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UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING THE SAME 1-POINTS

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ABSTRACT. We prove a uniqueness theorom for meromrphic functions which share the same 1-points.

1. Introduction and Definitions

Let f, g be two nonconstant meromorphic functions defined on the open complex plane $\mathbb{C}$. If f and g have the same a-points with the same multiplicities, we say that f and g share the value a CM (country multiplicities). We do not explain the standard notations and definitions of Nevanlinna's theory of meromorphic functions becuase these are available in [4]. We denote by E a set of real numbers with finite linear measure, not the same at each occurrence.

Ozawa [6] initiated the problem of uniqueness of entire functions on the basis of sharing the 1-points. His result can be stated as follows:

THEOREM A [6]. *Let f and g be two nonconstant entire functions. If f, g share 1 CM with $\delta(0; f) > 0$ and 0 is lacunary for g, then either $f \equiv g$ or $f \cdot g \equiv 1$.*

Extending this problem to meromorphic functions Yi proved the following theorems.

THEOREM B [7]. *Let f and g be two nonconstant meromorphic functions satisfying $\delta(\infty; f) = \delta(\infty; g) = 1$. If f, g share 1 CM and $\delta(0; f) + \delta(0; g) > 1$ then either $f \equiv g$ or $f \cdot g \equiv 1$.*

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THEOREM C [8]. *Let f and g be two nonconstant meromorphic functions such that f and g share 1, ∞ CM. If $\delta(0;f) + \delta(0;g) + 2\Theta(\infty;f) > 3$ then either $f \equiv g$ or $f.g \equiv 1$.*

THEOREM D [9]. *Let f and g be two nonconstant meromorphic fuctions such that f, g share 1, ∞ CM. If $N(r,0;f)+N(r,0;g)+2\bar{N}(r,f) < (\lambda + 0(1)) \times \max\{T(r,f), T(r,g)\}$ for $r \notin E$, where $\lambda < 1$ then either $f \equiv g$ or $f \cdot g \equiv 1$.*

Gangdi [2] proved the following uniqueness theorem for meromorphic functions which involves sharing of functions.

THEOREM E [2]. *Let f, g be nonconstant meromorphic functions and μ, λ be two meromorphic functions such that $T(r,\mu) = S(r,f)$, $T(r,\lambda) = S(r,g)$ If f, g share ∞ CM, $f - \mu, g - \lambda$ share 0 CM and $\delta(0;f) + \Theta(\infty;f) > 3/2, \delta(0;g) + \Theta(\infty;g) > 3/2$ then either $\lambda \cdot f \equiv \mu \cdot g$ or $f \cdot g \equiv \mu \cdot \lambda$.*

We note that for $\lambda \equiv \mu \equiv 1$, theorem E is weaker than Theorem C. Improving Theorem B recently Yi and Yang [10] proved the following result.

THEOREM F. (cf. [10]) *Let f and g be two nononstant meromorphic functions saitsfying $\Theta(\infty;f) = \Theta(\infty;g) = 1$. If f, g share 1 CM and $\delta(0;f) + \delta(0;g) > 1$ then either $f \equiv g$ or $f \cdot g \equiv 1$.*

The purpose of this paper is to make some further investigations on the problem of uniqueness of meromorphic functions sharing same 1-points.

Following definitions will be reuqired in the sequel.

DEFINITION 1 [1]. For a meromorphic function f and a positive integer p, $N_p(r,a;f)$ denotes the counting function of a-points of f where an a-point with multiplicity m is counted m times if $m \leq p$ and p times if $m > p$.

DEFINITION 2 [9]. For a meromorphic function f we put $\delta_p(a;f) = 1 - \limsup\limits_{r\to\infty} \dfrac{N_p(r,a;f)}{T(r,f)}$. Then clearly $0 \leq \delta(a;f) \leq \delta_p(a;f) \leq \delta_{p-1}(a;f) \leq \cdots \leq \delta_2(a;f) \leq \delta_1(a;f) = \Theta(a;f) \leq 1$.

In all the theorems from A to F we see that one of the following two conditions is necessary:
(i) f, g share ∞ CM,
(ii) $\delta(\infty; f) = \delta(\infty; g) = 1$ or $\Theta(\infty; f) = \Theta(\infty; g) = 1$.
In the paper we prove uniqueness theorems for meromorphic functions without considering the above two conditions.

2. Lemmas

In this section we present some lemmas which will be used to prove the main results.

LEMMA 1 [8]. *Let f_1, f_2, f_3 be nonconstant meromorphic functions satisfying $f_1 + f_2 + f_3 \equiv 1$. If f_1, f_2, f_3 are linearly independent then $g_1 = -f_3/f_2$, $g_2 = 1/f_2$ and $g_3 = -f_1/f_2$ are also linearly independent.*

LEMMA 2. *Let f_1, f_2 be nonconstnat meromorphic functions such that $af_1 + bf_2 \equiv 1$ where a, b are nonzero constants. Then*

$$T(r, f_1) \leq \bar{N}(r, 0; f_1) + \bar{N}(r, 0; f_2) + \bar{N}(r, f_1) + S(r, f_1).$$

Proof. By the second fundamental theorem we get

$$\begin{aligned} T(r, f_1) &\leq \bar{N}(r, o; f_1) + \bar{N}(r, a^{-1}; f_1) + \bar{N}(r, f_1) + S(r, f_1) \\ &= \bar{N}(r, o; f_1) + \bar{N}(r, 0; f_2) + \bar{N}(r, f_1) + S(r, f_1) \end{aligned}$$

and this proves the lemma. □

LEMMA 3 [3],[5]. *Let $f_1, f_2, \cdots, f_p$ be linearly independent meromorphic functions satisfying $\sum_{j=1}^{p} f_j \equiv 1$. Then for $i = 1, 2, \cdots, p$ and for $r \notin E$*

$$\begin{aligned} T(r, f_i) < &\sum_{j=1}^{p} N(r, 0; f_j) + N(r, f_i) + N(r, D) - \sum_{j=1}^{p} N(r, f_j) \\ &- N(r, 0; D) + o\{T(r)\}, \end{aligned}$$

where D is the wronskian determinant of $f_1, f_2, \cdots, f_p$ and
$T(r) = \max_{1\leq j\leq p}\{T(r, f_j)\}$.

LEMMA 4. *Let f_1, f_2, f_3 be nonconstnat meromorphic functions such that $f_1+f_2+f_3 \equiv 1$. If f_1, f_2, f_3 are linearly independent then for $r \notin E$*

$$T(r, f_i) < \sum_{j=1}^{3} N_2(r, 0; f_j) + \max_{\substack{1\leq i,j\leq 3\\(i\neq j)}}\{N_2(r, \infty; f_i) + N_2(r, \infty; f_j)\} + o\{T(r)\},$$

where $T(r) = \max_{1\leq j\leq 3}\{T(r, f_j)\}$.

Proof. By Lemma 3 we get

$$T(r, f_1) < \sum_{j=1}^{3} N(r, 0; f_j) - N(r, f_2) - N(r, f_3) + N(r, D) - N(r, 0; D) + o\{T(r)\}, \tag{1}$$

where D is the wronskian determinant of f_1, f_2, f_3.

We prove the following two inequalities which combined with (1) will prove the lemma:

$$\sum_{j=1}^{3} N(r, 0; f_j) - N(r, 0; D) \leq \sum_{j=1}^{3} N_2(r, 0; f_j) \tag{2}$$

and

$$N(r, D) \leq N(r, f_2) + N(r, f_3) + \max_{\substack{1\leq i,j\leq 3\\(1\neq j)}}\{N_2(r, \infty; f_i) + N_2(r, \infty; f_j)\}. \tag{3}$$

If z_0 is neither a zero nor a pole of meromorphic function, we agree to call it a zero of the function with multiplicity zero.

Now if z_0 is a zero of some $f_j (1 \le j \le 3)$ with multiplicity p then it is a zero of D with multiplicity at least $\max\{0, p-2\}$. So the inequality (2) is proved.

To prove inequality (3) we first note that a pole z_0 of D is a pole of at least one of f_1, f_2, f_3 and conversely. We now consider following cases.

Case 1. Let z_0 be not a pole of f_1. Since $f_2 + f_3 \equiv 1 - f_1$, it follows that z_0 is not a pole of $f_2 + f_3$. Since z_0 is a pole of at least one of f_1, f_2, f_3, it follows that z_0 is a pole of f_2 and f_3 of the same multiplicity m, say (becuase the singularities of f_2 and f_3 at z_0 cancel each other). Since $D = \begin{vmatrix} f_2' + f_3' & f_3' \\ f_2'' + f_3'' & f_3'' \end{vmatrix}$, z_0 is a pole of D with multiplicity not exceeding

$$m + 2 \le m + m + (1 + 1). \tag{4}$$

Case 2. Let z_0 be a pole of f_1 with multiplicity $m (\ge 1)$. Since $f_2 + f_3 \equiv 1 - r_1$, we see that z_0 is a pole of $f_2 + f_3$ with multiplicity m. We further consider the following subcases.

Subcase (i). Let z_o be a pole of f_2 with multiplicity m and a pole of f_3 with multiplicity $q (1 \le q < m)$. Since $D = \begin{vmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{vmatrix}$, z_0 is a pole of D with multiplicity not exceeding

$$m + q + 3 = m + q + (2 + 1). \tag{5}$$

Subcase (ii). Let z_0 be a pole of f_2 and f_3 with the same multiplicity m. Then there exist two functions ϕ, Ψ which are analytic at z_0 and $\phi(z_0) \ne 0, \Psi(z_0) \ne 0$ such that in some neighbourhood of z_0, $F_2(z) = (z - z_0)^{-m} \phi(z)$ and $f_3(z) = (z - z_0)^{-m} \Psi(z)$. Also $D = f_2' \, f_3'' - f_2'' \, f_3'$ shows that z_0 is a pole of D with multiplicity not exceeding $2m + 3$; but by actual calculation we see that the coefficient of $(z - z_0)^{-(2m+3)}$ is $m^2(m+1)\phi\Psi - m^2(m+1)\phi\Psi \equiv 0$. So s_0 is a pole of D with multiplicity not exceeding

$$2m + 2 = m + m + (1 + 1) \tag{6}$$

Subcase (iii). Let z_0 be a pole of f_2 with multiplicity m but z_0 is not a pole of f_3. We note that z_0 is a pole of f_1 with multiplicity m. Since $D = f_2' \, f_3'' - f_2'' \, f_3'$, z_0 is pole of D with multiplicity not exceeding

$$(7) \qquad m + 2 = m + 0 + (1 + 1).$$

Subcase (iv). Let z_0 be a pole of f_2 with multiplexity $m+p (p \geq 1)$. Then z_0 is also a pole of f_3 with multiplicity $m + p$ and the terms containing $(z-z_0)^{-(m+1)}, (z-z_0)^{-(m+2)}, \cdots, (z-z_0)^{-(m+p)}$ in Laurent expansion of f_2 and f_3 about z_0 cancel each other because $f_2 + f_3$ has a pole at z_0 with multiplicity m. Since $D = \begin{vmatrix} f_2' + f_3' & f_3' \\ f_2'' + f_3'' & f_3'' \end{vmatrix}$, it follows that z_0 is a pole of D with multiplicity not exceeding

$$(8) \qquad 2m + p + 3 \leq (m + p) + (m + p) + (1 + 1).$$

Combining (4), (5), (6), (7) and (8) the inequality (2) can be obtained. This proves the lemma. □

3. Theorem

In this section we discuss the main results.

THEOREM 1. *Let f, g be two nonconstant meromorphic functions sharing 1 CM. If $N_2(r,0;f)+N_2(r,0;g)+2N_2(r,\infty;f)+2N_2(r,\infty;g) < \{\lambda + o(1)\} \times \max\{T(r,f), T(r,g)\}$ for $r \notin E$ where $\lambda < 1$ then either $f \equiv g$ of $f \cdot g \equiv 1$.*

COROLLARY. *Let f, g be two nonconstant meromorphic functions sharing 1 CM. If $\delta_2(0;f) + \delta_2(0;g) + 2\delta_2(\infty;f) + 2\delta_2(\infty;g) > 5$ then either $f \equiv g$ or $f \cdot g \equiv 1$.*

Following example shows that the theorem and the corollary are sharp.

EXAMPLE. Let $f = \exp(z)$, $g = 2 - \exp(z)$. Then f, g share 1 CM and $N_2(r,0;f) \equiv 0, N_2(r,\infty,g) \equiv 0, N_2(r,\infty;f) \equiv 0, N_2(r,0;g) \sim T(r, \exp(z))$, $T(r,f) = T(r, \exp(z))$, $T(r,f) = T(r, \exp(z))$, $T(r,g) = T(r, \exp(z)) + 0(1)$, but neither $f \equiv g$ nor $f \cdot g \equiv 1$.

Proof of Theorem 1. Let

$$h = \frac{f-1}{g-1}. \tag{9}$$

Since f, g share 1 CM, it follows that poles and zeros of h occur only at the poles of f and g respectively. Also we note that $N_2(r,\infty;h) \leq N_2(r,\infty;f)$ and $N_2(r,0;h) \leq N_2(r,\infty;g)$.

We put $f_1 = f,\ f_2 = h,\ f_3 = -gh$ so that

$$f_1 + f_2 + f_3 \equiv 1. \tag{10}$$

Let $h = k$, a constant. If $k \neq 1$ from (10) we get $\frac{1}{1-k}f - \frac{k}{1-k}g \equiv 1$ and so by Lemma 2 it follows that

$$T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + S(r,f)$$

and

$$T(r,g) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;g) + S(r,g).$$

This shows in view of the given condition that

$\max\{T(r,f),T(r,g)\} < \{\lambda + o(1)\}\max\{T(r,f),T(r,g)\}$,which is a contradiction because $\lambda < 1$. Hence $k = 1$ and so $f \equiv g$.

Now let h be nonconstant. If possible, suppose that f_1, f_2, f_3 are linearly independent. Then by lemma 4 we get

$$\begin{aligned}
T(r,f) = T(r,f_1) \leq & \sum_{j=1}^{3} N_2(r,0;f_j) \\
& + \max_{\substack{1\leq i,j\leq 3\\(i\neq j)}} \{N_2(r,0;f_i) + N_2(r,0;f_j)\} + o\{T(r)\}. \\
\leq & N_2(r,0;f) + N_2(r,0;g) + 2N_2(r,0;h) \\
& + \max_{\substack{1\leq i,j\leq 3\\(i\neq j)}} \{N_2(r,0;f_i) + N_2(r,0;f_j)\} + o\{T(r)\} \\
\leq & N_2(r,0;f) + N_2(r,0;g) + 2N_2(r,\infty;g) \\
& + \max_{\substack{1\leq i,j\leq 3\\(i\neq j)}} \{N_2(r,0;f_i) + N_2(r,0;f_j)\} + o\{T(r)\}
\end{aligned} \tag{11}$$

Now by (9)

$$N_2(r,\infty;f_1)+N_2(r,\infty,f_3)=N_2(r,\infty;f)+N_2(r,\infty;h(g-1))$$
$$=N_2(r,\infty;f)+N_2(r,\infty;f-1)=sN_2(r,\infty;f),$$
$$N_2(r,\infty;f_3)+N_2(r,\infty;f_2)+N_2(r,\infty;h(g-1))+N_2(r,\infty;h)$$
$$\le N_2(r,\infty;f-1)+N_2(r,\infty;f)=2N_2(r,\infty;f),\text{and}$$
$$N_2(r,\infty;f_2)+N_2(r,\infty;f_1)=N_2(r,\infty;h)+N_2(r,\infty;f)\le 2N_2(r,\infty;f)$$

So from (11) we obtain

(12)
$$T(r,f)\le N_2(r,0;f)+N_2(r,0;g)+2N_2(r,\infty;f)+2N_2(r,\infty;g)+o\{T(r)\}.$$

Now we put $g_1=-f_3/f_2,\ g_2=1/f_2$ and $g_3=-f_1/f_2$. Then by lemma 1 and Lemma 4 we get similarly

(13)
$$T(r,g)\le N_2(r,0;f)+N_2(r,0;g)+2N_2(r,\infty;f)+2N_2(r,\infty;g)+o\{T(r)\}.$$

By the given condition we get from (12) and (13)

$$\max\{T(r,f),T(r,g)\}<\{\lambda+o(1)\}\max\{T(r,f),T(r,g)\},$$

which is a contradiction becuase $\lambda<1$.

Hence there exist constants c_1,c_2,c_3, not all zero, such that

$$c_1f_1+c_2f_2+c_3f_3\equiv 0. \tag{14}$$

If possible, let $c_1=0$. Then from(14) we get $(c_2-c_3g)h\equiv 0$. Since $h\not\equiv 0$, it follows that g is a constant which is a contradiction. So $c_1\ne 0$. Now eliminating f_1 from (10) and (14) we get

$$cf_2+df_3\equiv 1, \tag{15}$$

where $c=1-c_2/c_1$ and $d=1-c_3/c_1$. We consider the following cases.

Case 1. Let $c\cdot d\ne 0$. Then from(15) we get $\frac{1}{ch}+\frac{d}{c}g\equiv 1$ and so by Lemma 2 it follows that

$$T(r,g)\le N_2(r,0;g)+N_2(r,\infty;h)+N_2(r,\infty;g)+S(r,g)$$
$$\le N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+S(r,g). \tag{16}$$

since f, g share 1 CM, we get by the second fundamental theorem in view of (16) that

$$\begin{aligned} T(r,f) &\leq \bar{N}(r,0;f) + \bar{N}(r,1;f) + \bar{N}(r,\infty;f) + S(r,f) \\ &\leq N_2(r,0;f) + \bar{N}(r,1;g) + N_2(r,\infty;f) + S(r,f) \\ &\leq N_2(r,0;f) + T(r,g) + N_2(r,\infty;f) + S(r,f) \\ &\leq N_2(r,0;f) + N_2(r,0;g) + 2N_2(r,\infty;f) \\ &\qquad + N_2(r,\infty;g) + S(r,f) + S(r,g). \end{aligned}$$

So by the given condition we see that

$$\max\{T(r,f), T(r,g)\} < \{\lambda + o(1)\} \cdot \max\{T(r,f), T(r,g)\}.$$

Which is a contradiction because $\lambda < 1$. Hence the case $c \cdot d \neq 0$ does not arise.

Case 2. Let $c \cdot d = 0$. From (15) we see that c and d are not simultaneously zero. We consider the following subcases.

Subcase (i). Let $d = 0$. Then from (15) we get

$$cf - g \equiv c - 1 \tag{17}$$

If $c \neq 1$we obtain from (17), $\frac{c}{c-1}f - \frac{1}{c-1}g \equiv 1$. So by Lemma 2 we see that

$$T(r,f) \leq N_2(r,0;f) + N_2(r,o;g) + N_2(r,\infty;f) + S(r,f)$$

and

$$T(r,g) \leq N_2(r,0;f) + N_2(r,o;g) + N_2(r,\infty;g) + S(r,g).$$

This implies by the given condition that

$$\max\{T(r,f), T(r,g)\} < \{\lambda + o(1)\} \max\{T(r,f), T(r,g)\},$$

which is a contradiction because $\lambda < 1$. Hence $c = 1$ and so from (17) we get $f \equiv g$.

Subcase (ii). Let $c = 0$. Then from (15) we get

$$df - 1/g \equiv d - 1. \tag{18}$$

If $d \neq 1$we obtain from (18) that

$$\frac{d}{d-1} \cdot f - \frac{1}{d-1} \cdot \frac{1}{g} \equiv 1.$$

So by Lemma 2 and the first fundamental theorem it follows that

$$T(r,f) \leq N_2(r,0;f) + N_2(r,\infty;g) + N_2(r,\infty;f) + S(r,f)$$

and

$$T(r,g) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;g) + S(r,g).$$

This implies by the given condition that

$$\max\{T(r,f),T(r,g)\} < \{\lambda + o(1)\} \max\{T(r,f),T(r,g)\},$$

which is a contradiction because $\lambda < 1$. Hence $d = 1$ and so from (18) we get $f \cdot g \equiv 1$. This proves the theorem. □

In the line of Theorem 1 we can prove the following more general result.

THEOREM 2. *Let f, g be two nonconstant meromorphic functions and $a(z)(\equiv\not 0), b(z)(\equiv\not 0)$ be two meromorphic functions such that*

$$T(r,a) = o\{T(r)\}, \quad T(r,b) = o\{T(r)\}$$

as $r \to \infty$ $(r \notin E)$ where $T(r) = \max\{T(r,f),T(r,g)\}$. If $f-a, g-b$ share 0 CM and $N_2(r,0;f)+N_2(r,0;g)+2N_2(r,\infty;f)+2N_2(r,\infty;g) < \{\lambda+o(1)\}\cdot T(r)$ for $r \notin E$ where $\lambda < 1$ then either $bf \equiv ag$ or $f\cdot g \equiv a\cdot b$.

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