. Then since K and k are continuous maps, H is continuous map. From the following commutative diagram

$$\begin{array}{ccc} V_\alpha & \xrightarrow{i'_\alpha} & X' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ h(A')\cap k^{-1}(U_\alpha) & \xrightarrow{i_\alpha} & X, \end{array}$$

$H(x,0) = K(k(x),0) = f\circ h\circ i'_\alpha\circ k(x) = f\circ i_\alpha\circ h\circ k(x)$. Since $1_X \sim hk : X\to X$, there exists a continuous function $R: X\times I\to X$ such that $R(\ ,0) = 1_X$, $R(\ ,1) = h\circ k$. Define $G: h(A')\cap k^{-1}(U_\alpha)\times I\to Y$ by

$$G(x,t) = \begin{cases} f\circ R(x,2t) & \text{if} \quad 0\leq t\leq \frac{1}{2}, \\ H(x,2t-1) & \text{if} \quad \frac{1}{2}\leq t\leq 1. \end{cases}$$

For $t=\frac{1}{2}$, $f\circ R(x,1) = f\circ h\circ g(x) = f\circ i_\alpha\circ h\circ g(x) = H(x,0)$. Thus G is well-defined and continuous. Since $G(x,0) = f\circ R(x,0) = f(x) = f\circ i_\alpha(x)$, $G(x,1) = H(x,1) = K(k(x),1) = c_*$, $f\circ i_\alpha \overset{G}{\sim} c_* : h(A')\cap k^{-1}(U_\alpha)\to Y$. Hence $\{h(A')\cap k^{-1}(U_\alpha)\}$ is an open covering of $h(A')$ and $f\circ i_\alpha \sim * : h(A')\cap k^{-1}(U_\alpha)\to Y$. Therefore $cat\ f_{h(A')}\leq cat\ f\circ h_{A'}$. This proves the theorem. $\square$

In Theorem 2.11, taking $f = 1_X : X \to X$ and applying Corollary 2.10 and Remark 2.2(1), we have the following corollary.

COROLLARY 2.12. [4] *Let $h : X' \to X$ be a homotopy equivalence and A' a subset of X'. Then $cat_{X'}\ A' = cat_X\ h(A')$. In particular, $cat\ X' = cat\ X$.*

From now on we turn our attention to path connected spaces with base point $*$. If there is an open neighborhood of $*$ which is contractible in a space, then we describe the space as *categorically well based.* In the n-fold topological product $\Pi^n X$ of a pointed space X with itself, let $T^n X$ be the subspace of $\Pi^n X$ consisting of the n-tuples $(x_1, \cdots, x_n)$ such that a least one of the x_i equals $*$. Then we have the following theorem which is a generalized result of James[4].

THEOREM 2.13. *Suppose that A is a normal subspace of X and Y is a path-connected and categorically well based space, and $f : X \to Y$ is a continuous map. Then $cat_Y\ f_A \leq n$ if and only if there exist a continuous function $g : A \to T^n Y$ such that the following diagram is homotopy commutative ;*

$$\begin{array}{ccc} A & \xrightarrow{g} & T^n Y \\ {\scriptstyle fi_A}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{\Delta} & \Pi^n Y. \end{array}$$

Proof. Since $\Delta \circ f \circ i_A$ and $j \circ g$ are homotopic , there exists a continuous function $h_t : A \to \Pi^n Y$ such that $h_0 = \Delta \circ f \circ i_A$, $h_1 = j \circ g$. Since Y is categorically well-based, there exist an open neighborhood N of $*$ which N is contractible in Y, that is, $i \sim * : N \hookrightarrow Y$. For all $k = 1, \cdots, n$, let $U_k = h_1^{-1} \circ p_k^{-1}(N)$, where $p_k : \Pi^n Y \to Y$ is the projection. Then U_k is open in A. Since $i \sim * : N \to Y$, there exists a continuous map $K_t : N \to Y$ such that $K_0 = i$ and $K_1 = c_*$. Let $\gamma_t = K_t \circ p_k \circ h_1 : U_k \xrightarrow{p_k \circ h_1} N \xrightarrow{K_t} Y$. Then $\gamma_t : U_k \to Y$ is continuous and

$$\gamma_0 = K_0 \circ p_k \circ h_1 = i \circ p_k \circ h_1 = p_k \circ h_1,$$
$$\gamma_1 = K_1 \circ p_k \circ h_1 = c_* \circ p_k \circ h_1 = c_*.$$

Define $R_t : U_k \to Y$ by

$$R_t = \begin{cases} p_k \circ h_{2t} & \text{if} \quad 0 \le t \le \frac{1}{2}\,, \\ \gamma_{2t-1} & \text{if} \quad \frac{1}{2} \le t \le 1. \end{cases}$$

By the pasting lemma, R_t is continuous and

$$R_0(x) = p_k \circ h_0(x) = p_k \circ (\Delta \circ f \circ i_A)(x) = f(x) = f \circ i_k(x),$$
$$R_1(x) = \gamma_1(x) = c_*(x) = *,$$

where $i_k : U_k \to X$ is the inclusion. Thus $f \circ i_k \sim * : U_k \to Y$. Moreover, since $\bigcup U_k = \bigcup [h_1^{-1} \circ p_k^{-1}(N)] = h_1^{-1}[\bigcup p_1^{-1}(N)] = h_1^{-1}[(N \times X \times \cdots \times X) \bigcup (X \times N \times \cdots \times X) \bigcup \cdots \bigcup (X \times X \times \cdots \times N)] = h_1^{-1}[\Pi^n X] = A$, $\{U_k\}$ is a covering of A. Hence $cat_Y\ f_A \le n$. Conversely, suppose $\{V_k | V_k$ is open in $A, 1 \le k \le n\}$ is a covering of A each of which f is homotopic to a constant map. For each $k = 1, \cdots, n$, there exists a continuous function $g_k : V_k \times I \to Y$ such that

$$g_k(\ ,0) = f \circ i_k,\ g_k(\ ,1) = c_{x_k},$$

where $i_k : V_k \to X$ is the inclusion. Since Y is path-connected, for each $k = 1, \cdots, n$,there exists a path $p_k : I \to Y$ such that $p_k(0) = x_k,\ p_k(1) = *$. Let $h_k : V_k \times I \to Y$ be given by

$$h_k(x,t) = \begin{cases} g_k(x, 2t) & \text{if} \quad 0 \le t \le \frac{1}{2}\,, \\ p_k(2t-1) & \text{if} \quad \frac{1}{2} \le t \le 1. \end{cases}$$

Then h_k is continuous and $h_k(\ ,0) = g_k(\ ,0) = f \circ i_k$, $h_k(\ ,1) = p_k(1) = c_*$. Since A is normal, there exists a covering $\{A_1, \cdots, A_n \mid A_k$ is closed in A, for all $k\}$ of A with the property that for each $k = 1, \cdots, n$, there exists an open set W_k in A such that $A_k \subset W_k \subset \bar{W}_k \subset V_k$. Since A_k and $A - W_k$ are disjoint sets, by the Urysohn's lemma, there exists a map $\gamma_t : A \to I$ such that $\gamma_k(A_k) = 1,\ \gamma_k(A - W_k) = 0$. Define a map $d_k : A_k \times I \to Y$ by

$$d_k(a,t) = \begin{cases} f(a) & \text{if} \quad a \in A - \bar{W}_k, \\ h_k(a, t\gamma_k(a)) & \text{if} \quad a \in V_k. \end{cases}$$

Then d_k is well defined and continuous. Let $d : A \times I \xrightarrow{\Delta} (A \times I) \times \cdots \times (A \times I) \xrightarrow{d_1 \times \cdots \times d_n} Y \times \cdots \times Y$. Then d is continuous and $d(a,0) = (d_1(a,0), \cdots, d_n(a,0)) = (f(a), \cdots, f(a)) = \Delta \circ f \circ i_A(a)$. Let $a \in A = \bigcup A_k$. Then there exists a subset A_{k_0} such that $a \in A_{k_0}$ and $\gamma_{k_0}(a) = 1$. Since $a \in A_{k_0} \subset V_{k_0}$,

$$d_{k_0}(a,1) = h_{k_0}(a, \gamma_{k_0}(a)) = h_{k_0}(a,1) = c_*(a) = *.$$

Thus $d(a,1) = (d_1(a,1), \cdots, d_n(a,1)) \in T^nY$. Let $g = d(\ ,1) : A \to T^nY$. Then $\Delta \circ f \circ i_A \overset{d}{\sim} j \circ g$. This proves the theorem. □

COROLLARY 2.14. [4] *Suppose that X is normal, and Y is a path-connected and categorically well based space, and $f : X \to Y$ is a continuous map. Then cat $f \leq n$ if and only if there exist a continuous function $g : X \to T^nY$ such that the following diagram is homotopy commutative ;*

$$\begin{array}{ccc} X & \xrightarrow{g} & T^nY \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{\Delta} & \Pi^nY. \end{array}$$

THEOREM 2.15. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and $B' \subset B$ a path connected categorically well based such that $E' = p^{-1}(B')$ is a normal categorically well based path connected space. Then $cat_E\ E' \leq cat_{E'}\ i \cdot cat_{B'}\ p_{E'}$. In particular, $cat_E\ E' \leq cat_{E'} F \cdot cat B'$.*

Proof. Let $cat_{B'}\ p_{E'} = n$. Then by Theorem 2.13, there exists a map $\phi : E' \to T^nB'$ such that the following diagram is homotopy commutative ;

$$\begin{array}{ccc} E' & \xrightarrow{\phi} & T^n B' \\ {\scriptstyle pi'}\downarrow & & \downarrow{\scriptstyle j_{B'}} \\ B' & \xrightarrow{\Delta^n_{B'}} & \Pi^n B', \end{array}$$

where $i' : E' \to E$ is the inclusion. Thus there exists a continuous function $H : E' \times I \to \Pi^n B'$ such that

$$H(\ ,0) = \Delta^n_{B'} \circ (p \circ i') = \Pi^n (p \circ i') \Delta^n_{E'},$$
$$H(\ ,0) = j_{B'} \circ \phi.$$

Consider the following commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{\Delta^n_{E'}} & E' \times \cdots \times E' \\ {\scriptstyle pi'}\downarrow & & \downarrow{\scriptstyle \Pi^n(pi)} \\ B' & \xrightarrow{\Delta^n_{B'}} & B' \times \cdots \times B'. \end{array}$$

Since $p \circ i' : E' \to B'$ is a fibration, $\Pi^n (p \circ i') : \Pi^n E' \to \Pi^n B'$ is a fibration. For the commutative diagram

$$\begin{array}{ccc} E' \times \{0\} & \xrightarrow{\Delta^n_{E'}} & \Pi^n E' \\ \downarrow{\scriptstyle inj} & & \downarrow{\scriptstyle \Pi^n(pi')} \\ E' \times I & \xrightarrow{H} & \Pi^n B', \end{array}$$

there exists a continuous function $G : E' \times I \to \Pi^n E'$ such that

$$G(\ ,0) = \Delta^n_{E'},\ \ \Pi^n (p \circ i') \circ G = H.$$

Let $\phi' = G(\ ,1) : E' \times \{1\} \to \Pi^n E'$. Then $\Pi^n (p \circ i') \circ \phi' = \Pi^n (p \circ i') \circ G(\ ,1) = H(\ ,1) = j_{B'} \circ \phi : E' \to \Pi^n B'$. Therefore

$$\Pi^n (p \circ i) \circ \phi' = j_{B'} \circ \phi \tag{1}$$

Let $cat_{E'}\, i_F = m$. By Theorem 2.13, there exists a function $\theta : F \to T^m E'$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} F & \xrightarrow{\theta} & T^m C' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j_{E'}} \\ E' & \xrightarrow{\Delta^m_{E'}} & \Pi^m E'. \end{array}$$

Thus there exists a continuous function $K : F \times I \to \Pi^m E'$ such that $K(\ ,0) = \Delta^m_{E'} \circ i, K(\ ,1) = j_{E'} \circ \theta$. Since $(B', *)$ is a closed cofibred pair, by the Strϕm's theorem [11], (E', F) is a cofibred pair. That is, $F \xrightarrow{i} E'$ is cofibration. Thus for a space $\Pi^m E'$, a map $g = \Delta^m_{E'} \circ i$ and a continuous function $K : F \times I \to \Pi^m E'$ such that $K(\ ,0) = g$, there is a continuous function $\bar{K} : E' \times I \to \Pi^m E'$ such that

$$\bar{K}(\ ,0) = \Delta^m_{E'},\ \ \bar{K} \circ (i \times 1) = K.$$

Let $\tau = \bar{K}(\ ,1) : E' \to \Pi^m E'$. Then $\tau \circ i = \bar{K}\,|_{F\times\{1\}} = K(\ ,1) = j_{E'} \circ \theta$. That is,

$$\tau \circ i = j_{E'} \circ \theta \tag{2}$$

Since the following diagram is commutative

$$\begin{array}{ccc} E' & \xrightarrow{\Delta^n_{E'}} & E' \times \cdots \times E' \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \Pi^n(\Delta^m_{E'})} \\ E' & \xrightarrow{\Delta^{mn}_{E'}} & (E' \times \cdots \times E') \times \cdots \times (E' \times \cdots \times E'), \end{array}$$

$\Pi^n(\tau) \circ \phi' \overset{\bar{K}\phi'}{\sim} \Pi^n(\Delta^m_{E'}) \circ \phi' \overset{\Pi^n(\Delta^m_{E'})\circ G}{\sim} \Pi^n(\Delta^m_{E'}) \circ \Delta^n_{E'} = \Delta^{mn}_{E'}$. That is,

$$\Pi^n(\tau) \circ \phi' \sim \Delta^{mn}_{E'}.$$

Let $x \in E'$. We know, by (1), that

$$\Pi^n(p \circ i) \circ \phi'(x) = j_{B'} \circ \phi(x) = \phi(x) \in T^n B'.$$

Thus there is an element $x_f \in F$ such that $\phi'(x) = (x_1, x_2, \cdots, x_f, \cdots, x_n)$, where F is subset of E'. From (2), we know that $\tau \circ i = j_{E'} \circ \theta$. Thus

$$\begin{aligned}\Pi^n(\tau) \circ \phi'(x) &= \Pi^n(\tau)(x_1, x_2, \cdots, x_f, \cdots, x_n) \\ &= (\tau(x_1), \tau(x_2), \cdots, \tau(x_f), \cdots, \tau(x_n)) \\ &= (\tau(x_1), \tau(x_2), \cdots, \theta(x_f), \cdots, \tau(x_n)) \\ &= (\tau(x_1), \tau(x_2), \cdots, (\cdots, *, \cdots), \cdots, \tau(x_n)),\end{aligned}$$

where $\theta : F \to T^m E'$ and $(\cdots, *, \cdots) \in T^m E'$. Therefore at least one coordinates of $\Pi^n(\tau) \circ \phi'(x)$ is the base point of E'. That is, there is a map $\Pi^n(\tau) \circ \phi' : E' \to T^{mn} E'$ such that

$$j\Pi^n(\tau) \circ \phi' \sim \Delta_{E'}^{mn}.$$

Hence by Theorem 2.13, $cat_E\ E' \leq mn = cat_{E'}\ i_F \cdot cat_{B'}\ p_{E'}$.

In particular, we have, from Corollary 2.9, that

$$\begin{aligned}&cat_{E'}\ i_F \leq cat_{E'}\ i(F) = cat_{E'}\ F \\ &cat_{B'}\ p_{E'} \leq cat_{B'}\ p(E') = cat_{B'}\ B' = cat\ B'.\end{aligned}$$

Thus $cat_E\ E' \leq cat_{E'} F \cdot cat\ B'$. □

In Theorem 2.15, taking $B' = B$, we have the following corollary.

COROLLARY 2.16. [4] *Let* $F \xrightarrow{i} E \xrightarrow{p} B$ *be a fibration, and* B *a path connected categorically well based space and* E *a normal categorically well based path connected space. Then* $cat\ E \leq cat\ i \cdot cat\ p$. *In particular,* $cat\ E \leq cat\ F \cdot cat\ B$.

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