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A PROBLEM FOR ANALYTIC FUNCTIONS OF BOUNDED AND VANISHING MEAN OSCILLATION

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ABSTRACT. In this note we consider some characterizations of analytic functions of bounded and vanishing mean oscillation on the unit disk in $\mathbb{C}$ and answer a question about them in the negative.

1. Introduction

Let $\mathcal{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane, $z = x + iy$, and denote by $dxdy$ the usual area measure on $\mathcal{D}$. For $w \in \mathcal{D}$, let the Möbius transformation $\varphi_w : \mathcal{D} \to \mathcal{D}$ be defined by

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}, \quad w \in \mathcal{D}.$$

For $0 < r < 1$, let $\Delta(w, r) = \{z \in \mathcal{D} : |\varphi_w(z)| < r\}$ be the pseudohyperbolic disk with center w and radius r. The space $BMOA$ ("Bounded Mean Oscillation", see [1]) is the set of all analytic functions on $\mathcal{D}$ for which $\|f\|_{BMOA} < \infty$, where

$$\|f\|_{BMOA} = \sup_{w\in\mathcal{D}} \left(\int_0^{2\pi} |f(\varphi_w(e^{i\theta})) - f(w)|^2 d\theta \right)^{1/2}.$$

Contained in $BMOA$ is the subspace $VMOA$ ("Vanishing Mean Oscillation"), the set of all analytic functions f on $\mathcal{D}$ for which

$$\lim_{|w|\to 1-} \int_0^{2\pi} |f(\varphi_w(e^{i\theta})) - f(w)|^2 d\theta = 0.$$

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It is well-known that for a function f analytic on D we have (see [3])

$$(1.1)\qquad f \in BMOA \Longleftrightarrow \sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^2(1-|\varphi_w(z)|^2)dxdy < \infty$$

and (see [5])

$$(1.2)\qquad f \in VMOA \Longleftrightarrow \lim_{|w|\to 1-} \iint_{\mathcal{D}} |f'(z)|^2(1-|\varphi_w(z)|^2)dxdy = 0.$$

The Bloch space $\mathcal{B}$ is the set of all analytic functions f on $\mathcal{D}$ for which $\|f\|_{\mathcal{B}} = \sup_{z\in\mathcal{D}} |f'(z)|(1-|z|^2) < \infty$, and the little Bloch space $\mathcal{B}_0$ is contained in $\mathcal{B}$ for which $\lim_{|z|\to 1-} |f'(z)|(1-|z|^2) = 0$. For $0 < p < \infty$ and $1 < \eta < \infty$, we know that (see [6])

$$f \in \mathcal{B} \Longleftrightarrow \sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^{\eta}dxdy < \infty$$

and

$$f \in \mathcal{B}_0 \Longleftrightarrow \lim_{|w|\to 1-} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^{\eta}dxdy = 0.$$

The Bloch space $\mathcal{B}$ and the space $BMOA$ share many analogous properties, as do the little Bloch space $\mathcal{B}_0$ and the space $VMOA$. Motivated by these facts and the observation of the equivalents (1.1) and (1.2) for $BMOA$ and $VMOA$, respectively, Stroethoff asked in [6] the following:

Question. Let f be an analytic function on $\mathcal{D}$ and $0 < p < \infty$. Are the following statements true?

$$f \in BMOA \Longleftrightarrow \sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)dxdy < \infty.$$

$$f \in VMOA \Longleftrightarrow \lim_{|w|\to 1-} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)dxdy = 0.$$

Choa and Miao settled the question above in the negative, respectivly, that is:

THEOREM A ([2],[4]).

(i) *If* $0 < p < 2$, *then there exists an analytic function* $f \in BMOA$ *such that*

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)\,dxdy = \infty.$$

(ii) *If* $0 < p < 2$, *then there exists an analytic function* $f \in VMOA$ *such that*

$$\lim_{|w|\to 1-} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)\,dxdy \neq 0.$$

Moreover, Stroethoff proved in [6] the following result:

THEOREM B. *Let* f *be an analytic function* f *on* $\mathcal{D}$ *and* $0 < p < \infty, 0 < \sigma < 1$. *Then*

(i) $\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\sigma dxdy < \infty \Longrightarrow f \in BMOA$,

(ii) $\lim_{|w|\to 1} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\sigma dxdy = 0 \Longrightarrow f \in VMOA$.

However, Theorem B gave only sufficient conditions for an analytic function f to belong to the spaces $BMOA$ and $VMOA$. A natural question then arises: *for* $0 < p < \infty, 0 < \sigma < 1$, *are the conditions (i) and (ii) in Theorem B necessary for* f *to belong to* $BMOA$ *and* $VMOA$, *respectively?* In this paper we answer this question in the negative. Our result is the following:

THEOREM. *Let* $0 < p < \infty$ *and* $0 < \sigma < 1$.

(A) *There exists an analytic function* $f \in BMOA$ *such that*

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\sigma dxdy = \infty.$$

(B) *There exists an analytic function* $f \in VMOA$ *such that*

$$\lim_{|w|\to 1-} \iint_{\mathcal{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\sigma dxdy \neq 0.$$

2. Lemma

Before embarking into the proof of Theorem, let us state the definition of α-Carleson measure and its some equivalent conditions, which will be used in our proof.

For a subarc $I \subset \partial\mathcal{D}$, where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$, let

$$S(I) = \{z \in \mathcal{D} : 1 - |I| \leq |z| < 1, z/|z| \in I\}.$$

If $|I| \geq 1$ then we put $S(I) = \mathcal{D}$. For $0 < \alpha < \infty$, we say that a positive measure μ defined on $\mathcal{D}$ is an α-Carleson measure if

$$\sup\{\mu(S(I))/|I|^\alpha : I \subset \partial\mathcal{D}\} < \infty.$$

If $\alpha = 1$, we get the classical Carleson measure (see [3]).

LEMMA [7]. *Let μ be a positive measure and $\alpha > 1$. Then the following statements are equivalent:*

(a) *μ is an α-Carleson measure.*

(b) *for $0 < r < 1$, there exists constant C such that*

$$\mu(\Delta(w,r)) \leq C(1 - |w|)^\alpha, w \in \mathcal{D}.$$

(c)

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} \left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^\alpha d\mu(z) < \infty.$$

REMARK. Lemma above is not true for the case $0 < \alpha \leq 1$.

3. The proof of Theorem

(A) For $0 < p < 2$, by (i) in Theorem A there exsits $f \in BMOA$ such that

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^p (1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)dxdy = \infty, \tag{3.1}$$

it follows that for $0 < \sigma < 1$

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_w(z)|^2)^\sigma dxdy = \infty. \tag{3.2}$$

By Hölder's inequality, for $0 < p < 2$ and $0 < \sigma < 1$ we get

$$\begin{aligned} &\iint_{\mathcal{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_w(z)|^2) dxdy \\ &\leq \left(\iint_{\mathcal{D}} |f'(z)|^2 (1-|\varphi_w(z)|^2)^\sigma dxdy \right)^{p/2} \times \\ &\times \left(\iint_{\mathcal{D}} (1-|z|^2)^{-2} (1-|\varphi_w(z)|^2)^{\frac{2-p\sigma}{2-p}} dxdy \right)^{(2-p)/2} \\ &= \left(\iint_{\mathcal{D}} |f'(z)|^2 (1-|\varphi_w(z)|^2)^\sigma dxdy \right)^{p/2} \times \\ &\times \left(\iint_{\mathcal{D}} (1-|z|^2)^{-2+\frac{2-p\sigma}{2-p}} \left(\frac{1-|w|^2}{|1-\overline{w}z|^2} \right)^{\frac{2-p\sigma}{2-p}} dxdy \right)^{(2-p)/2}. \end{aligned} \tag{3.3}$$

Since the differential form $(1-|z|^2)^{-2+\frac{2-p\sigma}{2-p}} dxdy$ is $\frac{2-p\sigma}{2-p}$-Carleson measure and $\frac{2-p\sigma}{2-p} > 1$, by Lemma we have

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} (1-|z|^2)^{-2+\frac{2-p\sigma}{2-p}} \left(\frac{1-|w|^2}{|1-\overline{w}z|^2} \right)^{\frac{2-p\sigma}{2-p}} dxdy = M \neq 0. \tag{3.4}$$

Therefore, by (3.3) and (3.4) we obtain

$$\begin{aligned} &\iint_{\mathcal{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_w(z)|^2) dxdy \\ &\leq M' \left(\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^2 (1-|\varphi_w(z)|^2)^\sigma dxdy \right)^{p/2}. \end{aligned} \tag{3.5}$$

Thus, by (3.1) and (3.5) we have

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^2 (1-|\varphi_w(z)|^2)^\sigma dxdy = \infty. \tag{3.6}$$

By (3.2) and (3.6) Theorem is valid for cases $0 < p \le 2$ and $0 < \sigma < 1$

Now we consider the cases $2 < p < \infty$ and $0 < \sigma < 1$. Let $q = 1 + \frac{2}{p}$ By (i) in Theorem A there exsits $g \in BMOA$ such that

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |g'(z)|^q (1-|z|^2)^{q-2}(1-|\varphi_w(z)|^2)dxdy = \infty.$$

By Hölder's inequality we have

$$\begin{aligned}
\infty &= \sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |g'(z)|^q (1-|z|^2)^{q-2}(1-|\varphi_w(z)|^2)dxdy \\
&= \sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |g'(z)|^{1+\frac{2}{p}} (1-|z|^2)^{\frac{2}{p}-1}(1-|\varphi_w(z)|^2)^{\frac{\sigma}{p}+1-\frac{\sigma}{p}}dxdy \\
&\le \sup_{w\in\mathcal{D}} \left(\iint_{\mathcal{D}} |g'(z)|^{2+p} (1-|z|^2)^{p}(1-|\varphi_w(z)|^2)^{\sigma}dxdy\right)^{1/p} \times \\
&\times \sup_{w\in\mathcal{D}} \left(\iint_{\mathcal{D}} (1-|z|^2)^{-2}(1-|\varphi_w(z)|^2)^{\frac{p-\sigma}{p-1}}dxdy\right)^{(p-1)/p}.
\end{aligned} \tag{3.7}$$

Since $g \in BMOA \subset \mathcal{B}$, we set $\sup_{z\in\mathcal{D}} |g'(z)|(1-|z|^2) = K$. Hence

$$\begin{aligned}
\infty &= \sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |g'(z)|^q (1-|z|^2)^{q-2}(1-|\varphi_w(z)|^2)dxdy \\
&\le \sup_{w\in\mathcal{D}} \left(\iint_{\mathcal{D}} |g'(z)|^{2+p} (1-|z|^2)^{p}(1-|\varphi_w(z)|^2)^{\sigma}dxdy\right)^{1/p} \times \\
&\times \sup_{w\in\mathcal{D}} \left(\iint_{\mathcal{D}} (1-|z|^2)^{-2}(1-|\varphi_w(z)|^2)^{\frac{p-\sigma}{p-1}}dxdy\right)^{(p-1)/p} \\
&\le K^{2/p} \sup_{w\in\mathcal{D}} \left(\iint_{\mathcal{D}} |g'(z)|^{p} (1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^{\sigma}dxdy\right)^{1/p} \times \\
&\times \sup_{w\in\mathcal{D}} \left(\iint_{\mathcal{D}} (1-|z|^2)^{-2+\frac{p-\sigma}{p-1}}\left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^{\frac{p-\sigma}{p-1}} dxdy\right)^{(p-1)/p} \\
&\le K' \sup_{w\in\mathcal{D}} \left(\iint_{\mathcal{D}} |g'(z)|^{p} (1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^{\sigma}dxdy\right)^{1/p}
\end{aligned} \tag{3.8}$$

since $(1-|z|^2)^{-2+\frac{p-\sigma}{p-1}}dxdy$ is $\frac{p-\sigma}{p-1}$ -Carleson measure and $\frac{p-\sigma}{p-1} > 1$. Therefore, from (3.8) we know that there exsits $g \in BMOA$ such that

$$\sup_{w\in\mathcal{D}} \iint_{\mathcal{D}} |g'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_w(z)|^2)^\sigma dxdy = \infty$$

for $2 < p < \infty, 0 < \sigma < 1$. This shows that (A) holds for all cases.

Similar to the proof of (A), we can get (B) by (ii) in Theorem A. Thus the proof of Theorem is complete.

References

[1] A. Baernstein, *Analytic functions of bounded mean oscillation*, Aspects of Contemporary Complex Analysis, Academic Press, New York, 1979, pp. 3–36.

[2] J. S. Choa, *Note on the space BMOA*, Canad. Math. Bull. **35** (1992), 40–45.

[3] J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[4] J. Miao, *A property of analytic functions with Hadamard gaps*, Bull. Austral. Math. Soc. **45** (1992), 105–112.

[5] S. C. Power, *Vanishing Carleson measuer*, Bull. London Math. Soc. **12** (1980), 207-210.

[6] K. Stroethoff, *Besov-type characterizations for the Bloch spaces*, Bull. Austral. Math. Soc. **39** (1989), 405–420.

[7] H. Wulan, *Carleson measure and the derivatives of functions in BMO*, J. Inner Mongolia Normal University **2** (1993), 1–9.

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