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# A NOTE ON GENERALIZATIONS OF PREECE'S IDENTITY AND OTHER CONTIGUOUS RESULTS

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ABSTRACT. The aim of this paper is to establish generalizations of the well known Preece's identity and other identities involving product of generalized hypergeometric series by using new and very short method.

## 1. Introduction and results required

The generalized hypergeometric function with $p$ numerator and $q$ denominator parameters is defined by

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p;\\ \beta_1,\dots,\beta_q;\end{matrix}\, z\right] = {}_pF_q\left[\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, $$

where $(\alpha)_n$ denotes the Pochhammer symbol (or *the shifted factorial*, since $(1)_n = n!$) defined by

$$ (\alpha)_0 = 1 \qquad \text{and} \qquad (\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1) \ \ (n = 1,2,3,\dots), $$

for any complex number $\alpha$.

From the theory of differential equations, Professor Preece [3] established the following very interesting identity involving product of generalized hypergeometric series:

$$ \begin{aligned} &{}_1F_1(\alpha\,;2\alpha;\,x)\times{}_1F_1(1-\alpha\,;2-2\alpha;\,x) \\ (1.1)\qquad &= e^x\,{}_1F_2\left(\frac{1}{2};\,\alpha+\frac{1}{2},\,\frac{3}{2}-\alpha;\,\frac{x^2}{4}\right). \end{aligned} $$

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Very recently the first author [4] has given a very short proof of (1.1). In another paper the first author [5] has also obtained the following two results contiguous to (1.1):

$$\begin{aligned}&{}_1F_1(\alpha\,;2\alpha+1;\,x)\times{}_1F_1(1-\alpha\,;2-2\alpha;\,x)\\ &=e^x\left\{{}_1F_2\left(\frac{1}{2};\,\alpha+\frac{1}{2},\,\frac{3}{2}-\alpha;\,\frac{x^2}{4}\right)\right.\\ &\left.-\frac{x}{2(2\alpha+1)}{}_2F_3\left(\frac{3}{2},\,1;\,\frac{3}{2}-\alpha,\,\alpha+\frac{3}{2},\,2;\,\frac{x^2}{4}\right)\right\}\end{aligned}\tag{1.2}$$

and

$$\begin{aligned}&{}_1F_1(\alpha\,;2\alpha-1;\,x)\times{}_1F_1(1-\alpha\,;2-2\alpha;\,x)\\ &=e^x\left\{{}_1F_2\left(\frac{1}{2};\,\alpha-\frac{1}{2},\,\frac{3}{2}-\alpha;\,\frac{x^2}{4}\right)\right.\\ &\left.+\frac{x}{2(2\alpha-1)}{}_1F_2\left(\frac{1}{2};\,\frac{3}{2}-\alpha,\,\alpha+\frac{1}{2};\,\frac{x^2}{4}\right)\right\}.\end{aligned}\tag{1.3}$$

The well-known Kummer's first and second theorems [2] are

$$ {}_1F_1(\alpha;\,\gamma;\,x)=e^x\times{}_1F_1(\gamma-\alpha;\,\gamma;\,-x), \tag{1.4}$$

$$ e^{-x/2}\times{}_1F_1(\alpha;\,2\alpha;\,x)={}_0F_1\left(-\,;\,\alpha+\frac{1}{2};\,\frac{x^2}{16}\right). \tag{1.5}$$

Recently the first author and Nagar [6] have obtained the following two interesting results contiguous to (1.5):

$$\begin{aligned}&e^{-x/2}\times{}_1F_1(\alpha;\,2\alpha+1;\,x)\\ &={}_0F_1\left(-\,;\,\alpha+\frac{1}{2};\,\frac{x^2}{16}\right)-\frac{x}{2(2\alpha+1)}{}_0F_1\left(-\,;\,\alpha+\frac{3}{2};\,\frac{x^2}{16}\right)\end{aligned}\tag{1.6}$$

and

$$\begin{aligned}&e^{-x/2}\times{}_1F_1(\alpha;\,2\alpha-1;\,x)\\ &={}_0F_1\left(-\,;\,\alpha-\frac{1}{2};\,\frac{x^2}{16}\right)+\frac{x}{2(2\alpha-1)}{}_0F_1\left(-\,;\,\alpha+\frac{1}{2};\,\frac{x^2}{16}\right).\end{aligned}\tag{1.7}$$

The following is an interesting result due to Bailey [1]:

$$\begin{aligned}(1.8)\qquad &{}_0F_1(-;\rho;x)\times{}_0F_1(-;\sigma;x)\\ &={}_2F_3\left(\frac{1}{2}(\rho+\sigma),\,\frac{1}{2}(\rho+\sigma-1);\,\rho,\,\sigma,\,\rho+\sigma-1;\,4x\right).\end{aligned}$$

We also recall the following interesting formula (see Srivastava *et al.* [7, p. 322, Eq.(186)]):

$$\begin{aligned}(1.9)\qquad &{}_1F_1(\alpha;2\alpha;x)\times{}_1F_1(\beta;2\beta;-x)\\ &={}_2F_3\left(\frac{1}{2}(\alpha+\beta),\,\frac{1}{2}(\alpha+\beta+1);\,\alpha+\frac{1}{2},\,\beta+\frac{1}{2},\,\alpha+\beta;\,\frac{x^2}{4}\right).\end{aligned}$$

The aim of this paper is to establish a generalization of the well-known Preece's identity (1.1) by a similar method given by the first author [4]. In the last, two interesting results are also given.

## 2. Main results

In this section, we shall establish the following generalization of the well-known Preece's identity (1.1) and other results:

$$\begin{aligned}(2.1)\qquad &{}_1F_1(\alpha;2\alpha;x)\times{}_1F_1(\beta;2\beta;x)\\ &=e^x\,{}_2F_3\left(\frac{1}{2}(\alpha+\beta),\,\frac{1}{2}(\alpha+\beta+1);\,\alpha+\frac{1}{2},\,\beta+\frac{1}{2},\,\alpha+\beta;\,\frac{x^2}{4}\right).\end{aligned}$$

(2.2)

$$\begin{aligned}&{}_1F_1(\alpha;2\alpha+1;x)\times{}_1F_1(\beta;2\beta;x)\\ &=e^x\left\{{}_2F_3\left(\frac{1}{2}(\alpha+\beta+1),\,\frac{1}{2}(\alpha+\beta);\,\alpha+\frac{1}{2},\,\beta+\frac{1}{2},\,\alpha+\beta;\,\frac{x^2}{4}\right)-\frac{x}{2(2\alpha+1)}\right.\\ &\quad\left.\times\,{}_2F_3\left(\frac{1}{2}(\alpha+\beta+2),\,\frac{1}{2}(\alpha+\beta+1);\,\alpha+\frac{3}{2},\,\beta+\frac{1}{2},\,\alpha+\beta+1;\,\frac{x^2}{4}\right)\right\}.\end{aligned}$$

(2.3)

$$\begin{aligned}&{}_1F_1(\alpha;2\alpha-1;x)\times{}_1F_1(\beta;2\beta;x)\\ &=e^x\left\{{}_2F_3\left(\frac{1}{2}(\alpha+\beta-1),\,\frac{1}{2}(\alpha+\beta);\,\beta+\frac{1}{2},\,\alpha-\frac{1}{2},\,\alpha+\beta-1;\,\frac{x^2}{4}\right)\right.\\ &\quad\left.+\frac{x}{2(2\alpha-1)}\,{}_2F_3\left(\frac{1}{2}(\alpha+\beta+1),\,\frac{1}{2}(\alpha+\beta);\,\beta+\frac{1}{2},\,\alpha+\frac{1}{2},\,\alpha+\beta;\,\frac{x^2}{4}\right)\right\}.\end{aligned}$$

## 3. Proofs

Using (1.4) and (1.9), we immediately reach at (2.1). Indeed,

$$\begin{aligned}
&{}_1F_1(\alpha\,;2\alpha;\,x)\times{}_1F_1(\beta\,;2\beta;\,x)\\
=&e^x\,{}_1F_1(\alpha\,;2\alpha;\,x)\times{}_1F_1(\beta\,;2\beta;\,-x)\\
=&e^x\,{}_2F_3\left(\frac{1}{2}(\alpha+\beta),\,\frac{1}{2}(\alpha+\beta+1);\,\alpha+\frac{1}{2},\,\beta+\frac{1}{2},\,\alpha+\beta;\,\frac{x^2}{4}\right).
\end{aligned}$$

In order to prove (2.2), it is sufficient to show that

(3.1)

$$\begin{aligned}
&e^{-x}\,{}_1F_1(\alpha\,;2\alpha+1;\,x)\times{}_1F_1(\beta\,;2\beta;\,x)\\
=&{}_2F_3\left(\frac{1}{2}(\alpha+\beta+1),\,\frac{1}{2}(\alpha+\beta);\,\alpha+\frac{1}{2},\,\beta+\frac{1}{2},\,\alpha+\beta;\,\frac{x^2}{4}\right)\\
&-\frac{x}{2(2\alpha+1)}\,{}_2F_3\left(\frac{1}{2}(\alpha+\beta+2),\,\frac{1}{2}(\alpha+\beta+1);\,\alpha+\frac{3}{2},\,\beta+\frac{1}{2},\,\alpha+\beta+1;\,\frac{x^2}{4}\right).
\end{aligned}$$

Now, start with the left-hand side of (3.1): Using (1.4), we have

$$\begin{aligned}
\text{L.H.S.}&=e^{-x}\,{}_1F_1(\alpha\,;2\alpha+1;\,x)\times{}_1F_1(\beta\,;2\beta;\,x)\\
&=\left\{e^{-x/2}{}_1F_1(\alpha\,;2\alpha+1;\,x)\right\}\left\{e^{-x/2}{}_1F_1(\beta\,;2\beta;\,x)\right\}\\
&=\left\{e^{-x/2}{}_1F_1(\alpha\,;2\alpha+1;\,x)\right\}\left\{e^{x/2}{}_1F_1(\beta\,;2\beta;\,-x)\right\}
\end{aligned}$$

using (1.5) and (1.6) in the first expression and (1.8) in the second expression, we get

$$\begin{aligned}
&={}_0F_1\left(-\,;\beta+\frac{1}{2};\,\frac{x^2}{16}\right)\times\left\{{}_0F_1\left(-\,;\,\alpha+\frac{1}{2};\,\frac{x^2}{16}\right)\right.\\
&\qquad\left.-\frac{x}{2(2\alpha+1)}{}_0F_1\left(-\,;\,\alpha+\frac{3}{2};\,\frac{x^2}{16}\right)\right\}\\
&={}_2F_3\left(\frac{1}{2}(\alpha+\beta+1),\,\frac{1}{2}(\alpha+\beta);\,\alpha+\frac{1}{2},\,\beta+\frac{1}{2},\,\alpha+\beta;\,\frac{x^2}{4}\right)\\
&\quad-\frac{x}{2(2\alpha+1)}\,{}_2F_3\left(\frac{1}{2}(\alpha+\beta+2),\,\frac{1}{2}(\alpha+\beta+1);\,\alpha+\frac{3}{2},\,\beta+\frac{1}{2},\,\alpha+\beta+1;\,\frac{x^2}{4}\right)\\
&=\text{R.H.S.}
\end{aligned}$$

Similarly the result (2.3) can be deduced. So there is no need to give the proof of (2.3).

Note that the results (2.2) and (2.3) are new and contiguous to (2.1). The results (2.2) and (2.3) are in fact the generalizations of the results (1.2) and (1.3) respectively.

## 4. Special cases

If we take $\beta = 1 - \alpha$ in (2.1), we get the well-known identity due to Preece.

On the other hand, if we take $\beta = 2 - \alpha$ and $\beta = 2 - \alpha$ in (2.1), we get the following results:

$$\begin{aligned} &{}_1F_1(\alpha\,; 2\alpha;\, x) \times {}_1F_1(2-\alpha\,; 4-2\alpha;\, x) \\ &= e^x {}_2F_3\left(1,\, \frac{3}{2};\, \alpha+\frac{1}{2},\, \frac{5}{2}-\alpha,\, 2;\, \frac{x^2}{4}\right) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &{}_1F_1(\alpha\,; 2\alpha;\, x) \times {}_1F_1(3-\alpha\,; 6-2\alpha;\, x) \\ &= e^x {}_2F_3\left(\frac{3}{2},\, 2;\, \alpha+\frac{1}{2},\, \frac{7}{2}-\alpha+\frac{1}{2},\, 3;\, \frac{x^2}{4}\right). \end{aligned} \tag{4.2}$$

Clearly the results (4.1) and (4.2) are closely related to the well-known Preece's identity (1.1).

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