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ANALYSIS OF SOME NONLOCAL BOUNDARY VALUE PROBLEMS ASSOCIATED WITH FEEDBACK CONTROL

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ABSTRACT. Some nonlocal boundary value problems which arise from a feedback control problem are considered. We give a precise statement of the mathematical problems and then prove the existence and uniqueness of the solutions. We consider the Dirichlet type boundary value problem and the Neumann type boundary value problem with nonlinear boundary conditions. We also provide a regularity results for the solutions.

1. Introduction

Let us consider the following boundary value problem:

$$-\Delta u = \psi \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = \phi \quad \text{on} \quad \Gamma_s, \tag{1.2}$$

$$u = \phi + F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)\, g \quad \text{on} \quad \Gamma_c, \tag{1.3}$$

where Ω is a nonempty simply connected domain in R^N, $N = 2$ or 3, with a smooth boundary $\partial\Omega = \Gamma$; Γ_s and Γ_c, on which the sensors and actuators are located, respectively, are portions of Γ. For simplicity we let $\bar{\Gamma} = \bar{\Gamma}_s \cup \bar{\Gamma}_c$ and $\Gamma_s \cap \Gamma_c = \emptyset$. In (1.1)-(1.3), ψ and ϕ denote a given source and a given boundary condition respectively. The function g is a fixed function defined on boundary such that $g \in H^{1/2}(\Gamma)$. The function g has compact support on Γ_c. F is, in general, a linear or nonlinear

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functional on $H^{-1/2}(\Gamma_s)$. This problem motivated by feedback control theory. A typical optimal control problem is the following: find the best controller such that some observation $\gamma = Fu$ achieves a desired value γ_d or is at least as close as possible to γ_d, where F is a general linear or nonlinear operator which may involve integrals of u and/or derivatives of u, where u is the state of the system (See [3, 6, 7, 8]).

Because of the boundary condition (1.3), the above problem (1.1)-(1.3) is not an elliptic problem in the usual sense.

We shall study the existence and uniqueness of the boundary value problem for Dirichlet type and Neumann type in sections 2 and 3, respectively. Here, we introduce some of the notions and function spaces used in subsequent sections (for details see [1] and [9]). Let $H^s(\mathcal{D})$, $s \in R$, be the standard Sobolev space of order s with respect to the set $\mathcal{D}$, where $\mathcal{D}$ is either the domain $\Omega \subseteq R^N$, or its boundary Γ, or part of that boundary. Recall that $H^0(\mathcal{D}) = L^2(\mathcal{D})$. Let the space $H^m_D(\mathcal{D})$ be the closure in the $H^m(\mathcal{D})$ norm of the functions in $H^m(\mathcal{D})$ which have compact support in $\mathcal{D} \setminus D$. We close this subsection by introducing some theorems which will be useful later. The following theorems can be found in [2] and in the references cited there. Throughout, C will be a generic constant with different values on different places.

THEOREM 1.1. *Let $u \in H^k(\Omega)$, $k > 1/2$. Then there exists a trace of the function u on $\partial\Omega$ and*

$$||u||_{H^{k-1/2}(\partial\Omega)} \leq C||u||_{H^k(\Omega)} \tag{1.4}$$

where C does not depend on u.

THEOREM 1.2. *Let $u \in H^k(\Omega)$, $k > 3/2$. Then there exists a trace $\partial u/\partial n$ on $\partial\Omega$ and*

$$\left|\left|\frac{\partial u}{\partial n}\right|\right|_{H^{k-3/2}(\partial\Omega)} \leq C||u||_{H^k(\Omega)} \tag{1.5}$$

where C does not depend on u.

For the case $k \leq 3/2$, we have the following theorem. Let $\mathcal{Q}(\Omega) \subset H^1(\Omega)$ be the space of all functions which satisfy the equation

$$-\Delta u = 0 \tag{1.6}$$

in the weak sense; *i.e.* $Q(\Omega)$ be such a subspace of functions u that

$$\int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx = 0 \tag{1.7}$$

for every $v \in H_0^1(\Omega)$.

THEOREM 1.3. *Let $u \in Q(\Omega)$. Then we have $\partial u/\partial n \in H^{-1/2}(\partial\Omega)$ and*

$$\left|\left|\frac{\partial u}{\partial n}\right|\right|_{H^{-1/2}(\partial\Omega)} \leq C||u||_{H^1(\Omega)} \tag{1.8}$$

where C does not depend on u.

2. The Dirichlet Type Boundary Value Problem

Let Ω be an open, bounded and nonempty simply connected domain and the boundary Γ be Lipschitz-continuous. The boundary Γ consists of Γ_c and Γ_s such that $\Gamma = \bar{\Gamma}_s \cup \bar{\Gamma}_c$ and $\Gamma_s \cap \Gamma_c = \emptyset$. Throughout this section, we will assume $\psi \in L^2(\Omega)$, $\phi \in H^{1/2}(\Gamma)$, and $g \in H^{1/2}_{\Gamma_c}(\Gamma)$ whenever we do not specify the function spaces.

Let us consider the inhomogeneous Dirichlet type boundary value problem

$$-\Delta u = \psi \quad \text{in } \Omega, \tag{2.1}$$

$$u = \phi \quad \text{on } \Gamma_s, \tag{2.2}$$

$$u = F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big) g + \phi \quad \text{on } \Gamma_c, \tag{2.3}$$

where F is a functional on $H^{-1/2}(\Gamma_s)$.

Let us consider the following non-homogeneous Dirichlet's problem:

Given f in $H^{-1}(\Omega)$ and g in $H^{1/2}(\Gamma)$, find a function u such that:

$$-\Delta u = f \quad \text{in } \Omega \tag{2.4}$$

$$u = \phi \quad \text{on } \Gamma \tag{2.5}$$

PROPOSITION 2.1. *Problem (2.4)–(2.5) has one and only one solution $u \in H^1(\Omega)$ and there exists a constant $C = C(\Omega)$ such that*

$$||u||_{1,\Omega} \leq C(||f||_{-1,\Omega} + ||\phi||_{1/2,\Gamma}) \tag{2.6}$$

i.e., u depends continuously upon the data of (2.4) – (2.5).

Proof. For the proof, see [5]. □

THEOREM 2.2. *Let $\tilde{u}$ and $\hat{u}$ be the solution of the boundary value problems*

$$-\Delta\tilde{u} = 0 \quad in\,\Omega, \tag{2.7}$$

$$\tilde{u} = \phi \quad on\,\partial\Omega \tag{2.8}$$

and

$$-\Delta\hat{u} = 0 \quad in\,\Omega, \tag{2.9}$$

$$\hat{u} = g \quad on\,\partial\Omega \tag{2.10}$$

respectively, where $g \in H^{1/2}_{\Gamma_c}(\Gamma)$.

If a nonlinear functional F satisfies

$$|F(h) - F(\check{h})| \le \delta \,||\, h - \check{h} \,||_{-1/2,\Gamma_s} \quad \forall h, \check{h} \in H^{-1/2}(\Gamma_s) \tag{2.11}$$

where δ satisfies

$$0 \le \delta \left|\left| \frac{\partial \hat{u}}{\partial n} \right|\right|_{-1/2,\Gamma_s} < 1, \tag{2.12}$$

then the Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= 0 \quad in\,\Omega, && (2.13)\\ u &= \phi \quad on\,\Gamma_s, && (2.14)\\ u &= F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big) g + \phi \quad on\,\Gamma_c, && (2.15) \end{aligned}$$

has a unique solution.

Proof. By Proposition 2.1, given any $\alpha \in R$, the boundary value problem

$$-\Delta u = 0 \quad \text{in}\,\Omega, \tag{2.16}$$

$$u = \alpha g + \phi \quad \text{on}\,\partial\Omega \tag{2.17}$$

has a unique solution $u = \alpha\hat{u} + \tilde{u}$. Thus, we are looking for a function $u \in \mathcal{Q}(\Omega)$ such that

$$u = F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)\,\hat{u} + \tilde{u} \tag{2.18}$$

For the existence, it is sufficient to show that the equation (2.18) has a solution $\bar{u} \in \mathcal{Q}(\Omega)$. Fixed any $u_0 \in \mathcal{Q}(\Omega)$ and thereafter iteratively define

$$u_{k+1} = F\Big(\frac{\partial u_k}{\partial n}\Big|_{\Gamma_s}\Big)\,\hat{u} + \tilde{u} \tag{2.19}$$

for $k = 0, 1, 2, \ldots$. Then, obviously $u_k \in \mathcal{Q}(\Omega)$ for $k = 0, 1, 2, \ldots$ and

$$\begin{aligned}
&\Big|F\Big(\frac{\partial u_{k+1}}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial u_k}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \\
&\quad\le \delta \,\Big\|\frac{\partial u_{k+1}}{\partial n} - \frac{\partial u_k}{\partial n}\Big\|_{-1/2,\Gamma_s} \\
&\quad= \delta \,\Big\| F\Big(\frac{\partial u_k}{\partial n}\Big|_{\Gamma_s}\Big)\frac{\partial \hat{u}}{\partial n} + \frac{\partial \tilde{u}}{\partial n} - F\Big(\frac{\partial u_{k-1}}{\partial n}\Big|_{\Gamma_s}\Big)\frac{\partial \hat{u}}{\partial n} - \frac{\partial \tilde{u}}{\partial n}\Big\|_{-1/2,\Gamma_s} \\
&\quad= \delta \,\Big\|\frac{\partial \hat{u}}{\partial n}\Big\|_{-1/2,\Gamma_s} \Big| F\Big(\frac{\partial u_k}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial u_{k-1}}{\partial n}\Big|_{\Gamma_s}\Big)\Big|
\end{aligned}$$

and so

$$\begin{aligned}
&\Big|F\Big(\frac{\partial u_{k+1}}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial u_k}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \\
&\qquad\qquad\le \Big(\delta\,\Big\|\frac{\partial \hat{u}}{\partial n}\Big\|_{-1/2,\Gamma_s}\Big)^k \Big|F\Big(\frac{\partial u_1}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial u_0}{\partial n}\Big|_{\Gamma_s}\Big)\Big|
\end{aligned}$$

for $k = 1, 2, \ldots$. Consequently if $k \ge l$,

$$\begin{aligned}
&\|u_k - u_l\|_{1,\Omega} \\
&\quad= \Big| F\Big(\frac{\partial u_{k-1}}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial u_{l-1}}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \,\|\hat{u}\|_{1,\Omega} \\
&\quad\le \sum_{j=l-1}^{k-2} \Big| F\Big(\frac{\partial u_{j+1}}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial u_j}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \,\|\hat{u}\|_{1,\Omega} \\
&\quad\le \Big|F\Big(\frac{\partial u_1}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial u_0}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \,\|\hat{u}\|_{1,\Omega} \sum_{j=l-1}^{k-2} \Big(\delta\,\Big\|\frac{\partial \hat{u}}{\partial n}\Big\|_{-1/2,\Gamma_s}\Big)^j.
\end{aligned}$$

Hence $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{Q}(\Omega)$ provided (2.11)-(2.12), and so there exists a point $\bar{u} \in \mathcal{Q}(\Omega)$ with

$$u_k \to \bar{u} \quad \text{in } \mathcal{Q}(\Omega). \tag{2.20}$$

Clearly

$$\bar{u} = F\Big(\frac{\partial \bar{u}}{\partial n}\Big|_{\Gamma_s}\Big)\hat{u} + \tilde{u}. \tag{2.21}$$

For the uniqueness, let $\breve{u}$ be another solution to the equation (2.18) such that $\breve{u} \neq \bar{u}$. Then, from the hypothesis (2.11), we have

$$\begin{aligned}
||\breve{u} - \bar{u}||_{1,\Omega} &= \Big|F\Big(\frac{\partial \breve{u}}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial \bar{u}}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \, ||\hat{u}||_{1,\Omega} \\
&\leq \delta \Big|\Big| \frac{\partial \hat{u}}{\partial n} \Big|\Big|_{-1/2,\Gamma_s} \Big|F\Big(\frac{\partial \breve{u}}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial \bar{u}}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \, ||\hat{u}||_{1,\Omega} \\
&\vdots \\
&\leq \Big(\delta \Big|\Big| \frac{\partial \hat{u}}{\partial n} \Big|\Big|_{-1/2,\Gamma_s}\Big)^n \Big|F\Big(\frac{\partial \breve{u}}{\partial n}\Big|_{\Gamma_s}\Big) - F\Big(\frac{\partial \bar{u}}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \, ||\hat{u}||_{1,\Omega} \\
&\leq \Big(\delta \Big|\Big| \frac{\partial \hat{u}}{\partial n} \Big|\Big|_{-1/2,\Gamma_s}\Big)^n \delta \Big|\Big|\frac{\partial \breve{u}}{\partial n} - \frac{\partial \bar{u}}{\partial n}\Big|\Big|_{-1/2,\Gamma_s} ||\hat{u}||_{1,\Omega} \\
&\leq \Big(\delta \Big|\Big| \frac{\partial \hat{u}}{\partial n} \Big|\Big|_{-1/2,\Gamma_s}\Big)^n C||\breve{u} - \bar{u}||_{1,\Omega} \, ||\hat{u}||_{1,\Omega}
\end{aligned}$$

for any n. The last inequality can be obtained using Theorem 1.3 where we reset C as $C\delta$. Since the last term goes to zero as $n \to \infty$, it contradict to $\breve{u} \neq \bar{u}$.

Therefore, the proof is completed. □

REMARK 2.1. Note that (2.12) can be viewed as condition on the data g. We also note that if (2.13) is inhomogeneous one can always turn into a homogeneous problem with different ϕ.

We have a regularity results for the soultuion u of the boundary value problem (2.13)-(2.15).

THEOREM 2.3.

$$||u||_{1,\Omega} \leq \frac{\delta\Big|\Big|\frac{\partial \tilde{u}}{\partial n}\Big|\Big|_{-1/2,\Gamma_s} + \beta}{1 - \delta\Big|\Big|\frac{\partial \hat{u}}{\partial n}\Big|\Big|_{-1/2,\Gamma_s}} ||\hat{u}||_{1,\Omega} + ||\tilde{u}||_{1,\Omega} \tag{2.22}$$

where $\beta = |F(0)|$.

Proof. Let us consider the following inequalities

$$\begin{aligned}
\left|F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)\right| &\le \delta\Big\|\frac{\partial u}{\partial n}\Big\|_{-1/2,\Gamma_s} + \Big|F(0)\Big| \\
&= \delta\Big\| \frac{\partial \tilde{u}}{\partial n} + F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big) \frac{\partial \hat{u}}{\partial n} \Big\|_{-1/2,\Gamma_s} + \Big|F(0)\Big| \\
&\le \delta\Big(\Big\|\frac{\partial \tilde{u}}{\partial n}\Big\|_{-1/2,\Gamma_s} + \Big|F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)\Big| \Big\|\frac{\partial \hat{u}}{\partial n}\Big\|_{-1/2,\Gamma_s}\Big) + \Big|F(0)\Big|.
\end{aligned}$$

Let $\beta = |F(0)|$, then provided (2.11) we have

$$\left|F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)\right| \le \frac{\delta\Big\|\frac{\partial \tilde{u}}{\partial n}\Big\|_{-1/2,\Gamma_s} + \beta}{1-\delta\Big\|\frac{\partial \hat{u}}{\partial n}\Big\|_{-1/2,\Gamma_s}}. \tag{2.23}$$

Thus, we have

$$\begin{aligned}
||u||_{1,\Omega} &= \Big\|\tilde{u} + F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)\, \hat{u}\Big\|_{1,\Omega} \\
&\le ||\tilde{u}||_{1,\Omega} + \frac{\delta\Big\|\frac{\partial \tilde{u}}{\partial n}\Big\|_{-1/2,\Gamma_s} + \beta}{1-\delta\Big\|\frac{\partial \hat{u}}{\partial n}\Big\|_{-1/2,\Gamma_s}}\, ||\hat{u}||_{1,\Omega}
\end{aligned}$$

□

Now, let us consider for the case of continuous linear functional F.

THEOREM 2.4. *Let $\tilde{u}$ and $\hat{u}$ be the solution of the boundary value problems (2.7) – (2.8) and (2.9) – (2.10) respectively.*
If a continuous linear functional F satisfies

$$F\Big(\frac{\partial \hat{u}}{\partial n}\Big|_{\Gamma_s}\Big) \ne 1 \tag{2.24}$$

then the Dirichlet boundary value problem (2.13) – (2.15) has a unique solution.

Proof. By Proposition 2.1, given any $\alpha \in R$, the boundary value problem

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{2.25}$$

$$u = \alpha g + \phi \quad \text{on } \partial\Omega \tag{2.26}$$

has a unique solution $u = \alpha\hat{u} + \tilde{u}$.

Taking the normal derivative, from the linearity of F, we get

$$F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big) = F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)F\Big(\frac{\partial \hat{u}}{\partial n}\Big|_{\Gamma_s}\Big) + F\Big(\frac{\partial \tilde{u}}{\partial n}\Big|_{\Gamma_s}\Big) \tag{2.27}$$

Thus, providing (2.24) holds, $F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)$ is well defined and

$$F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big) = \frac{F\Big(\frac{\partial \tilde{u}}{\partial n}\Big|_{\Gamma_s}\Big)}{1 - F\Big(\frac{\partial \hat{u}}{\partial n}\Big|_{\Gamma_s}\Big)} \tag{2.28}$$

By the substitution method, we have shown the existence of the solution u. The uniqueness of the solution follows from the following Lemma and the linearity of the problem. □

REMARK 2.2. We note that for a fixed $g \in H^{1/2}_{\Gamma_c}(\Gamma)$ and a continuous linear functional F the solution u to the boundary value problem (2.13)-(2.15) depends linearly on the boundary condition ϕ and so $\tilde{u}$.

LEMMA 2.5. *For any $g \in H^{1/2}_{\Gamma_s}(\Gamma)$ such that*

$$F\Big(\frac{\partial \hat{u}}{\partial n}\Big|_{\Gamma_s}\Big) \neq 1 \tag{2.29}$$

where $\hat{u}$ is the solution of the boundary value problem

$$-\Delta \hat{u} = 0 \quad in\, \Omega, \tag{2.30}$$

$$\hat{u} = g \quad on\, \Gamma, \tag{2.31}$$

the boundary value problem

$$-\Delta u = 0 \quad in\, \Omega, \tag{2.32}$$

$$u = 0 \quad on\, \Gamma_s, \tag{2.33}$$

$$u = F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big)g \quad on\, \Gamma_c \tag{2.34}$$

has a unique solution $u \equiv 0$.

proof From the boundary conditions, we know that $\tilde{u} = 0$ in Ω. From Theorem 2.4, we only need to show the

$$F\Big(\frac{\partial u}{\partial n}\Big|_{\Gamma_s}\Big) = 0. \tag{2.35}$$

Let $\bar{u}$ be the nontrivial solution of (2.32)-(2.34), *i.e.*, $\bar{u}$ is not identically zero, which implies that

$$F\Big(\frac{\partial \bar{u}}{\partial n}\Big|_{\Gamma_s}\Big) \neq 0. \tag{2.36}$$

Taking the normal derivative and a bounded linear functional F in $\bar{u}$, we have

$$F\Big(\frac{\partial \bar{u}}{\partial n}\Big|_{\Gamma_s}\Big) = F\Big(\frac{\partial \bar{u}}{\partial n}\Big|_{\Gamma_s}\Big)F\Big(\frac{\partial \hat{u}}{\partial n}\Big|_{\Gamma_s}\Big) \tag{2.37}$$

From (2.29), $F\Big(\frac{\partial \bar{u}}{\partial n}\Big)$ must be zero which contradicts (2.36). □

3. The Neumann Type Boundary Value Problem

Let Ω be an open, bounded and nonempty simply connected domain and the boundary Γ be Lipschitz-continuous. The boundary Γ consists of Γ_c and Γ_s such that $\Gamma = \bar{\Gamma}_s \cup \bar{\Gamma}_c$ and $\Gamma_s \cap \Gamma_c = \emptyset$. Throughout this section, we will assume $\phi \in H^{-1/2}(\Gamma)$, and $g \in H^{-1/2}_{\Gamma_c}(\Gamma)$ whenever we do not specify the function spaces.

Let us consider the Neumann Type boundary value problems

$$-\Delta u = 0 \quad \text{in}\, \Omega, \tag{3.1}$$

$$\frac{\partial u}{\partial n} = \phi \quad \text{on}\, \Gamma_s, \tag{3.2}$$

$$\frac{\partial u}{\partial n} = F(u|_{\Gamma_s})g + \phi \quad \text{on}\, \Gamma_c. \tag{3.3}$$

where, in general, F is a nonlinear functional on $H^{1/2}(\Gamma_s)$.

First, we consider the Neumann boundary value problems

$$-\Delta u = 0 \quad \text{in}\, \Omega, \tag{3.4}$$

$$\frac{\partial u}{\partial n} = \phi \quad \text{on}\, \Gamma \tag{3.5}$$

where n is unit exterior normal on $\partial\Omega$. From the theory of the elliptic partial differential equations, we know that there is a solution $u \in H^1(\Omega)$ satisfying (3.4)-(3.5), and

$$\inf ||u||_{H^1(\Omega)} \leq C||\phi||_{H^{-1/2}(\partial\Omega)} \tag{3.6}$$

for some $C > 0$ independent of ϕ, with the infimum taken over all such u satisfying (3.4)-(3.5), if and only if the compatibility condition

$$\int_{\partial\Omega} \phi \, dS = 0 \tag{3.7}$$

is satisfied.

We circumvent the difficulty by seeking u in the quotient space $H^1(\Omega)/R$ equipped with the quotient norm

$$||\dot{u}||_{H^1(\Omega)/R} = \inf_{u \in \dot{u}} ||u||_{H^1(\Omega)}. \tag{3.8}$$

The theorem below states an important property of this space.

THEOREM 3.1. *Let Ω be a bounded, simply connected and Lipschitz continuous open subset of R^N. The space $H^1(\Omega)/R$ is a Hilbert space for the quotient norm* (3.8). *Moreover, on this space the functional $\dot{u} \to |u|_{1,\Omega}$ is a norm equivalent to* (3.8).

Proof. For the proof, see [10]. □

We will assume that the functions $\phi \in H^{-1/2}(\partial\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$ satisfy

$$\int_{\partial\Omega} \phi \, dS = 0 \quad \text{and} \quad \int_{\partial\Omega} g \, dS = 0 \tag{3.9}$$

Hereafter, we will assume that g has compact support on Γ_c.

THEOREM 3.2. *Let $\tilde{u}$ and $\hat{u}$ be the solution of the boundary value problems*

$$-\Delta\tilde{u} = 0 \quad \text{in } \Omega \tag{3.10}$$

$$\frac{\partial\tilde{u}}{\partial n} = \phi \quad \text{on } \partial\Omega \tag{3.11}$$

and

$$-\Delta\hat{u} = 0 \quad \text{in } \Omega \tag{3.12}$$

$$\frac{\partial\hat{u}}{\partial n} = g \quad \text{on } \partial\Omega \tag{3.13}$$

respectively.

If a nonlinear functional F satisfies

$$|F(h) - F(\check{h})| \le \delta \, || \, h - \check{h} \, ||_{1/2,\Gamma_s} \quad \forall h, \check{h} \in H^{1/2}(\Gamma_s) \tag{3.14}$$

where δ satisfies

$$0 \le \delta \left|\left| \hat{u}|_{\Gamma_s} \right|\right|_{1/2,\Gamma_s} < 1, \tag{3.15}$$

then the boundary value problem,

$$-\Delta u = 0 \quad in\, \Omega\ (\Omega^c), \tag{3.16}$$

$$\frac{\partial u}{\partial n} = \phi \quad on\, \Gamma_s, \tag{3.17}$$

$$\frac{\partial u}{\partial n} = F(u|_{\Gamma_s})g + \phi \quad on\, \Gamma_c. \tag{3.18}$$

has a unique solution in $H^1(\Omega)/R$.

Proof. By the theory of the elliptic partial differential equation, for given any $\alpha \in R$, the boundary value problem

$$-\Delta u = 0 \quad \text{in}\Omega, \tag{3.19}$$

$$\frac{\partial u}{\partial n} = \alpha g + \phi \quad \partial\Omega \tag{3.20}$$

has a unique solution $u = \alpha\hat{u} + \tilde{u}$ in $H^1(\Omega)/R$. Thus, we are looking for a function $u \in \mathcal{Q}(\Omega)$ such that

$$u = F(u|_{\Gamma_s})\, \hat{u} + \tilde{u} \tag{3.21}$$

For the existence, it suffices to show that the equation (3.21) has a solution $\tilde{u} \in \mathcal{Q}(\Omega)$. Fixed any $u_0 \in \mathcal{Q}(\Omega)$ and thereafter iteratively define

$$u_{k+1} = F(u_k|_{\Gamma_s})\, \hat{u} + \tilde{u} \tag{3.22}$$

for $k = 0, 1, 2, \ldots$. Then, obviously $u_k \in \mathcal{Q}(\Omega)$ for $k = 0, 1, 2, \ldots$ and

$$\begin{aligned} &|F(u_{k+1}|_{\Gamma_s}) - F(u_k|_{\Gamma_s})| \\ &\quad\le \delta \,||\, u_{k+1} - u_k \,||_{1/2,\Gamma_s} \\ &\quad= \delta \,||\, F(u_k|_{\Gamma_s})\, \hat{u} + \tilde{u} - F(u_{k-1}|_{\Gamma_s})\, \hat{u} - \tilde{u}||_{1/2,\Gamma_s} \\ &\quad= \delta \,||\, \hat{u} \,||_{1/2,\Gamma_s} \mid F(u_k|_{\Gamma_s}) - F(u_{k-1}|_{\Gamma_s}) \mid \end{aligned}$$

and so

$$\begin{aligned} &|F(u_{k+1}|_{\Gamma_s}) - F(u_k|_{\Gamma_s})| \\ &\qquad\qquad \le (\, \delta \,||\, \hat{u} \,||_{1/2,\Gamma_s}\,)^k \, |F(u_1|_{\Gamma_s}) - F(u_0|_{\Gamma_s})| \end{aligned}$$

for $k = 1, 2, \ldots$. Consequently if $k \geq l$,

$$\begin{aligned} |u_k - u_l|_{1,\Omega} \\ &= | F(u_{k-1}|_{\Gamma_s}) - F(u_{l-1}|_{\Gamma_s})| \, |\hat{u}|_{1,\Omega} \\ &\leq \sum_{j=l-1}^{k-2} | F(u_{j+1}|_{\Gamma_s}) - F(u_j|_{\Gamma_s})| \, |\hat{u}|_{1,\Omega} \\ &\leq |F(u_1|_{\Gamma_s}) - F(u_0|_{\Gamma_s})| \, |\hat{u}|_{1,\Omega} \sum_{j=l-1}^{k-2} \left(\delta \, \| \, \hat{u} \, \|_{1/2,\Gamma_s} \right)^j . \end{aligned}$$

Hence $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{Q}(\Omega)$ provided (3.14)-(3.15), and so there exists a point $\bar{u} \in \mathcal{Q}(\Omega)$ with

(3.23) $$u_k \to \bar{u} \quad \text{in } \mathcal{Q}(\Omega).$$

Clearly

(3.24) $$\bar{u} = F(\bar{u}|_{\Gamma_s})\hat{u} + \tilde{u}.$$

For the uniqueness, let $\check{u}$ be another solution to the equation (3.21) such that $\check{u} \neq \bar{u}$. Then, from the hypothesis (2.11), we have

$$\begin{aligned} |\check{u} - \bar{u}|_{1,\Omega} &= |F(\check{u}|_{\Gamma_s}) - F(\bar{u}|_{\Gamma_s})| \, |\hat{u}|_{1,\Omega} \\ &\leq \delta \, \| \, \hat{u} \, \|_{1/2,\Gamma_s} |F(\check{u}|_{\Gamma_s}) - F(\bar{u}|_{\Gamma_s})| \, |\hat{u}|_{1,\Omega} \\ &\;\;\vdots \\ &\leq \left(\delta \, \| \, \hat{u} \, \|_{1/2,\Gamma_s}\right)^n |F(\check{u}|_{\Gamma_s}) - F(\bar{u}|_{\Gamma_s})| \, |\hat{u}|_{1,\Omega} \\ &\leq \left(\delta \, \| \, \hat{u} \, \|_{1/2,\Gamma_s}\right)^n \delta \|\check{u} - \bar{u}\|_{1/2,\Gamma_s} |\hat{u}|_{1,\Omega} \\ &\leq \left(\delta \, \| \, \hat{u} \, \|_{1/2,\Gamma_s}\right)^n C|\check{u} - \bar{u}|_{1,\Omega} \, |\hat{u}|_{1,\Omega} \end{aligned}$$

for any n. The last inequality can be obtained using Theorem 1.1 where we reset C as $C\delta$. Since the last term goes to zero as $n \to \infty$, it contradict to $\check{u} \neq \bar{u}$.

Therefore, the proof is completed. □

Note that (3.15) again can be viewed as a condition on g.

We have a regularity results for the soultuion u of the boundary value problem (3.16)-(3.18).

THEOREM 3.3.

$$|u|_{1,\Omega} \leq \frac{\delta||\tilde{u}||_{-1/2,\Gamma_s} + \beta}{1-\delta||\hat{u}||_{1/2,\Gamma_s}} |\hat{u}|_{1,\Omega} + |\tilde{u}|_{1,\Omega} \tag{3.25}$$

where $\beta = |F(0)|$.

Proof. Let us consider the following inequalities

$$\begin{aligned} |F(u|_{\Gamma_s})| &\leq \delta||u||_{1/2,\Gamma_s} + |F(0)| \\ &= \delta||\ \tilde{u} + F(u|_{\Gamma_s})\ \hat{u}\ ||_{1/2,\Gamma_s} + |F(0)| \\ &\leq \delta(||\tilde{u}||_{1/2,\Gamma_s} + |F(u|_{\Gamma_s})|\ ||\hat{u}||_{1/2,\Gamma_s}) + |F(0)|. \end{aligned}$$

Let $\beta = |F(0)|$, then provided (3.14) we have

$$|F(u|_{\Gamma_s})| \leq \frac{\delta||u||_{1/2,\Gamma_s} + \beta}{1-\delta||\hat{u}||_{1/2,\Gamma_s}}. \tag{3.26}$$

Thus, we have

$$\begin{aligned} |u|_{1,\Omega} &= |\tilde{u} + F(u|_{\Gamma_s})\ \hat{u}|_{1,\Omega} \\ &\leq |\tilde{u}|_{1,\Omega} + \frac{\delta||\tilde{u}||_{1/2,\Gamma_s} + \beta}{1-\delta||\hat{u}||_{1/2,\Gamma_s}} |\hat{u}|_{1,\Omega} \end{aligned}$$

□

For the case of continuous linear functional F, we have the following results.

THEOREM 3.4. *Let* $\tilde{u}$ *and* $\hat{u}$ *be the solution of the boundary value problems* (3.10) – (3.11) *and* (3.12) – (3.13) *respectively.*

If a continuous linear functional F *satisfies*

$$F(\hat{u}|_{\Gamma_s}) \neq 1, \tag{3.27}$$

then the Neumann boundary value problem (3.1) – (3.3) *has a unique solution in* $H^1(\Omega)/R$.

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