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NORMS FOR COMPACT OPERATORS ON HILBERTIAN OPERATOR SPACES

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ABSTRACT. For Hilbert spaces H, K, a compact operator $T : H \to K$, and column, row, operator Hilbert spaces H_c, K_c, H_r, K_r, H_o, K_o, we show that $\|T_{co}\|_{cb}=\|T_{ro}\|_{cb}=\|T_{oc}\|_{cb}=\|T_{or}\|_{cb}=\|T\|_4$.

1. Introduction

The theory of operator spaces and their completely bounded maps has provided a powerful tool for studying operator algebras. In the theory of operator space, bounded operators are replaced by completely bounded operators, isometries by complete isometries, and Banach spaces by operator spaces.

E. Effros and Z. J. Ruan, D. P. Blecher, D. Y. Shin, and G. Pisier study Hilbert spaces as operator spaces. E. Effros and Z. J. Ruan [4], D. P. Blecher [1] study column and row Hilbert spaces, and show that $H_c^* \cong H_r$ and $H_r^* \cong H_c$, G. Pisier [6] studies the operator Hilbert spaces and shows that $H_o^* \cong H_o$, and D. Y. Shin [7] studies column, row and operator Hilbert spaces and shows that $H_c^* \cong H_r$, $H_r^* \cong H_c$ and $H_o^* \cong H_o$ differently.

Let T be a bounded linear operator from a Hilbert space H to a Hilbert space K. We may induce a linear operator from the row Hilbert space H_r to the column Hilbert space K_c defined by $T_{rc}(x) = T(x)$. Similarly, we may induce T_{oc}, T_{cc}, T_{cr}, T_{or}, T_{rr}, T_{co}, T_{ro}, and T_{oo}.

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D. P. Blecher [1], E. Effros and Z.-J. Ruan [4], and D. Y. Shin [7] show that $\|T_{cc}\|_{cb} = \|T_{rr}\|_{cb} = \|T_{oo}\|_{cb} = \|T\|$ and $\|T_{rc}\|_{cb} = \|T_{cr}\|_{cb} = \|T\|_2$.

In this paper we show that $\|T_{co}\|_{cb} = \|T_{ro}\|_{cb} = \|T_{oc}\|_{cb} = \|T_{or}\|_{cb} = \|T\|_4$ for a compact operator.

2. Operator Spaces and Three Hilbertian Operator Spaces

Let E be a vector space over the complex field C, let $M_n(E)$ denote the vector space of $n \times n$ matrices with entries from E, let M_n denote the set of all $n \times n$ complex matrices with C^*-norm.

For $x = [x_{ij}] \in M_m(E)$, $y = [y_{ij}] \in M_n(E)$, $\alpha = [\alpha_{ij}]$, $\beta = [\beta_{ij}] \in M_m$, we write

$$x \oplus y = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in M_{m+n}(E),$$

$$\alpha x = [z_{ij}], x\beta = [w_{ij}] \in M_m(E),$$

where $z_{ij} = \sum_{p=1}^{m} \alpha_{ip} x_{pj}$ and $w_{ij} = \sum_{p=1}^{m} \beta_{pj} x_{ip}$. Here we use the symbol 0 for a rectangular matrix of zero element over E.

If there is a norm $\|\cdot\|_n$ on $M_n(E)$ for each positive integer n, the family of the norms $\{\|\cdot\|_n\}$ is called a matrix norm on E. E is called a space with a matrix norm. If there no danger of confusion, we set $\|\cdot\| = \|\cdot\|_n$.

A space E with a matrix norm is called an operator space if for α, $\beta \in M_n$, $x \in M_n(E)$, it satisfies the following :

$$\|\alpha x \beta\|_n \leq \|\alpha\| \, \|x\|_n \|\beta\| \tag{1}$$

$$\|x \oplus y\|_{m+n} = \max\{\|x\|_m, \|y\|_n\} \tag{2}$$

A subspace of an operator space as a vector space is also operator space.

If H is a Hilbert space, we set $H^n = \overbrace{H \oplus \cdots \oplus H}^{n \text{ times}}$. Let $B(H,K)$ denote the set of all bounded linear operators from a Hilbert space H

to a Hilbert space K and $B(H) = B(H, H)$. We may identify $B(H^n)$ with $M_n(B(H))$. Then $B(H)$ is an operator space.

Suppose that E and F are operator spaces and $\phi : E \to F$ is a linear map. We define the map $\phi_n : M_n(E) \to M_n(F)$ by $\phi_n([x_{ij}]) = [\phi(x_{ij})]$ for $[x_{ij}] \in M_n(E)$. We write $\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in N\}$, where $\|\phi_n\| = \sup\{\|\phi_n(x)\| : x \in M_n(E), \|x\| = 1\}$. We call ϕ completely bounded if $\|\phi\|_{cb} < \infty$. We call ϕ a complete isometry if for every positive integer n, $\phi_n : M_n(E) \to M_n(F)$ is an isometry.

Let $CB(E, F)$ denote the set of all completely bounded linear maps from E to F and $CB(E) = CB(E, E)$. If we identify $M_n(CB(E, F))$ with $CB(E, M_n(F))$ for every positive integer n, then the normed space $CB(E, F)$ with this matrix norm becomes an operator space. Since every bounded linear functional is completely bounded with the same norm, we may identify $B(E, C)$ and $CB(E, C)$. With this identification, $B(E, C)$ becomes an operator space. We call this the operator space dual of E, and denote it by E^*.

Two operator spaces are completely isometrically isomorphic if there is a complete isometry of the first space onto the second.

Let C^n be the n- dimensional, canonical Hilbert space. Given Hilbert space H, we may identify $M_n(H)$ with $B(C^n, H^n)$ for every positive integer n. Then the Hilbert space H with this matrix norm can be an operator space which is called a column Hilbert space and is indicated by H_c and the norm on $M_n(H_c)$ is indicated by $\| \cdot \|_c$. Secondly, we may identify $M_n(H)$ with $B(H^n, C^n)$ for $n \in N$. This gives an operator space structure on H, which is called a row Hilbert space and is indicated by H_r and the norm on $M_n(H_r)$ is indicated by $\| \cdot \|_r$.

G. Pisier [6] shows the following :

THEOREM 1. *For any index set I, there is a Hilbert space H and an operator space $OH(I)$ included in $B(H)$ such that*

(a) $OH(I)$ is isometric to $l_2(I)$ as a Banach space,

(b) the canonical identification between $OH(I)$ and $\overline{OH(I)^*}$ (corresponding to the canonical identification between $l_2(I)$ and $\overline{l_2(I)^*}$) is completely isometric.

Moreover, the space $OH(I)$ is the unique operator space (up to com-

plete isometry) possessing these properties (a) and (b).

This operator space $OH(I)$ is called an operator Hilbert space. Hence given Hilbert space H, we can give it the operator Hilbert space structure. This operator Hilbert space is denoted by H_o and the norm on $M_n(H_o)$ is indicated by $\|\cdot\|_o$.

We say that an operator space E is Hilbertian if the underlying space E is isomorphic to a Hilbert space.

Let $\{E^n_{i,j}\}$ denote the standard matrix units for M_n, that is, $E^n_{i,j}$ is 1 in the (i,j)-entry and 0 elsewhere.

Let $a = [a_{ij}]$ and c be $n \times n$ matrices, let b and d be $m \times m$ matrices, and let $a \otimes b$ be the $mn \times mn$ matrix $[a_{ij}b]$. Then $(a \otimes b)^* = a^* \otimes b^*$, $a \otimes b + a \otimes d = a \otimes (b+d)$, $a \otimes b + c \otimes b = (a+c) \otimes b$, $\|a \otimes b\| = \|a\|\|b\|$, $(a \otimes b)(c \otimes d) = ac \otimes bd$, and $E^n_{i,j} \otimes E^m_{k,l} = E^{mn}_{(i-1)m+k,(j-1)m+l}$.

For an orthonormal basis $\{e_i\}_{i \in I}$ of H and $x = [x_{kl}] \in M_n(H)$ with $x_{kl} = \sum_{i \in I} x^i_{kl} e_i$, we set $x_i = [x^i_{kl}] \in M_n$. We formally write $x = \sum_{i \in I} x_i e_i$. For $a = [a_{kl}] \in M_n$, we denote $\overline{a} = [\overline{a}_{kl}]$, where $\overline{a}_{kl}$ is the complex conjugation of a_{kl}.

D. Y. Shin [7] shows the following.

THEOREM 2. *For* $x = \sum_{i \in I} x_i e_i \in M_n(H)$, *we have* $\|x\|_c = \|\sum_{i \in I} x_i^* x_i\|^{\frac{1}{2}}$, $\|x\|_r = \|\sum_{i \in I} x_i x_i^*\|^{\frac{1}{2}}$, *and* $\|x\|_o = \|\sum_{i \in I} x_i \otimes \overline{x}_i\|^{\frac{1}{2}}$.

For $H = C^m$, we set $C_m = C^m_c$, $R_m = C^m_r$, $O_m = C^m_o$.

PROPOSITION 3. *Let* $E = \{\sum_{i=1}^m a_i E^m_{i,1} : a_i \in C\} \subseteq M_m$, *let* $F = \{\sum_{i=1}^m a_i E^m_{1,i} : a_i \in C\} \subseteq M_m$, *and let* $\{e_i\}_{i=1}^m$ *be the canonical basis for* C^m. *Let* $\phi : C_m \to E$ *defined by* $\phi(\sum_{i=1}^m a_i e_i) = \sum_{i=1}^m a_i E^m_{i,1}$, *and let* $\psi : R_m \to F$ *defined by* $\psi(\sum_{i=1}^m a_i e_i) = \sum_{i=1}^m a_i E^m_{1,i}$. *Then* ϕ *and* ψ *are complete isometries.*

Proof. Let $a = [a_{ij}] \in M_n$, $b = [b_{kl}] \in M_m$. Then we have $a \otimes b = \sum_{i,j=1}^n \sum_{k,l=1}^m a_{ij} b_{kl} E^{mn}_{(i-1)m+k,(j-1)m+l}$. Hence, for $x = \sum_{i=1}^m x_i e_i \in M_n(C_m)$, we have $\phi_n(x) = \sum_{i=1}^k x_i \otimes E_{i,1}$. By elementary calculation, $\|(\sum_{i=1}^m x_i \otimes E_{i,1})^* (\sum_{j=1}^m x_j \otimes E^m_{j,1})\| = \|\sum_{i,j=1}^m x_i^* x_j \otimes E^m_{1,i} E^m_{1,j}\| = \|\sum_{i=1}^m x_i^* x_i \otimes E^m_{i,1}\| = \|\sum_{i=1}^m x_i^* x_i\| = \|x\|_c$ namely, $\|x\|_c = \|\phi_n(x)\|$.

Hence ϕ is a complete isometry. Similarly, we can show that ψ is complete isometry. □

3. Main Results

For Hilbert spaces H, K and a compact operator $T : H \to K$, $|T| = (T^*T)^{\frac{1}{2}}$, there is an orthonormal basis $\{e_i\}_{i\in I}$ for H and non-negative real numbers λ_i $(i \in I)$ such that $|T|(e_i) = \lambda_i e_i$. Put $\|T\|_4 = (\sum_{i\in I} \lambda_i^4)^{\frac{1}{4}}$. It can be infinite.

LEMMA 4. *Let $\{e_i\}_{i=1}^n$ be the canonical basis for C^n, let λ_i be non-negative real numbers for $1 \le i \le n$, and let $T : C^n \to C^n$ be a linear map with $T(e_i)=\lambda_i e_i$. Then $\|T_{co}\|_{cb}=\|T_{ro}\|_{cb} =\|T_{oc}\|_{cb}=\|T_{or}\|_{cb}= \|T\|_4$.*

Proof. By Proposition 3, R_n can be identified with $\{\sum_{i=1}^n a_i E_{1,i}^n : a_i \in C\} \subseteq M_n$. Hence we have $\|T_{or}\|_{cb}=\| \sum_{i=1}^n \lambda_i E_{1i} \otimes \overline{\lambda_i E_{1i}}\|^{\frac{1}{2}}$ by [6, Proposition 1.4] and by elementary caculations $\| \sum_{i=1}^n \lambda_i E_{1i} \otimes \overline{\lambda_i E_{1i}}\|^{\frac{1}{2}}$ $=(\sum_{i=1}^n \lambda_i^4)^{\frac{1}{4}}$. Therefore $\|T_{or}\|_{cb} = (\sum_{i=1}^n \lambda_i^4)^{\frac{1}{4}}$. Similarly we can show that $\|T_{oc}\|_{cb}=\|T\|_4$. Since $R_n^*=C_n$, $C_n^*=R_n$, $O_n^*=O_n$ and λ_i are non-negative real numbers for $1 \le i \le n$, we have $T_{or}^*=T_{co}$, $T_{oc}^*=T_{ro}$. Hence $\|T_{co}\|_{cb}= \|T_{ro}\|_{cb} =\|T\|_4$ by [2, Proposition 2.3]. □

LEMMA 5. *For non-negative real numbers λ_i for $1 \le i \le n$ and $x_i \in M_k$, the following hold.*

(1) *If $\| \sum_{i=1}^n x_i^* x_i\| = 1$, then $\| \sum_{i=1}^n \lambda_i x_i \otimes \overline{\lambda_i x_i}\| \le (\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}}$. In particular, if $k \ge n$, then*
$(\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}} = \sup\{\| \sum_{i=1}^n \lambda_i x_i \otimes \overline{\lambda_i x_i}\| : \| \sum_{i=1}^n x_i^* x_i\| = 1\}$.

(2) *If $\| \sum_{i=1}^n x_i x_i^*\| = 1$, then $\| \sum_{i=1}^n \lambda_i x_i \otimes \overline{\lambda_i x_i}\| \le (\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}}$. In particular, if $k \ge n$, then*
$(\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}} = \sup\{\| \sum_{i=1}^n \lambda_i x_i \otimes \overline{\lambda_i x_i}\| : \| \sum_{i=1}^n x_i x_i^*\| = 1\}$.

(3) *If $\| \sum_{i=1}^n x_i \otimes \overline{x_i}\| = 1$, then $\| \sum_{i=1}^n \lambda_i^2 x_i^* x_i\| \le (\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}}$. In particular, if $k \ge n$, then*
$(\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}} = \sup\{\| \sum_{i=1}^n \lambda_i^2 x_i^* x_i\| : \| \sum_{i=1}^n x_i \otimes \overline{x_i}\| = 1\}$.

(4) *If $\| \sum_{i=1}^n x_i \otimes \overline{x_i}\| = 1$, then $\| \sum_{i=1}^n \lambda_i^2 x_i x_i^*\| \le (\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}}$. In particular, if $k \ge n$, then*
$(\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}} = \sup\{\| \sum_{i=1}^n \lambda_i^2 x_i x_i^*\| : \| \sum_{i=1}^n x_i \otimes \overline{x_i}\| = 1\}$.

Proof. Let $\{e_i\}_{i=1}^n$ be the canonical basis for C^n and $T : C^n \to C^n$ be a linear map with $T(e_i)=\lambda_i e_i$. Then by Lemma 4, $\|T_{ro}\|_{cb}=\|T\|_4$. Let $T_{ro}= \phi$. Then for a positive integer k, $\phi_k : M_k(R_n) \to M_k(O_n)$ and

$\sum_{i=1}^n x_ie_i \in M_k(R_n)$, $\|\sum_{i=1}^n \lambda_i x_i \otimes \overline{\lambda_i x_i}\|^{\frac{1}{2}} = \|\sum_{i=1}^n \lambda_i x_i e_i\|_o$ by Theorem 2 and $\|\sum_{i=1}^n \lambda_i x_i e_i\|_o = \|\sum_{i=1}^n \phi_k(x_ie_i)\|_o \leq \|\phi\|_{cb}\|\sum_{i=1}^n x_ie_i\|_r$. For $k \geq n$, putting $x_i = E_{1,i}^k$, we have $\|\sum_{i=1}^n x_i^* x_i\| = 1$ and $\|\sum_{i=1}^n \lambda_i x_i \otimes \overline{\lambda_i x_i}\| = (\sum_{i=1}^n \lambda_i^4)^{\frac{1}{2}}$. Hence (1) holds. Similarly we can show the others. □

LEMMA 6. *Let H be a Hilbert space, let $\{e_i\}_{i\in I}$ be an othonormal basis for H, let λ_i $(i \in I)$ be non-negative real numbers, and let $T : H \to H$ be a linear map with $T(e_i)=\lambda_i e_i$. Then $\|T_{co}\|_{cb}=\|T_{ro}\|_{cb}$ $=\|T_{oc}\|_{cb}=\|T_{or}\|_{cb}= \|T\|_4$.*

Proof. Let $x = \sum_{i\in I} x_ie_i \in M_k(H)$ and let J be a finite subset of I. By Lemma 5, we have $\|\sum_{i\in J} \lambda_i x_i \otimes \overline{\lambda_i x_i}\| \leq (\sum_{i\in J}\lambda_i^4)^{\frac{1}{2}}\|\sum_{i\in J} x_i^*x_i\|$. Hence $\|\sum_{i\in I} \lambda_i x_i \otimes \overline{\lambda_i x_i}\| \leq (\sum_{i\in I}\lambda_i^4)^{\frac{1}{2}}\|\sum_{i\in I} x_i^*x_i\|$. Therefore $\|T_{co}\|$ $\leq \|T\|_4$. For fixed k, l $(l \leq k)$ and a subset $J=\{i_1, \cdots, i_l\}\subseteq I$, we set $x_{i_1} = E_{1,1}^k, \cdots, x_{i_l} = E_{1,l}^k$ and for $i \notin J$, $x_i = 0$. Then $\|\sum_{i\in I} x_i^*x_i\| = 1$ and $\|\sum_{i\in I} \lambda_i x_i \otimes \overline{\lambda_i x_i}\| = (\sum_{i\in J}\lambda_i^4)^{\frac{1}{2}}$. Hence $\|T_{co}\| \geq \|T\|_4$ and $\|T_{co}\|=$ $\|T\|_4$. Similarly we can show that $\|T_{ro}\| = \|T\|_4$ and by the same reason in the proof of Lemma 4, $\|T_{oc}\|=\|T_{or}\|=\|T\|_4$. □

THEOREM 7. *Let H, K be Hilbert spaces and let $T : H \to K$ be a compactor operator. Then $\|T_{co}\|_{cb}=\|T_{ro}\|_{cb}=\|T_{oc}\|_{cb}=\|T_{or}\|_{cb}= \|T\|_4$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Then $U^*T=|T|$ and there is an orthonormal basis $\{e_i\}_{i\in I}$ of H and non-negative real numbers λ_i $(i \in I)$ such that $|T|(e_i)=\lambda_i e_i$. Hence by Lemma 6 we have $\|\,|T|_{oc}\,\|_{cb}=\|\,|T|\,\|_4=\|T\|_4$.

Since $T_{oc} : H_o \to K_c$ is decomposed $H_o \xrightarrow{|T|_{oc}} H_c \xrightarrow{U_{cc}} K_c$, we have $\|T_{oc}\|_{cb} \leq \|\,|T|_{oc}\,\|_{cb}\|U\|=\|T\|_4$ and since $|T|_{oc} : H_o \to H_c$ is decomposed $H_o \xrightarrow{T_{oc}} K_c \xrightarrow{U_{cc}^*} H_c$, we have $\|\,|T|_{oc}\,\|_{cb} \leq \|T_{oc}\|_{cb}$. Hence $\|T_{oc}\|_{cb} =\|T\|_4$. Similarly we can show the others. □

References

[1] D. P. Blecher, *Tensor product of operator spaces II*, Can. J. Math. **44** (1992), 75–90.

[2] ———, *The standard dual of operator spaces*, Pacific J. Math. **153** (1992), 15–30.

[3] D. P. Blecher and V. I. Paulsen, *Tensor product of operator spaces*, J. Funct. Anal. **99** (1991), 262–291.

[4] E. Effros and Z.-J. Ruan, *Self duality for the Haagerup tensor product and Hilbert space factorizations*, J. Funct. Anal. **100** (1991), 257–284.

[5] V. I. Paulsen, *Completely Bounded Maps and Dilations,* Pitman Research Notes in Math. 146, Longman, London, 1986.

[6] G. Pisier, *The operator Hilbert space OH, complex interpolation and tensor norms*, Memoirs of A. M. S. 585.

[7] D. Y. Shin, *On the three operator space structures of Hilbert spaces,* Comm. Korean Math. Soc. **11** (1996), 983–996.

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