Bull. Korean Math. Soc. **35** (1998), No. 2, pp. 279–292

MAXIMAL INEQUALITIES WITH AN APPLICATION TO THE WEAK CONVERGENCE FOR 2-PARAMETER ARRAYS OF POSITIVELY DEPENDENT RANDOM VARIABLES

TAE-SUNG KIM AND EUN-YANG SEOK

ABSTRACT. We derive maximal inequalities of linearly positive quadrant dependent ($LPQD$) random variables for $d = 1, 2$. With an application we also obtain the weak convergence for 2-parameter arrays of $LPQD$ random variables.

1. Introduction

Lehmann(1966) introduced a simple and natural definition of positive dependence: Two random variables X and Y are said to be positive quadrant dependent(PQD) if for any real x, y $P(X > x, Y > y) \geq P(X > x)P(Y > y)$ and a sequence $\{X_j : j \geq 1\}$ of random variables is called pairwise PQD if for any real r_i, r_j and $i \neq j$, $P(X_i > r_i, X_j > r_j) \geq P(X_i > r_i)P(X_j > r_j)$. A much stronger concept than pairwise PQD was considered by Esary, Proschan, and Walkup(1967): A finite family of random variables is said to be associated if

$$Cov(f(X_1, \cdots, X_m), g(X_1, \cdots, X_m)) \geq 0 \tag{1}$$

for any coordinatewise nondecreasing functions f and g on R^m whenever the covariance exists. A family of infinite number of random variables is associated if every finite subfamily is associated. A sequence $\{X_j : j \geq 1\}$

Received April 4, 1997. Revised September 27, 1997.
1991 Mathematics Subject Classification: 60F05, 60E15.
Key words and phrases: associated, linearly positive quadrant dependent, maximal inequality, weak convergence.
This paper was partially supported by Won Kwang University in 1997.

of random variables is said to be linearly positive quadrant dependent ($LPQD$) if for any disjoint A, B and positive r_j's

$$\sum_{i\in A} r_i X_i \text{ and } \sum_{j\in B} r_j X_j \text{ are PQD.} \tag{2}$$

Let us remark that this concept of positive dependence is between pairwise PQD and association and it is well known(see, for example, [11, p131]) that neither pairwise PQD nor $LPQD$ nor association implies the others. A d-parameter array $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ of random variables is called stationary if for all m and for all $\underline{j}, \underline{k}_1, \cdots, \underline{k}_m \in Z^d (X_{\underline{k}_1}, \cdots, X_{\underline{k}_m})$ has the same distribution as $(X_{\underline{j}+\underline{k}_1}, \cdots, X_{\underline{j}+\underline{k}_m})$. Throughout this paper we will deal with the stationary arrays. Newman(1980) derived a weak convergence of finite dimensional distributions for the d-parameter arrays of associated random variables. When $d = 1$ and 2 this convergence can be strengthened to yield the invariance principles(see [12] and [13]). Dabrowski(1985) proved a functional law of iterated logarithm for $d = 1$. Burton and Kim(1988) showed an invariance principle for the d-parameter arrays under a stronger moment condition than Newman and Wright (1982). In the nonstationary associated case, Birkel(1988) obtained the invariance principle for $d = 1$ under the second moment condition and Kim(1996) derived an invariance principle for d-parameter arrays under $2 + \delta$ moment condition. For linearly positive quadrant dependence Birkel(1993) proved a functional central limit theorem for a nonstationary $LPQD$ sequence under $2 + \delta$ moment condition and Kim and Seo(1996) extended it to the d-parameter arrays.

Newman(1980) had already mentioned that the weak convergence of finite dimensional distributions for stationary d-parameter arrays of associated random variables, still holds for the LPQD case instead of association(see [10, p122]) under second moment condition. The following theorem is a modified version of Theorem 2 in Newman(1980):

Theorem A (Newman, 1980.) Let $\{X_{\underline{j}} : \underline{j} = (j_1, \cdots, j_d) \in Z^d\}$ be a stationary d-parameter array of $LPQD$ random variables with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume

$$0 < \sigma^2 = \sum_{\underline{j}\in Z^d} Cov(X_{\underline{0}}, X_{\underline{j}}) < \infty. \tag{3}$$

For $\underline{t} \in [\underline{0}, \underline{1}]$ define

(4)
$$W_n(\underline{t}) = n^{-\frac{d}{2}} \sum_{j_1=1}^{[nt_1]} \cdots \sum_{j_d=1}^{[nt_d]} X_{\underline{j}},$$

where $[\cdot]$ denotes the usual greatest integer function, and let $W_n(\underline{t})$ be the d-parameter Wiener process, a mean zero Gaussian process with $Cov(W(\underline{t}), W(\underline{s})) = \sigma^2 \prod_{i=1}^{d} \min(t_i, s_i)$. Then the finite dimensional distributions of W_n converges in distribution to those of W.

By requiring the stronger concept of association instead of $LPQD$ Newman and Wright (1982) obtained a weak convergence for a stationary 2-parameter array of associated random variables.

THEOREM B (Newman, Wright 1982.) *Let $\{X_{\underline{j}} : \underline{j} \in Z^2\}$ be a stationary 2-parameter array of associated random variables with $EX_{\underline{j}} = 0$, $EX_{\underline{j}}^2 < \infty$. Let $W_n(t_1, t_2) = n^{-1} \sum_{j_1=1}^{[nt_1]} \sum_{j_2=1}^{[nt_2]} X_{\underline{j}}$ and let $W(\underline{t})$ be the 2-parameter Wiener process, a mean zero Gaussian process with $Cov(W(\underline{t}), W(\underline{s})) = \sigma^2 \prod_{i=1}^{2} \min(t_i, s_i)$. Assume that (3) holds for $d = 2$. Then $W_n(\underline{t})$ converges weakly to $W(\underline{t})$.*

In this paper, we show maximal inequalities of $LPQD$ random variables for $d = 1$ and 2 and obtain a weak convergence for 2-parameter arrays of $LPQD$ random variables by using the maximal inequality together with Theorem A.
In Section 2 maximal inequalities for one dimensional $LPQD$ random variables are presented and in Section 3 maximal inequalities for the 2-dimensional arrays are derived by applying the methods of the proofs of the maximal inequalities for associated random variables in Newman and Wright(1982). We also obtain the weak convergence for a stationary 2-parameter array of $LPQD$ random variables by using the maximal inequality and Theorem A in Section 4.
The problem of proving the maximal inequality of associated random variables for $d \geq 3$ is presently an open question(see [13]). Hence to show the maximal inequality of $LPQD$ random variables for $d \geq 3$ is also open.

2. Preliminaries

We starts this section with introducing Lemma 1 of Lehmann(1966).

LEMMA 2.1. (Lehmann, 1966.) *If* X_1 *and* X_2 *are PQD then* $Cov(f(X_1)\, g(X_2)) \geq 0$ *for any nondecreasing(nonincreasing) functions* f, g.

Let $S_n^* = \max(S_1, \cdots, S_n)$ *and* $S_n^{**} = \min(S_1, \cdots, S_n)$, *where* $S_n = X_1 + \cdots + X_n$.

THEOREM 2.1. *Let* $\{X_j : j \geq 1\}$ *be a sequence of stationary LPQD random variables with* $EX_j = 0$ *and let* m *be a nonnegative and nondecreasing function with* $m(0) = 0$. *Then for any* n

$$E\left(\int_0^{S_n^*} udm(u)\right) \leq E(S_n m(S_n^*)), \tag{5}$$

$$E\left(\int_0^{S_n^{**}} udm(u)\right) \leq E(S_n m(S_n^{**})), \tag{6}$$

and for any $\lambda > 0$,

$$\lambda P(S_n^* \geq \lambda) \leq \int_{\{S_n^* \geq \lambda\}} S_n dP, \tag{7}$$

$$\lambda P(S_n^{**} \geq \lambda) \leq \int_{\{S_n^{**} \geq \lambda\}} S_n dP. \tag{8}$$

Proof. Following the lines of the proof of Theorem 3 of Newman and Wright(1982) we will prove this theorem. Let $S_0^* = 0$. Then

$$S_n m(S_n^*) = \sum_{k=0}^{n-1} S_{k+1}(m(S_{k+1}^*) - m(S_k^*)) + \sum_{k=1}^{n-1}(S_{k+1} - S_k)m(S_k^*). \tag{9}$$

Note that either $S_k^* = S_{k+1}^*$ or $S_{k+1} = S_{k+1}^* > S_k^*$. Thus for any k,

$$\begin{aligned} S_{k+1}(m(S_{k+1}^*) - m(S_k^*)) &= S_{k+1}^*(m(S_{k+1}^*) - m(S_k^*)) \\ &\geq \int_{S_k^*}^{S_{k+1}^*} u\, dm(u). \end{aligned} \tag{10}$$

From (9) and (10) we obtain

$$(11)\quad \begin{aligned} S_n m(S_n^*) &\geq \sum_{k=0}^{n-1} \int_{S_n^*}^{S_{k+1}^*} u\, dm(u) + \sum_{k=1}^{n-1}((S_{k+1} - S_k)m(S_k^*)) \\ &= \int_0^{S_n^*} u\, dm(u) + \sum_{k=1}^{n-1}((S_{k+1} - S_k)m(S_k^*)). \end{aligned}$$

Next we note that X_{k+1} and $S_j, 1 \leq j \leq k$ are PQD by the definition of $LPQD$(see (2)) and that $m(S_k^*)$ is a nondecreasing function of $S_j, 1 \leq j \leq k$, since S_k^* is a nondecreasing function of $S_j, 1 \leq j \leq k$. Thus by Lemma 2.1 and assumption $EX_j = 0$

$$(12)\qquad E((S_{k+1} - S_k)m(S_k^*)) = Cov(X_{k+1}, m(S_k^*)) \geq 0.$$

By taking the expection of (11) and using (12) we obtain

$$E(S_n m(S_n^*)) \geq E(\int_0^{S_n^*} u\, dm(u)),$$

which yields (5). Take $m(u) = 1_{\{u \geq \lambda\}}$ in (5). Then it follows from (5) that

$$(13)\quad \begin{aligned} L.H.S \text{ of } (5) &= E(\int_0^{S_n^*} u\, dm(u)) = E[u 1_{\{u\geq\lambda\}}]_0^{S_n^*} = E[S_n^* 1_{\{S_n^* \geq \lambda\}}] \\ &\geq E[\lambda 1_{\{S_n^*\geq\lambda\}}] = \lambda P\{S_n^* \geq \lambda\}, \end{aligned}$$

and

$$(14)\quad \begin{aligned} R.H.S \text{ of } (5) &= E(S_n m(S_n^*)) = E[S_n 1_{\{S_n^*\geq\lambda\}}] \\ &= \int_{\{S_n^*\geq\lambda\}} S_n dP. \end{aligned}$$

Similarly, (6) and (8) are proved. □

The following corollary is corresponding to Corollary 4 of Newman and Wright(1982).

COROLLARY 2.3. *Let $\{X_j : j \geq 1\}$ be a stationary sequence of $LPQD$ random variables with $EX_j = 0$. Then*

$$(15)\qquad E((S_n^* - S_n)^2) \leq E(S_n^2)$$

$$(16)\qquad E((S_n^{**} - S_n)^2) \leq E(S_n^2)$$

Proof. By taking $m(u) = u1_{\{u\geq 0\}}$ in (5) we have $E(S_n^*)/2 \leq E(S_nS_n^*)$, which yields $ES_n^{*2} - 2(ES_nS_n^*) + ES_n^2 \leq ES_n^2$ and thus (15) holds. Similarly, (16) is obtained. □

Applying the methods of the proofs in Corollary 5 and Corollary 6 of [10] we will prove Corollary 2.4 and Corollary 2.5.

COROLLARY 2.4. *Let $\{X_j : j \geq 1\}$ be a stationary sequence of LPQD random variables with $EX_j = 0$. Then*

$$E(S_n^{*2}) \leq E(S_n^2), \tag{17}$$

$$E(S_n^{**2}) \leq E(S_n^2). \tag{18}$$

Proof. Define $T_1 = 0$ and $T_k = \sum_{i=n-k+2}^{n} X_i$ for $k = 2, 3, \cdots, (n+1)$ and let $T_n^{**} = min(T_1, \cdots, T_n)$. Clearly, $T_2 = X_n, T_3 = X_n + X_{n-1}, \cdots, T_n = X_n + \cdots + X_2, T_{n+1} = X_n + \cdots + X_1$. Since $\{X_n, X_{n-1}, \cdots, X_2, X_1\}$ is a sequence of $LPQD$ random variables by (6) with $m(u) = u1_{\{u\geq 0\}}$

$$E(T_n^{**2}/2) \leq E(T_nT_n^{**}). \tag{19}$$

Note that X_1 and $T_j, 1 \geq j \geq n$ are PQD by definition of $LPQD$(see (2)) and that T_n^{**} is a nondecreasing function of $T_j, 1 \leq j \leq n$. Hence $E(X_1T_n^{**}) = Cov(X_1, T_n^{**}) \geq 0$ by Lemma 2.1 and

$$\begin{aligned} E(T_nT_n^{**}) &\leq E(T_nT_n^{**}) + E(X_1T_n^{**}) \\ &= E(T_{n+1}T_n^{**}). \end{aligned} \tag{20}$$

From (19) and (20) we obtain $E(T_{n+1} - T_n^{**})^2 \leq E(T_{n+1}^2)$, which is the same as (17) since $T_{n+1} = S_n$ and

$$\begin{aligned} T_{n+1} - T_n^{**} &= \max(T_{n+1} - T_n, T_{n+1} - T_{n-1}, \cdots, T_{n+1} - T_1) \\ &= \max(S_1, S_2, \cdots, S_n) = S_n^*. \end{aligned}$$

Next, to show (18) let $T_n^* = max(T_1, \cdots, T_n)$ and take $m(u) = u1_{\{u\geq 0\}}$ in (6). Then

$$E(T_n^{*2}/2) \leq E(T_nT_n^*). \tag{21}$$

Note that X_1 and $T_j, 1 \leq j \leq n$ are PQD by definition of $LPQD$ and that T_n^* is a nondecreasing function of T_j. Hence $E(X_1T_n^*) = Cov(X_1, T_n^*) \geq 0$

by Lemma 2.1 and

$$(22) \qquad \begin{aligned} E(T_n T_n^*) &\le E(T_n T_n^*) + E(X_1 T_n^*) \\ &= E(T_{n+1} T_n^*). \end{aligned}$$

From (21) and (22) we obtain $E(T_{n+1} - T_n^*)^2 \le E(T_{n+1}^2)$, which is the same as (18) since $T_{n+1} = S_n$ and $T_{n+1} - T_n^* = \min(S_1, S_2, \cdots, S_n) = S_n^{**}$. □

The next corollary will be used to obtain the maximal inequality for $d = 2$ in Section 3 below.

COROLLARY 2.5. *Let $\{X_j : j \ge 1\}$ be a stationary sequence of LPQD random variables with $EX_j = 0$. Then for $0 \le \lambda_1 < \lambda_2$,*

$$(23) \qquad P(S_n^* \ge \lambda_2) \le \left(\frac{ES_n^2}{(\lambda_2 - \lambda_1)^2} \right)^{\frac{1}{2}} (P(S_n \ge \lambda_1))^{\frac{1}{2}},$$

$$(24) \quad P(\max(|S_1|, \cdots, |S_n|) \ge \lambda_2) \le \sqrt{2} \left(\frac{ES_n^2}{(\lambda_2 - \lambda_1)^2} \right)^{\frac{1}{2}} (P(|S_n| \ge \lambda_1))^{\frac{1}{2}}.$$

Proof. Starting from (7) with $\lambda = \lambda_2$ we have

$$\begin{aligned} \lambda_2 P(S_n^* \ge \lambda_2) &\le \int_{\{S_n^* \ge \lambda_2\}} S_n dP \le \int_{\{S_n^* \ge \lambda_1\}} S_n dP + \int_{\{S_n^* \ge \lambda_2, S_n < \lambda_1\}} S_n dP \\ &\le \int_{\{S_n^* \ge \lambda_1\}} S_n dP + \lambda_1 P(S_n^* \ge \lambda_2), \end{aligned}$$

which immediately yields

$$(25) \qquad P(S_n^* \ge \lambda_2) \le (\lambda_2 - \lambda_1)^{-1} E(S_n 1_{\{S_n \ge \lambda_1\}}).$$

The Cauchy-Schwarz inequality applied to the right hand side of (25) then yields (23). To obtain (24) we add to (23) the analogous inequality with all X_i's replaced by their negatives(which also are *LPQD*) and use the fact that for $A, B \ge 0$, $\sqrt{A} + \sqrt{B} \le \sqrt{2(A + B)}$. □

3. A maximal inequality of 2-parameter arrays

Throughout this section we deal with a stationary 2-parameter array $\{X_{\underline{j}} : \underline{j} = (j_1, j_2) \in Z^2\}$ of $LPQD$ random variables with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$, and the partial sum

$$S_{\underline{j}} = S_{(j_1,j_2)} = \sum_{i=1}^{j_1} \sum_{k=1}^{j_2} X_{(i,k)}.$$

We also define for $m, n \geq 1$,

$$S^*_{(m,n)} = \max\{S_{\underline{j}} : 1 \leq j_1 \leq m, 1 \leq j_2 \leq n\}.$$

In order to strengthen Theorem A to obtain weak convergnce for $d = 2$, we need a maximal inequality which controls the tail of $S^*_{(m,n)}$ in terms of the tail of $S_{(m,n)}$ as done for $d = 1$ by (23). Our $d = 2$ result will in fact be based on (23) and the key step is the following lemma; our approach is modelled after previous work on maximal inequalities for 2-parameter associated random variables.

LEMMA 3.1. *For fixed m, let*

$$\overline{S}_j = \max\{S_{(k,j)} : 1 \leq k \leq m\}, \tag{26}$$

$$\overline{S}^*_j = \max(\overline{S}_1, \cdots, \overline{S}_j). \tag{27}$$

Then for all nonnegative and nondecreasimg function m with $m(0)$,

$$E(\overline{S}_{j+1} - \overline{S}_j)m(\overline{S}^*_n) \geq 0. \tag{28}$$

Proof. Note that $\overline{S}^*_j = S^*_{(m,j)}$. We apply the arguments of the proof of (37) of Lemma 9 in Newman and Wright(1982). Define, K_j by

$$K_j = \min\{k : S_{(k,j)} = \max(S_{(1,j)}, \cdots, S_{(m,j)})\}.$$

Then

$$\overline{S}_{j+1} - \overline{S}_j = \overline{S}_{j+1} - \overline{S}_{(K_j,j)} \geq S_{(K_j,j+1)} - S_{(K_j,j)} = \sum_{k=1}^{K_j} X_{(k,j+1)},$$

and so since $m(\overline{S}_j^*) \geq 0$ we have

$$(29) \qquad (\overline{S}_{j+1} - \overline{S}_j)m(\overline{S}_j^*) \geq \sum_{k=1}^{K_j} X_{(k,j+1)} m(\overline{S}_j^*).$$

Taking expectation on both side of (29)

$$(30) \qquad \begin{aligned} E((\overline{S}_{j+1} - \overline{S}_j)m(\overline{S}_j^*)) &\geq E\left(\sum_{k=1}^{K_j} X_{(k,j+1)} m(\overline{S}_j^*)\right) \\ &= \sum_{k=1}^{K_j} E[X_{(k,j+1)} m(\overline{S}_j^*)] \\ &= \sum_{k=1}^{K_j} Cov(X_{(k,j+1)}, m(\overline{S}_j^*)) \geq 0 \end{aligned}$$

where we have used the fact that $EX_{(k,j)} = 0$ for every k and j. The non-negativity of $Cov(X_{(k,j+1)}, m(\overline{S}_j^*))$ follows from the definition of $LPQD$ and Lemma 2.1 since $m(\overline{S}_j^*)$ is a nondecreasing function of $S_{ij}, 1 \leq i \leq m$ not including $X_{(k,j+1)}$ and $X_{(k,j+1)}$ and S_j^* are PQD as sums of disjoint subsets of X_{ij}'s. □

REMARK 1. Lemma 3.1 can not be extended to $d \geq 3$ and thus the maximal inequality for $d \geq 3$ is not obtained in this way.

From Theorem 2.2 and Lemma 3.1 we obtain the following lemma:

LEMMA 3.2. *Let $\overline{S}_j$ and $\overline{S}_j^*$ be as in (26) and (27), respectively. For any nonnegative and nondecreasing function m with $m(0) = 0$ and any $j \geq 1$*

$$(31) \qquad E\left(\int_0^{\overline{S}_n^*} u \, dm(u)\right) \leq E(\overline{S}_j m(\overline{S}_j^*))$$

and for any $\lambda > 0$,

$$(32) \qquad \lambda P(\overline{S}_j^* \geq \lambda) \leq \int_{\{\overline{S}_j^* \geq \lambda\}} \overline{S}_j dP.$$

Proof. We will use the method of the proof of Theorem 2.2 and relation (28). Note that

$$(33)\quad \overline{S}_j m(\overline{S}_j^*) = \sum_{i=0}^{j-1} \overline{S}_{i+1}(m(\overline{S}_{i+1}^*) - m(\overline{S}_i^*)) + \sum_{i=1}^{j-1}(\overline{S}_{i+1} - \overline{S}_i)m(\overline{S}_i^*)$$

and that either $\overline{S}_{i+1}^* = \overline{S}_i^*$ or $\overline{S}_{i+1}^* = \overline{S}_{i+1}$. Thus

$$(34)\quad \begin{aligned} \overline{S}_{i+1}(m(\overline{S}_{i+1}^*) - m(\overline{S}_i^*)) &= \overline{S}_{i+1}^*(m(\overline{S}_{i+1}^*) - m(\overline{S}_i^*)) \\ &\geq \int_{\overline{S}_i^*}^{\overline{S}_{i+1}^*} u\, dm(u). \end{aligned}$$

Let $\overline{S}_o^* = 0$. Then (33) and (34) yield

$$(35)\quad \begin{aligned} \overline{S}_j m(\overline{S}_j^*) &\geq \sum_{i=0}^{j-1} \int_{\overline{S}_i^*}^{\overline{S}_{i+1}^*} u\, dm(u) + \sum_{i=1}^{j-1}((\overline{S}_{i+1} - \overline{S}_i)m(\overline{S}_i^*)) \\ &= \int_0^{\overline{S}_i} u\, dm(u) + \sum_{i=1}^{j-1}((\overline{S}_{i+1} - \overline{S}_i)m(\overline{S}_i^*)). \end{aligned}$$

By taking the expectation of (35) and using (28) we obtain (31). In (31) by taking $m(u)$ to be the indicator function $1_{\{u \geq \lambda\}}$ (32) follows. □

From Corollary 2.5 and (32) we obtain the following lemma.

LEMMA 3.3. *For* $0 \leq \lambda_1 < \lambda_2$,

$$(36)\quad P(\overline{S}_n^* \geq \lambda_2) \leq \left(\frac{E(\overline{S}_n^2)}{(\lambda_2 - \lambda_1)^2}\right)^{\frac{1}{2}} (P(\overline{S}_n \geq \lambda_1))^{\frac{1}{2}}.$$

We will use Corollary 2.4 and Corollary 2.5 and Lemma 3.3 to prove the following theorem, which generalizes (23) and (24) to $d = 2$.

THEOREM 3.4. *For* $0 \leq \lambda_1 < \lambda_2$,

$$(37)\quad P(S_{(m,n)}^* \geq \lambda_2) \leq (3^{\frac{3}{2}})(2^{-1}) \left(\frac{ES_{(m,n)}^2}{(\lambda_2 - \lambda_1)^2}\right)^{\frac{3}{4}} (P(S_{(m,n)} \geq \lambda_1))^{\frac{1}{4}},$$

$$(38)\qquad \begin{aligned} &P(\max\{|S_{(i,j)}| : 1 \le i \le m, 1 \le j \le n\} \\ &\le \left(3^{\frac{3}{2}}\right)\left(2^{-\frac{1}{4}}\right)\left(\frac{ES^2_{(m,n)}}{(\lambda_2-\lambda_1)^2}\right)^{\frac{3}{4}} (P(|S_{(m,n)}| \ge \lambda_1))^{\frac{1}{4}}. \end{aligned}$$

Proof. We will use the ideas in the proof of Theorem 10 of Newman and Wright(1982). It follows immediately from (28), (36) and the fact that with $\overline{S_j}$ as defined in (3.1), $S^*_{(m,n)} = \overline{S^*_n}$, that for $0 \le \lambda < \lambda_2$,

$$(39)\qquad P(S^2_{(m,n)} \ge \lambda_2) \le \left(\frac{E(\overline{S}^2_n)}{(\lambda_2-\lambda)^2}\right)^{\frac{1}{2}} (P(\overline{S}_n \ge \lambda))^{\frac{1}{2}}.$$

Let $T_k = S_{(k,n)}$. Then $T_k = X_1 + \cdots + X_k$ where $X_i = X_{(i,1)} + \cdots + X_{(i,n)}$ so that the X_i's are $LPQD$ with $EX_i = 0$. Since $\overline{S}_n = T^*_m$ and $S_{(m,n)} = T_m$, it follows from (17) of Corollary 2.4

$$(40)\qquad E(\overline{S}^2_n) = E(T^{*2}_m) \ge E(T^2_m) = E(S^2_{(m,n)})$$

and from (23) of Corollary 2.5 that for $0 \le \lambda_1 < \lambda$,

$$(41)\qquad P(\overline{S}_n \ge \lambda) \le \left(\frac{E(S^2_{(m,n)})}{(\lambda-\lambda_1)^2}\right)^{\frac{1}{2}} (P(S_{(m,n)} \ge \lambda_1)^{\frac{1}{2}}.$$

Combining (39), (40), and (41), we obtain

$$(42)\qquad P(S^*_{(m,n)} \ge \lambda_2) \le \frac{[E(S^2_{(m,n)})]^{\frac{3}{4}}}{(\lambda_2-\lambda)(\lambda-\lambda_1)^{\frac{1}{2}}} (P(S_{(m,n)} \ge \lambda_1))^{\frac{1}{4}}$$

choosing $\lambda = \frac{(2\lambda_1+\lambda_2)}{3}$ to minimize the right hand side of (42) leads to (37). To obtain (38), we add to (37) the analogous inequality obtained when all the $X_{(i,j)}$'s are replaced by their negatives, and use the fact that for $u, v \ge 0$,

$$u^{\frac{1}{4}} + v^{\frac{1}{4}} \le 2^{\frac{3}{4}}(u+v)^{\frac{1}{4}}. \qquad \square$$

Remark 2. Theorem 3.4 is an extension of Theorem 10 of Newman and Wright(1982) for 2-parameter array of associated random variables to the $LPQD$ case.

4. A weak convergence of 2-parameter arrays

The next theorem gives two-parameter weak convergence as a consequence of Theorem A and Theorem 3.4. We choose to consider weak convergence in the sense of [14] for the sake of convenience.

THEOREM 4.1. *Let* $\{X_{\underline{j}} : \underline{j} \geq 1\}$ *be a stationary 2-dimensional array of LPQD random variables with* $EX_{\underline{j}} = 0$ *and satisfy* (3) *in* Theorem A *for* $d = 2$. *Let* $W_n(\underline{t}), W(\underline{t})$ *be as in* Theorem A *with* $d = 2$ *and with* $(t_1, t_2) \in [0,1]^2$. *Then* $W_n(t_1, t_2)$ *converges weakly to* $W(t_1, t_2)$.

Proof. As in the proof of Theorem 11 in Newman and Wright [13] by Theorem A and [14, Theorem 2] it can be proved that

$$\forall \varepsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} \delta^{-2} P(\omega(W_n, \delta) > \varepsilon) = 0, \tag{43}$$

where $\omega(W_n, \delta) = \sup\{|W_n(\underline{t}) - W_n(\underline{s})| : \underline{s}, \underline{t} \in [0,1]^2, |\underline{s} - \underline{t}| < \delta\}$ and $|\underline{s} - \underline{t}| = \max(|s_1 - t_1|, |s_2 - t_2|)$. For the sake of completeness we repeat it here. Simple estimates show that to obtain (43) it suffices to have

$$\forall \varepsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} \delta^{-2} P(\overline{\omega}(W_n, \delta) \geq \varepsilon) = 0, \tag{44}$$

where

$$\begin{aligned} \overline{\omega}(W_n, \delta) &= \sup\{|W_n(\underline{t})| : \underline{t} \in [0, \delta]^2\} \\ &= n^{-1} \max\{|S_{(k,j)}| : 1 \leq k \leq n\delta, 1 \leq j \leq n\delta\}. \end{aligned}$$

Now from the fact that $E(S^2_{([n\delta],[n\delta])})/n^2 \to \sigma^2\delta^2$ (See [10]), and Theorem A we have that by putting $\lambda_1 = n\varepsilon$ and $\lambda_2 = n\varepsilon/2$ in (38) of Theorem 3.4

$$\begin{aligned} \limsup_{n\to\infty} P(\overline{\omega}(W_n, \delta) &\leq (3^{\frac{3}{2}})(2^{-\frac{1}{4}}) \left(\frac{2^2 E S^2_{([n\delta],[n\delta])}}{n^2 \varepsilon^2} \right)^{\frac{3}{4}} \lim_{n\to\infty} \left[P \left(\frac{S_{([n\delta],[n\delta])}}{n} \geq \right.\right. \\ &\leq C \left(\frac{\sigma^2\delta^2}{\varepsilon^2} \right)^{\frac{3}{4}} \lim_{n\to\infty} \left[P \left(\frac{S_{([n\delta],[n\delta])}}{n} \geq \frac{\varepsilon}{2} \right) \right]^{\frac{1}{4}} \\ &\leq C \left(\frac{\sigma^2\delta^2}{\varepsilon^2} \right)^{\frac{3}{4}} \left[\lim_{n\to\infty} P \left(W_n(\delta, \delta) \geq \frac{\varepsilon}{2} \right) \right]^{\frac{1}{4}} \qquad (45) \\ &\leq \left(\frac{\sigma^2\delta^2}{\varepsilon^2} \right)^{\frac{3}{4}} \left[\int_{\varepsilon/2}^{\infty} (2\pi\sigma^2\delta^2)^{-\frac{1}{2}} \exp\left(\frac{-u^2}{2\sigma^2\delta^2} \right) du \right]^{\frac{1}{4}} \end{aligned}$$

where C is a universal constant. Thus for fixed σ and ε, we have for some constants a and b

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{n\to\infty}\delta^{-2}P(\overline{\omega}(W_n,\delta)\geq\varepsilon) &\leq \lim_{\delta\to 0} a\delta^{-\frac{1}{2}}\left(\int_{b\delta^{-1}}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{u^2}{2}}du\right)^{\frac{1}{4}}\\ &= 0\end{aligned}$$

which yields (44) and completes the proof. □

REMARK 3. Theorem 4.1 is an extension of Theorem 11 of Newman and Wright(1982) for 2-parameter associated sequence to the *LPQD* case.

ACKNOWLEDGEMENT. The authors are grateful to the referee for carefully reading the manuscript and for offering some comments for improving the paper.

References

[1] T. Birkel, *The invariance principle for associated processes Stochastic Process,* Appl. **27** (1988), 57-71.

[2] —, *A note on the strong law of large numbers for positively dependent random variables,* Statist. Probab. Lett. **7** (1989), 17-20.

[3] —, *A functional central limit theorem for positively dependent random variable,* J. Multi. Anal. **44** (1993), 314-320.

[4] R. M. Burton and T. S. Kim, *An invariance principle for associated random fields,* Pacific J. Math. **132** (1988), 11-19.

[5] A. R. Dabrowski, *A functional law of the iterated logarithm for associated sequences,* Statist. Probab. Lett. **3** (1985), 209-212.

[6] J. Esary, F. Proschan and D. Walkup, *Association of random variables with applications,* Ann. Math. Statist. **38** (1967), 1466-1474.

[7] T. S. Kim, *The invariance principle for associated random fields,* Rocky Mountain J. Math. **26** (1996), 1443-1454.

[8] T. S. Kim and H. Y. Seo, *The invariance principle for linearly positive quadrant dependent random fields,* J. Korean Math. Soc. **33** (1996), 801-811.

[9] E. L. Lehmann, *Some concepts of dependence,* Ann. Math. Statist. **37** (1966), 1137-1153.

[10] C. M. Newman, *Normal flutuations and FKG inequalities,* Comm. Math. Phys. **74** (1980), 119-128.

[11] —, *Asymptotic independence and limit theorems for positively and negatively dependent radom variables. Inequalities in statistics and probability,* IMS Lectures Notes **15** (Ed. Y. L. Tong) (1984), 127-140.

[12] C. M. Newman and A. L. Wright, *An invariance principle for certain dependent sequences,* Ann. Probab. **9** (1981), 671-675.
[13] ——, *Associated random variables and martingale inequalities,* Z. Wahrsch. verw. Geb. **59** (1982), 361-371.
[14] M. Wichura, *Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters,* Ann. Math. Statist. **40** (1969), 681-687.

DEPARTMENT OF STATISTICS, WONKWANG UNIVERSITY, IK-SAN, CHONBUK 570-749, KOREA
E-mail: starkim@wonnms.wonkwang.ac.kr