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A HARNACK INEQUALITY FOR THE HEAT EQUATION ON A COMPACT RIEMANNIAN MANIFOLD WITH NON-CONVEX BOUNDARY

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ABSTRACT. In this article we prove a Harnack inequality for the positive solutions of the heat equation with Neumann boundary condition for a compact Riemannian manifold with possibly *non-convex* boundary.

1. Introduction

Let M be an n-dimensional compact Riemannian manifold with boundary ∂M. Let $g = (g_{ij})$ be a Riemannian metric on M. Then in local coordinates $(x_1, \dots, x_n)$ the Laplace operator Δ is given by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j}).$$

In this paper we are going to consider positive solutions of the heat equation

$$(1) \qquad (\Delta - \frac{\partial}{\partial t}) u(x,t) = 0,$$

on $M \times [0, \infty)$.

In their paper [2] P. Li and S. T. Yau proved the Harnack inequality for the positive solutions of (1) with Neumann boundary condition, i.e., $\frac{\partial u}{\partial \nu} = 0$ on $\partial M \times (0, \infty)$, for M with convex boundary. The purpose of this paper is to show a Harnack inequality for positive solutions of (1)

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with Neumann boundary condition for M with boundary ∂M satisfying weaker condition than convexity, i.e., interior rolling ϵ-ball condition.

Following the idea in [2] (See also [3]), the basic strategy of getting our Harnack inequality is to use the auxiliary function introduced by R. Chen [1] and a modified function G in order to get a new gradient estimate for M with boundary ∂M satisfying interior rolling ϵ-ball condition (See Section 3 for more details).

More precisely, we show the following results:

THEOREM 1.1. *Assume that $Ricci(M) \geq -k$ ($k \geq 0$), the second fundamental form elements of $\partial M \geq -H$ (H is non-negative constant), and the positive solution u of the heat equation* (1) *satisfies the Neumann boundary condition, i.e., $\frac{\partial u}{\partial \nu} = 0$ on $\partial M \times (0, \infty)$. Let $\alpha > 1$ be a constant. Then, we have for any constant β such that $\alpha > \beta > 1$ and ϵ a sufficiently small positive constant less than or equal to $\frac{\beta - 1}{2}$*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{n(1+H)\alpha^2}{2t} + \frac{n\alpha^2}{2}(C_2 + \frac{n\alpha^2 H^2}{\epsilon^2(\alpha-\beta)} + \frac{\beta k(1+H)}{\alpha - \beta}),$$

where

$$C_2 = \frac{2(n-1)H(3H+1)}{\epsilon} + \frac{H(8H+1)}{\epsilon^2}.$$

REMARK. When the boundary is convex, i.e., $H = 0$, this gradient estimate implies the estimate obtained by P. Li and S. T. Yau [2], [3] by letting β approach 1.

Using this gradient estimate, it is immediate to get the following Harnack inequality:

THEOREM 1.2. *Under the same assumption as in* Theorem 1.1, *we have the following: for $\alpha > \beta > 1$, $x_1, x_2 \in M$, $0 < t_1 < t_2 < \infty$,*

$$u(x_1,t_1) \leq u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{n\alpha(1+H)}{2}} \exp[\frac{\alpha d^2(x_1,x_2)}{4(t_2-t_1)} + \frac{n\alpha}{2}(C_2 + \frac{n\alpha^2H^2}{\epsilon^2(\alpha-\beta)} + \frac{\beta k(1+H)}{(\alpha-\beta)})(t_2-t_1)].$$

REMARK. When the boundary is convex, i.e., $H = 0$, our Harnack inequality again implies the Harnack inequality obtained by P. Li and S.T. Yau [2], [3] by letting β approach 1.

This paper is organized as follows. In Section 2, we give the definition of interior rolling ϵ-ball condition and a variant of Laplacian Comparison Theorem which is necessary to show Theorem 1.1. In Section 3, we give a proof of a gradient estimate (Theorem 1.1) which will be crucially used to get the Harnack inequality in Theorem 1.2, and a proof of Theorem 1.2.

2. Definition and Laplacian Comparison Theorem

In this section we recall the definiton of interior rolling ϵ-ball condition and prove two lemmas which are necessary to show Theorem 1.1.

We begin with the definition of interior rolling ϵ-ball condition. We say that ∂M satisfies the *interior rolling ϵ-ball condition* if for each point $p \in \partial M$, there is an open geodesic ball $B_q(\epsilon/2)$ at $q \in M$ such that $\{p\} = \overline{B_q(\epsilon/2)} \cap M$ and $B_q(\epsilon/2) \subset M$.

Next we state a lemma whose statement and proof are simliar to those in [2]. Thus, we leave its proof to the reader.

LEMMA 2.1. *Assume that $Ricci(M) \geq -k$ and u is a positive solution of (1) on $M \times [0,\infty)$. For $\alpha, \beta > 0$, let*

$$F(x,t) = t(\beta|\nabla f|^2 - \alpha f_t),$$

where $f = \log u$. Then, we have

$$\begin{aligned}(\Delta - \frac{\partial}{\partial t})F \geq & -2\nabla f \cdot \nabla F + \frac{2\beta t}{n}(|\nabla f|^2 - f_t)^2 \\ & - (\beta|\nabla f|^2 - \alpha f_t) - 2k\beta t|\nabla f|^2.\end{aligned}$$

Finally, in order to prove Theorem 1.1 we need a variant of Laplacian Comparison Theorem. This lemma was already stated in [1] without proof. For the sake of completeness, we give its detail proof here.

LEMMA 2.2. *Let M be an n-dimensional compact Riemannian manifold with boundary ∂M and let N be the n-dimensional simply-connected space of constant sectional curvature $K_\delta > 0$ with boundary ∂N satisfying constant mean curvature $-H$ ($H \geq 0$). Assume that the sectional* *curvature K_M of M is less than or equal to K_δ and the second fundamental form elements of $\partial M \geq -H$. Let ρ_M and ρ_N be the distance functions with respect to some points $p_M \in \partial M$ and $p_N \in \partial N$, respectively. If $x \in M$ and ρ_M is differentible at x, then for any $y \in N$ with $\rho_N(y) = \rho_M(x)$,*

$$\Delta\rho_M(x) \geq -(n-1)\frac{H + \sqrt{K_\delta}\tan(t_0\sqrt{K_\delta})}{1 - \frac{H}{\sqrt{K_\delta}}\tan(t_0\sqrt{K_\delta})},$$

provided that $0 \leq t_0 < \frac{\pi}{2\sqrt{K_\delta}}$ is a real number such that $\frac{H}{\sqrt{K_\delta}}\tan(t_0\sqrt{K_\delta})$ is not equal to 1, where t_0 is the distance from p_N to y.

Proof. We first note that the extension of the index theorem to submanifolds enables us to prove Laplacian Comparison Theorem for the distance function from some fixed point on the boundary [5]. Hence, the problem of computing $\Delta\rho_N$ can be reduced to that of finding a Jacobi field along a geodesic.

Now, let $\{e_i\}_{i=1}^n$ be an orthonormal basis at p_N such that $\frac{\partial}{\partial\gamma} = e_n$ and $S_{\gamma_*(0)}$ is diagonalized, where $\gamma : [0, t_0] \longrightarrow N$ is a geodesic parametrized by arc-length from p_N to y, and S denotes the second fundamental form. Denote by $\{e_i(t)\}_{i=1}^n$ parallel translate of $\{e_i\}_{i=1}^n$ along γ.

As in the proof of Laplacian Comparison Theorem [3], we can find Jacobi fields $\tilde{X}_i$ along γ such that

(a) $\tilde{X}_i(\gamma(t_0)) = e_i(\gamma(t_0))$
(b) $\tilde{X}_i(\gamma(0)) \in T_{p_N}\partial N$
(c) $S_{\gamma_*(0)}(\tilde{X}_i(0)) - (\frac{D}{dt}\tilde{X}_i)(0) \in (T_{p_N}\partial N)^\perp$

for each $i = 1, \dots, n-1$.

Since N has constant sectional curvature $K_\delta > 0$ and $< \tilde{X}_i, e_j >'' = -K_\delta < \tilde{X}_i, e_j >$ $(j = 1, \dots, n-1)$ for each $i = 1, \dots, n-1$, its general solution of $\tilde{X}_i$ is given by

$$\sum_{j=1}^{n-1}(a_j \sin(\sqrt{K_\delta}t) + b_j\cos(\sqrt{K_\delta}t))e_j(t). \tag{2}$$

Under initial conditions for $\tilde{X}_i$, we get

$$\begin{aligned} a_j &= b_j = 0, \; j \neq i, \\ a_i &= \frac{-H}{\sqrt{K_\delta}\cos(\sqrt{K_\delta}t_0) - H\sin(\sqrt{K_\delta}t_0)}, \\ b_i &= \frac{\sqrt{K_\delta}}{\sqrt{K_\delta}\cos(\sqrt{K_\delta}t_0) - H\sin(\sqrt{K_\delta}t_0)}, \end{aligned}$$

provided that $0 \leq t_0 < \frac{\pi}{2\sqrt{K_\delta}}$ is a real number such that $\frac{H}{\sqrt{K_\delta}}\tan(t_0\sqrt{K_\delta})$ is not equal to 1. Set

$$A = \sqrt{K_\delta}\cos(\sqrt{K_\delta}t_0) - H\sin(\sqrt{K_\delta}t_0).$$

Now, at y,

$$\begin{aligned} \mathrm{Hess}(\rho_N)(e_i, e_i) &= \int_0^{t_0} |\frac{D}{dt}\tilde{X}_i|^2 - < R(\tilde{X}_i, \gamma')\gamma', \tilde{X}_i > dt \\ &\quad + < S_{\gamma_*(0)}(\tilde{X}_i(0)), \tilde{X}_i(0) > \\ &= \int_0^{t_0} \frac{1}{A^2}(H^2 K_\delta \cos(2\sqrt{K_\delta}t) + 2HK_\delta^{3/2}\sin(2\sqrt{K_\delta}t) \\ &\quad - K_\delta^2\cos(2\sqrt{K_\delta}t))dt - \frac{HK_\delta}{A^2} \\ &= -\frac{H + \sqrt{K_\delta}\tan(t_0\sqrt{K_\delta})}{1 - \frac{H}{\sqrt{K_\delta}}\tan(t_0\sqrt{K_\delta})}. \end{aligned}$$

Hence, we get

$$\Delta\rho_N(y) = -(n-1)\frac{H + \sqrt{K_\delta}\tan(t_0\sqrt{K_\delta})}{1 - \frac{H}{\sqrt{K_\delta}}\tan(t_0\sqrt{K_\delta})}.$$

The index comparison theorem in [5] completes the proof. □

3. Proof of Theorems

In this section we prove Theorems 1.1 and 1.2. Since their proofs are similar to those in [2], we will indicate only major steps which are essential in understanding our proof.

Proof of Theorem 1.1 and 1.2. To overcome the non-convexity of the boundary, we will use the auxiliary function which was introduced in [1]. Thus, choose ψ as a non-negative C^2 function defined on $[0,\infty)$ such that ψ is less than or equal to H on $[0,1/2)$ and is H on $[1/2,\infty)$ satisfying

$$\psi(0)=0,\ 0\le\psi'(r)\le 2H,\ \psi'(0)=H,\ \psi''(r)\ge -H.$$

Set

$$\varphi(x)=\psi\left(\frac{r(x)}{\epsilon}\right),$$

where $r(x)$ denotes the distance function between $x\in M$ and boundary ∂M.

We define for $\alpha>\beta>1$

$$G(x,t)=(1+\varphi(x))F(x,t),$$

where $F(x,t)=t(\beta|\nabla f|^2-\alpha f_t)$, and $f=\log u$.

We assume that $|\nabla f|^2-\alpha f_t$ is positive (Otherwise, the theorem holds trivially). By the compactness of $M\times[0,T]$, $G(x,t)$ attains its maximun at some point $p=(x_0,t_0)\in M\times[0,T]$. First we show that $x_0\notin\partial M$. Suppose that $x_0\in\partial M$. At p we may choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ at x_0 such that $e_n=\frac{\partial}{\partial\nu}$. Then, by the maximun principle, we get

$$\frac{\partial G}{\partial\nu}(p)>0.$$

This implies, at p,

$$\begin{aligned}0&<\frac{1}{G}\cdot\frac{\partial G}{\partial\nu}\\&=\varphi_n+\frac{2\sum_{j=1}^n f_jf_{jn}}{\beta|\nabla f|^2-\alpha f_t}\\&=-\frac{H}{\epsilon}+\frac{-2\sum_{i,j=1}^{n-1}h_{ij}f_if_j}{\beta|\nabla f|^2-\alpha f_t}\\&=-\frac{H}{\epsilon}+\frac{-2\sum h_{ij}\frac{f_if_j}{|\nabla f|^2}}{\beta-\frac{\alpha f_t}{|\nabla f|^2}}\\&\le-\frac{H}{\epsilon}+\frac{2H}{\beta-1}\le 0,\end{aligned}$$

provided we choose $0 < \epsilon \leq \frac{\beta-1}{2}$. But this is a contradiction, which implies that $x_0 \notin \partial M$.

Now, we are going to get a gradient estimate for the positive solution u at the interior point p. From now on, all computaions will be at p, unless stated otherwise. Since G attains its maximun at p, we have

$$\begin{aligned} &0 = \nabla G = F\nabla\varphi + (1+\varphi)\nabla F, \\ &0 \geq \Delta G = \Delta((1+\varphi)F), \\ &0 \leq \frac{\partial}{\partial t}G = (1+\varphi)F_t. \end{aligned} \tag{3}$$

Let $\partial M(\epsilon) = \{x \in M | \rho(x) \leq \epsilon\}$ and K_δ be the upper bound of the sectional curvature in $\partial M(\epsilon)$. As in [1], if we choose ϵ so that $\sqrt{K_\delta}\tan(\epsilon\sqrt{K_\delta}) \leq \frac{1+H}{2}$ and $\frac{H}{\sqrt{K_\delta}}\tan(\epsilon\sqrt{K_\delta}) \leq \frac{1}{2}$ and we use Lemmas 2.1 and 2.2, starting from the second equation of (3) it is not difficult to get

$$\begin{aligned} 0 \geq (1+\varphi)F(-C_1 - \frac{2|\nabla\varphi|^2}{1+\varphi} - \frac{1+\varphi}{t}) + 2(1+\varphi)F\nabla(1+\varphi)\cdot\nabla f \\ + (1+\varphi)^2[\frac{2\beta t}{n}(|\nabla f|^2 - f_t)^2 - 2k\beta t|\nabla f|^2], \end{aligned} \tag{4}$$

where

$$C_1 = \frac{2(n-1)H(3H+1)}{\epsilon} + \frac{H}{\epsilon^2}.$$

If we use $\frac{|\nabla\varphi|^2}{1+\varphi} \leq \frac{4H^2}{\epsilon^2}$ and multiply by t it follows from (4) that

$$\begin{aligned} 0 \geq (1+\varphi)F(-C_2 t - (1+H)) - \frac{4H}{\epsilon}t(1+\varphi)^{3/2}F|\nabla f| \\ + \frac{2\beta t^2}{n}\{[(1+\varphi)(|\nabla f|^2 - f_t)]^2 - nk(1+\varphi)^2|\nabla f|^2\}, \end{aligned} \tag{5}$$

where $C_2 = C_1 + \frac{8H^2}{\epsilon^2}$.

Set $y = (1+\varphi)\beta|\nabla f|^2$ and $z = (1+\varphi)f_t$. Using $\beta(1+\varphi)^2|\nabla f|^2 \leq (1+H)y$ and $y^{1/2}(y-\alpha z) = \frac{(1+\varphi)^{3/2}}{t}\beta^{1/2}F|\nabla f|$, from (5) we get

$$\begin{aligned} 0 \geq (1+\varphi)F(-C_2 t - (1+H)) + \frac{2t^2}{n}[(\beta^{-1}y - z)^2 - nk(1+H)y \\ - \frac{2nH}{\epsilon}\beta^{-1/2}y^{1/2}(y-\alpha z)]. \end{aligned} \tag{6}$$

Finally, using a simple relation $\frac{1}{\beta}y - z = \frac{1}{\alpha}(y - \alpha z) + (\frac{\alpha-\beta}{\alpha\beta})y$ and a simple inequality $ax^2 - bx \geq -\frac{b^2}{4a}$ $(a, b > 0)$, from (6) we get

$$\begin{aligned}
0 &\geq (1+\varphi)F(-C_2 t - (1+H)) + \frac{2t^2}{n}[\frac{1}{\alpha^2}(y-\alpha z)^2 \\
&\quad - \frac{n^2\alpha^2\beta^2k^2(1+H)^2}{4(\alpha-\beta)^2} - \frac{n^2\alpha^2H^2}{2(\alpha-\beta)\epsilon^2}(y-\alpha z)] \\
&= (1+\varphi)F(-C_2 t - (1+H)) + \frac{2}{n\alpha^2}((1+\varphi)F)^2 \\
&\quad - \frac{n\alpha^2\beta^2k^2(1+H)^2}{2(\alpha-\beta)^2} - \frac{n\alpha^2H^2t}{(\alpha-\beta)\epsilon^2}(1+\varphi)F \\
&= \frac{2}{n\alpha^2}G^2 - [(1+H) + C_2 t + \frac{n^2\alpha^2H^2t}{(\alpha-\beta)\epsilon^2}]G \\
&\quad - \frac{n\alpha^2\beta^2k^2(1+H)^2t^2}{2(\alpha-\beta)^2},
\end{aligned} \tag{7}$$

where we used the relation $t(y - \alpha z) = (1+\varphi)F$ in the second equality.

Using the relation $\sqrt{b^2+c^2} \leq b + c$ $(b, c > 0)$, (7) yields

$$G \leq \frac{n\alpha^2}{2}\{(1+H) + C_2 t + \frac{n\alpha^2H^2t}{(\alpha-\beta)\epsilon^2} + \frac{\beta k(1+H)t}{(\alpha-\beta)}\}.$$

Since $F(x,T) \leq (1+\varphi)F(x,T) \leq (1+\varphi)F(p)$ and T is arbitrary, we have the desired inequality.

For the proof of Theorem 1.2, using the newly made gradient estimate and the method in [2], it is easy to get the Harnack inequality for the positive solutions of the heat equation (1) on $M \times [0, \infty)$ with Neumann boundary condition in case of M having boundary ∂M satisfying the interior rolling ϵ-ball condition (See [2], [3] for details). □

Note added in proof. We have recently learned that J. Wang independently has proved similar results in [4]. But we believe that results in this paper are true generalizations of Li and Yau's results. □

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