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REFLECTED DIFFUSION WITH JUMP AND OBLIQUE REFLECTION

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ABSTRACT. Let (G, v) be a bounded smooth domain and reflection vector field on ∂G, which points uniformly into G. Under the condition that locally for some coordinate system, $|v^i| < cv^d$, $i = 1, \cdot, \cdot, d-1$, where c is a constant depending on the Lipschitz constant of G, we have tightness for reflected diffusion with jump on G with reflection v depending only on c. From this, we obtain some properties of L-harmonic function where L is a sum of Laplacian and integro one.

1. Introduction

In this paper, we consider reflected diffusion process with jumps in bounded Lipschitz domain G in R^d, $d \geq 3$ with oblique reflection. We will state the process which we are interested in later. Intuitively this process behaves like Brownian motion before it jumps, it reflects instantaneously when it hits ∂G and the jumps are governed by stochastic integrals with respect to a compensated Poisson random measure.

Let $(\Omega, F, P, F_t, W_t, \tilde{\nu})$ be a complete Wiener-Poisson space in $R^d \times R^m_*$, $R^m_* = R^m \setminus \{0\}$ with Levy measure π, i.e, (Ω, F, P) is a complete probability space with filtration $\{F_t\}$, W_t is a standard Brownian motion in R^d, and for any Borel set A of R^m_*, $\tilde{\nu}(A \times [0,t)) = \nu(A \times [0,t)) - \pi(A)t$ where ν is a Poisson random measure on $[0,\infty) \times R^m_*$ with $E[\nu(A \times [0,t))] = \pi(A)t$.

(1.1) Let $G \subset R^d$, $v : \partial G \to R^d$ and $b(x,u) : \overline{G} \times R^m_* \to R^d$ be given where $\overline{G}$ is the closure of G. A reflected diffusion X^x_t with respect to

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(G, v, b) is the cadlag process satisfying the following:

(A) $X_t^x \in \overline{G}$ and there exists an increasing continuous process L_t with $L_0 = 0$ P-a.s. such that

(B) $X_t^x = x + W_t + \int_0^t \int b(X_s^x, u)\tilde{\nu}(du, ds) + \int_0^t v(X_s^x) dL_s$

(C) $L_t = \int_0^t 1_{(X_s^x \in \partial G)} dL_s$.

We assume for every $p \geq 2$, x, y in $\overline{G}$,

$$\int |b(x,u)|^p \pi(du) \leq C_p \tag{1.2}$$

$$\int |b(x,u) - b(y,u)|^p \pi(du) \leq C_p |x-y|^p \tag{1.3}$$

for some constant C_p depending only on p.

(1.4) $x + b(x,u) \in \overline{G}$ for any $x \in \overline{G}$ and $u \in R_*^m$ which means that all jumps from $\overline{G}$ are inside $\overline{G}$.

(1.5) support of $b(x,u) \subset \overline{G} \times U_0$ where $\pi(U_0) < \infty$.

Then Menaldi and Robin ([MR]) proved the following theorem.

THEOREM. (Theorem 2 of [MR]) *Assume the conditions* (1.2), (1.3) *and* (1.4). *If G is a bounded C^3 domain and v is C^2 on ∂G with $v \cdot n > 0$ where n is the unit inward normal to $\overline{G}$, then there exists a unique solution of the stochastic equation* (1.1).

Without $b(x,u)$, the process is called reflected Brownian motion (abbreviated as RBM). Kwon ([Kw]) showed that in bounded smooth domain G, if v on ∂G satisfies the condition that for each $x \in \partial G$, there exists $c > 0$ depending only on the Lipschitz coefficients of G such that on U, a neighborhood of x, $|v^i| < cv^d$ $i = 1, ..., d-1$ $v^d \geq 1$ on $\partial G \cap U$ with a coordinate system on U, RBM is tight and the coefficients for tightness depends only on c, not on the smoothness. Here we prove the same result for the process X_t in (1.1) under the same condition for v and (1.2)-(1.5). We can not remove the condition $\pi(U_0) < \infty$ in (1.5) and we may need another technique for the case of $\pi(U_0) = \infty$.

In Section 2, we give notations and conditions on G and v more specifically and oblique reflection.

In Section 3, we get tightness depending only on c and with the same argument in [BH], we get some properties of L-harmonic function where L is a operator of sum of Laplacian and integro one.

2. Conditions and notations

Let G be a bounded Lipschitz domain and v be the given oblique reflection vector field on ∂G. $B(x,r)$ denotes the open ball of radius r with center x.

(1) We say a vector $v(x)$ is oblique at $x \in \partial G$ when there are a Lipschitz function f and a constant $R > 0$ such that

$$G \cap B(x,R) = \{y = (y', y^d) \in R^d : f(y') < y^d, \quad |y| < R\}$$

in an orthonormal coordinate system centered at x for which $v(x)$ is parallel to the positive x^d-axis. The vector field v on ∂G is oblique means $v(x)$ is oblique for any $x \in \partial G$. When $\partial G \in C^1$, obliqueness of v means $v \cdot n > 0$ for the unit normal n pointing into G.

We assume the following conditions (2)-(4) for G and v.

(2) There exist a finite number of balls $B(a_k, r_k)$, $a_k \in \overline{G}$, $k = 1, 2, \cdots, N_G$, whose union contains $\overline{G}$ and for each $k = 1, 2, \cdots, N_G$, there exists a function $F : R^{d-1} \to R$ that is uniformly bounded and Lipschitz with constant γ,

$$|F(x_1) - F(x_2)| \le \gamma |x_1 - x_2| \quad x_1, x_2 \in R^{d-1}$$

and domain $O^k = B(a_k, r_k) \cap G$ is defined by

$$O^k = B(a_k, r_k) \cap \{(y', y^d) : y' \in R^{d-1}, \quad F(y') < y^d < \infty\}$$

for some coordinate system which is one centered at some $x \in \partial G$ and the positive x^d-axis is into G. From now on we mean the coordinate system of O^k as this one. On each $x \in \partial G \cap \overline{O^k}$, let $v = (v^1, \cdots, v^d)$ with the coordinate system of O^k. Then the key assumptions for v are

$$v^d \geq 1 \quad \text{and} \quad |v^i| < cv^d$$

for $i = 1, \cdots, d-1$ for some c such that $0 < c < 1/(2(d-1)\sqrt{d}\gamma)$. Without loss of generality, we may assume $\gamma \geq 1$ and $\sup_{x,y\in G} dist(x,y) > 3\gamma$.

By boundedness of G and (1), (2), without loss of generality, we may assume the following (3)-(4).

(3) For any $x \in \overline{G}$, we can take k and $\beta > 0$ such that $x \in \overline{O^k}$ and $dist(x,z) \geq \beta$ for any $z \in \partial\overline{O^k} \cap G$ and β does not depend on k, or x. In other words, any $x \in \overline{G}$ is away from boundary points of O^k which are in G for at least one k. From now on, we understand k for $x \in \overline{O^k}$ in this sense.

(4) By the property of Lipschitz domain, notice that for any $x \in \partial G \cap \overline{O^k}$, $x = (x', x^d)$ with the coordinate system of O^k, there is a cone and r_0 not depending on x such that

$$C = \{(y', y^d) : |y^d - x^d| \geq \gamma |x' - y'|\}$$

$$C \cap B(x, r_0) \setminus \{x\} \subset G.$$

3. Tightness

In Theorem 3.1, we obtain tightness. We assume that G is C^3 and v is C^2; but the coefficients with respect to tightness depends only on c in (2) in section 2, not on x, not on any addtional smoothness assumption on G and v.

THEOREM 3.1. *For given $0 < \epsilon < r_0$, $\eta > 0$, there exists $\delta > 0$ such that for $x \in \overline{G}$,*

$$P[\sup_{t<\delta} |X_t^x - x| > \epsilon] < \eta$$

where δ depends only on c, but not on x and r_0 in (4).

Proof. For $x \in G$ such that $dist(x, \partial G) \geq 2\epsilon$, there exists $\delta > 0$ such that

$$P[\sup_{t<\delta} |X_t^x - x| < 2\epsilon] = P[\sup_{t<\delta} |W_t + \int_0^t \int b(X_s^x, u)\tilde{\nu}(du, ds)| < 2\epsilon] > 1 - \eta$$

by (1.2). (cf: Stroock [St])

Now for $x \in G$ such that $dist(x, \partial G) < 2\epsilon$, let $\sigma = \inf\{t : X_t^x \in \partial G\}$, then

$$\begin{aligned}
&P[\sup_{t<\delta} |X_t^x - x| > 2\varepsilon] \\
&= P[\sup_{t<\delta} |X_t^x - x| > 2\varepsilon, \quad \sigma < \delta] + P[\sup_{t<\delta} |X_t^x - x| > 2\varepsilon, \quad \sigma \geq \delta] \\
&\leq P[P(\sup_{t<\delta} |X_t^x - x| > 2\varepsilon | F_\sigma), \quad \sigma < \delta] \\
&+ P[\sup_{t<\delta} |X_t^x - x| > 2\varepsilon, \quad X_s^x \in G \quad \text{for all } s < \delta.]
\end{aligned}$$

The second part of the last term is less than $\eta/2$ for some δ, therefore it suffices to show for $x \in \partial G$. Let $x \in \partial G \cap \overline{O^k}$ for some k. Let $x \in \overline{O}_k$ and $\sigma_k = \inf\{t : X_t^x \in \overline{O}_k\}$. Since X_t^x is contniuous at $t = 0$, $\sigma_k > 0$, we may write X^x on $\overline{O}_k$ with the coordinate sysem of O_k, P-a.s.,

$$(X_t^x)^i = x^i + W_t^i + \int_0^t \int b^i(X_s^x, u)\tilde{\nu}(du, ds) + \int_0^t v^i(X_s^x) dL_s$$

for $i = 1, 2, \cdots, d$ and $t < \sigma_k$. Therefore by the condition (2),

$$\begin{aligned}
|(X_t^x)^i - x^i| &\leq |W_t^i| + |\int_0^t \int b^i(X_s^x, u)\tilde{\nu}(du, ds)| + \int_0^t |v^i(X_s^x)| dL_s \\
&\leq |W_t^i| + c\int_0^t v^d(X_s^x) dL_s + |\int_0^t \int b^i(X_s, u)\tilde{\nu}(du, ds)| \\
&= c(W_t^d + \int_0^t v^d(X_s^x) dL_s) + |W_t^i| - cW_t^d \\
&+ |\int_0^t \int b^i(X_s^x, u)\tilde{\nu}(du, ds)|
\end{aligned}$$

$$= c((X_t^x)^d - x^d) + |W_t^i| - cW_t^d.$$
$$+ |\int_0^t \int b^i(X_s^x, u)\tilde{\nu}(du, ds)| - c\int_0^t \int b^d(X_s^x, u)\tilde{\nu}(du, ds)$$

Let $\tau = \inf\{t | \triangle X_t^x = |X_t^x - X_{t-}^x| > 0\}$.
Then $\triangle X_t = \int_{t-}^t \int b(X_s^x, u)\nu(du, ds)$ and

$$(5) \qquad P(\sup_{0\le t\le T} \triangle X_t^x > 0) \le P(\nu([0,T) \times U_0) > 0) = 1 - e^{-\pi(U_0)T}.$$

Hence given $\eta > 0$, there exists $\delta > 0$ such that for any $x \in \overline{G}$,

$$(6) \qquad \begin{aligned} P[X_t^x = x + W_t + &\int_0^t \int -b(X_s^x, u)\pi(du)ds \\ &+ \int_0^t v(X_s^x)dL_s \quad \text{for} \quad t < \delta] > \eta. \end{aligned}$$

Therefore for $t < \tau$,

$$\begin{aligned} |(X_t^x)^i - x^i| \le & c((X_t^x)^d - x^d) + |W_t^i| - cW_t^d \\ & + c\int_0^t \int b^d(X_s^x, u)\pi(du)ds + |\int_0^t \int b^i(X_s^x, u)\pi(du)ds| \\ \le & c((X_t^x)^d - x^d) + |W_t^i| - cW_t^d + c' \int_0^t \int |b(X_s^x, u)|\pi(du)ds \end{aligned}$$

for some constant $c' > 0$. Let $f(x) = \int |b(x,u)|\pi(du)$. Then $f(x) \le [\int |b(x,u)|^2\pi(du)]^{1/2}\pi(U_0)^{1/2} < M$ for some M. Hence

$$|(X_t^x)^i - x^i| \le c((X_t^x)^d - x^d) + |W_t^i| - cW_t^d + c' \int_0^t f(X_s^x)ds$$

where f is uniformly bounded on $\overline{G}$. In Thereom 3.1 of [Kw], the key argument are $|(X_t^x)^i - x^i| \le c((X_t^x)^d - x^d) + |W_t^i| - cW_t^d$ and some property of Brownian motion which also holds in this case since

f is uniformly bounded. Hence with (5),(6) and the strong Markov property, we have the same results by following the proof of Theorem 3.1 of [Kw]. □

Notice that if a is a constant, aG is the region above a Lipschitz function with the same constant γ as F. Let $v'(ax) = v(x)$ on $\partial(aG)$. Then under some condition of $b(x,u)$, we prove Lemma 3.1 and we refer this property as scaling.

LEMMA 3.1. *Assume* $b(ax,u) = \frac{1}{a}b(x,u)$ *for* $a > 0$ *and let* $Y_t \equiv aX^x_{t/a^2}$, *then* Y_t *is the reflected diffusion in* (1.1) *with respect to* (aG, v', b) $Y_0 = ax$, P a.s..

Proof. Recall that

$$X^x_t = x + W_t + \int_0^t \int b(X^x_s, u)\tilde{\nu}(du, ds) + \int_0^t v(X_s)dL_s.$$

Then

$$\begin{aligned} Y_t = aX^x_{t/a^2} =& ax + aW_{t/a^2} + \int_0^{t/a^2} \int ab(X^x_s, u)\tilde{\nu}(du, ds) \\ &+ \int_0^{t/a^2} av(X^x_s)dL_s \\ =& ax + W'_t + I + II \end{aligned}$$

where W'_t is a Brownian motion by the property of Brownian motion and

$$\begin{aligned} I =& \int_0^{t/a^2} \int ab(\frac{1}{a}Y_{a^2 s}, u)\tilde{\nu}(du, ds) \\ =& \int_0^t \int ab(\frac{1}{a}Y_r, u)\tilde{\nu}(du, \frac{dr}{a^2}) = \int_0^t \int ab(\frac{1}{a}Y_r, u)\tilde{\nu}'(du, dr)\frac{1}{a^2} \\ =& \int_0^t \int b(Y_r, u)\tilde{\nu}'(du, dr) \end{aligned}$$

$$II = \int_0^{t/a^2} av'(aX^x_s)dL_s = \int_0^t v'(aX^x_{\frac{r}{a^2}})adL_{\frac{r}{a^2}} = \int_0^t v'(Y_r)dL'_r$$

where ν' is a Poisson random measure with $E[\nu'(AX[0,t)] = \pi(A)t$ and $L'_r = \int_0^r 1_{(Y_s \in \partial(aG))} dL'_s$ by

$$\begin{aligned} L'_t = aL_{\frac{t}{a^2}} &= \int_0^{t/a^2} 1_{(X^x_s \in \partial G)} a dL_s \\ &= \int_0^t 1_{(X^x_{\frac{r}{a^2}} \in \partial G)} a dL_{\frac{r}{a^2}} \\ &= \int_0^t 1_{(aX_{\frac{r}{a^2}} \in \partial(aG))} a dL_{\frac{r}{a^2}} \\ &= \int_0^t 1_{(Y_r \in \partial(aG))} dL'_r. \end{aligned}$$

Hence Y_t satisfies

$$Y_t = ax + W'_t + \int_0^t \int b(Y_r, u)\tilde{\nu}(du, dr) + \int_0^t v'(Y_r) dL'_r. \qquad \square$$

Now we prove that with some positive probability, X is immediately in G with a uniform distance from ∂G.

THEOREM 3.2. *Assume that $b(ax, u) = \frac{1}{a} b(x, u)$ for $a > 0$. Then for given $t_0 > 0$, there are $\epsilon > 0$ and $\delta > 0$ such that for all $x \in \overline{G}$,*

$$P[X^x_s \notin G_\epsilon \quad \text{for some } s \leq t_0] > \delta$$

where $G_\epsilon = \{x \in G : dist(x, \partial G) < \epsilon\}$.

Proof. With $b = 0$, it is proved in Lemma 3.2 of [Kw]. The properties of W_t used there are the scaling property and $\int_{G_\epsilon \cap B(x,\lambda)} G(x, w) dw < \frac{1}{4}$ for sufficiently small ϵ and λ where G is the Green function of Brownian motion. But the latter property also holds with the Green function of the process X_t which is a sum of Brownian motion and $\int_0^t f(X_s) ds$ where f is uniformly bounded. By (6) before X^x_t hits the boundary and has jump, $X^x_t = x + W_t + \int_0^t \int -b(X^x_s, u)\pi(du) ds$. Therefore by Lemma 3.2 of [Kw] with Lemma 3.1, (5), (6) and the strong Markov property, we have the same result. $\square$

Let $\tau_r^x = \inf\{t : |X_t^x - x| > r\}$ and $T_C^x = \inf\{t : X_t^x \in C\}$ for Borel set C. Then by Lemma 3.1 and Theorem 3.2 we have the following Theorem 3.3, which is Proposition 3.6 of [BH] since these are the only properties to prove the proposition.

THEOREM 3.3. *Assume that there is $a' > 0$ sufficiently small such that $b(ax,u) = \frac{1}{a}b(x,u)$ for $a > a' > 0$. Let $x \in G$, $C \subset B(x,1) \cap G$ and $|C| > \eta > 0$ where $|C|$ is the Lebesque measure of C. Then given $\eta > 0$, there exists $\delta > 0$ depending only on η but not on x such that*

$$P(T_C^x < \tau_{3\gamma}^x) > \delta.$$

Let L be an opeartor on $C^2(\overline{G})$ such that

$$Lf(x) = \triangle f(x) + \int [f(x + b(x,u)) - f(x) - \nabla f(x) \cdot b(x,u)]\pi(du)$$

where $\triangle$ is the Laplacian, i.e., $\triangle f(x) = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i{}^2} f(x)$. By generalized Ito formula, it is easy to see that X in (1.1) *has L as the generator.* (*cf:* [GS] Part II,ch.2) *If $Lf = 0$, we say f is L-harmonic. By* Theorem *3.2 and 3.3, we have the following Harnack principle valid up to the boundary for L-harmonic functions by the same argument of* [BH].

THEOREM 3.4. *Assume* (1.2)-(1.5) *and $b(ax,u) = \frac{1}{a}b(x,u)$ for $a > a' > 0$. Then there exist $\alpha > 0$, depending only on γ such that if $z \in G$, $r > 0$, h is nonnegative and L-harmonic in $B(z,6r) \cap G$ and h has zero-v-directional derivative on $B(z,6r) \cap \partial G$, then*

$$\frac{1}{\alpha} \le h(x)/h(y) \le \alpha$$

for $x, y \in \overline{B(z, r/3\gamma) \cap G}$.

REMARK 1. The key step in the proof of Theorem 3.4 is that any non negative L-harmonic function can be described by $h(y) = E[h(X^y_{\tau_r^y \wedge T_C^y})]$ $\ge E[h(X_{T_C^y})]P(T_C^y < \tau_r^y)$ where $C \subset \{B(y, \frac{r}{3} \cap G\}$.

REMARK 2. Let G be a bounded Lipschitz domain with reflection v, denoted by (G, v). Let (G, v) satisfy the conditions (1)-(4) in Section 2 and (G_n, v_n), $n = 1, 2, \cdots$ be smooth domains and C^2 reflections approximating (G, v) in the sense that for any $x \in \overline{G}$, we can take $\{x_n\}$ such that $x_n \in \overline{G_n}$, $x_n \to x$ as $n \to \infty$. Moreover, if $x \in \partial G$, $x_n \in \partial G_n$ and $x_n \to x$, then $v(x_n) \to v(x)$. Without loss of generality, we may assume (G_n, v_n) satisfy the conditions (1)-(4) and $G_1 \supset G_2 \supset \cdots$. Let $P_n^{x_n}$ be the law of X^{x_n} in (1.1) with respect to (G_n, v_n, b). Then by Theorem 3.1, the sequence $\{P_n^{x_n}\}$ is tight on $(D_{\overline{G}_1}, \mathcal{F})$, the space of cadlag processes on $\overline{G_1}$, so there is a limit of a subsequence of $\{P_n^{x_n}\}$, say, P'. To show that another subsequential limit is same with P', that is to get the limit which we want to say reflected diffusion with jump on Lipschitz domain, we need the equicontinuity of the following class.

$$\{E \int_0^{T_r} f(X_t^x) dt | f \in C(\overline{G}), ||f|| \leq 1\}$$

where T_r is some suitable stopping time and we leave it for future research.

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