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PRÜFER v-MULTIPLICATION DOMAINS IN WHICH EACH t-IDEAL IS DIVISORIAL

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ABSTRACT. We give several characterizations of a TV-PVMD and we show that the localization $R[X;S]_{N_v}$ of a semigroup ring $R[X;S]$ is a TV-PVMD if and only if R is a TV-PVMD where $N_v = \{f \in R[X] | (A_f)_v = R\}$ and S is a torsion free cancellative semigroup with zero.

1. Introduction

Throughout this paper R will denote a commutative integral domain with identity having quotient field K. For a nonzero fractional ideal A of R, $A_v = (A^{-1})^{-1}$ and $A_t = \cup\{I_v | I$ is a finitely generated subideal of $A\}$. If $A_v = A$ (resp. $A_t = A$) then A is said to be divisorial (resp. a t-ideal). Since $A \subseteq A_t \subseteq A_v$, each divisorial ideal is a t-ideal. The fractional ideal A is said to be quasi-finite if $A^{-1} = J^{-1}$ for some finitely generated subideal J of A. An integral domain R is said to be a v-domain if every nonzero finitely generated ideal of R is v-invertible. An integral domain R is called a Prüfer v-multiplication domain (PVMD) if every finitely generated fractional ideal I of R is t-invertible, i.e., $(II^{-1})_t = R$. Thus a PVMD is a v-domain. It is well known that R is a PVMD if and only if for each maximal t-ideal M of R, R_M is a valuation domain [3, Theorem 5]. By an overring of R we mean a ring between R and K. A valuation overring V of R is called essential if $V = R_P$ for some prime ideal P of R. An integral domain R is called essential if it can be expressed as an intersection of essential valuation overrings of itself.

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We have several characterizations of a Krull domain (see [1], [7], [9]). One of them is that a Krull domain is a generalization of a Dedekind domain in terms of t-invertibility, i.e., each nonzero fractional ideal of R is t-invertible [9, Theorem 3.6]. In [7], Houston and Zafrullah defined a TV-domain ; an integral domain in which each t-ideal is divisorial. Note that a Mori domain is an integral domain which satisfies ACC on divisorial ideals. It follows that in a Mori domain each t-ideal is divisorial. Thus a Mori domain is a TV-domain. We know that an integral domain R is a Krull domain if and only if R is a Mori domain and R is a PVMD [9, Theorem 3.2]. Thus a PVMD which is also a TV-domain is a naturial generalization of a Krull domain. Throughout this paper, the term a TV-PVMD will denote a PVMD which is also a TV-domain.

In section 2, we study an independent ring of Krull type and using this we give a new characterization of a TV-PVMD. We prove in section 3 that $R[X;S]_{N_v}$ is a TV-PVMD if and only if R is a TV-PVMD where S is a torsion free cancellative semigroup with zero and $N_v = \{f \in R[X;S] | (A_f)_v = R\}$.

In general, our terminology and notation will follow that given in [1, 2]. The reader is referred to those for terms and notations not defined in this paper.

2. A PVMD which is a TV-domain.

Let R be an integral domain. A maximal t-ideal of R is a proper t-ideal of R which is maximal among proper t-ideals of R. We denote the set of maximal t-ideals of R by t-Max$R$. It is easy to see by Zorn's lemma that each maximal t-ideal is a prime ideal and t-Max$R$ is not empty. In [4], Griffin introduced a ring of Krull type; an integral domain which is a locally finite intersection of essential valuation overrings, equivalently a PVMD in which each nonzero element belongs to only finitely many maximal t-ideals. Thus a ring of Krull type is a PVMD. But a PVMD need not be a ring of Krull type (for example, an almost Dedekind domain which is not Noetherian [1, Example 42.6]). The ring of Krull type is an independent ring of Krull type if each prime t-ideal of R lies in a unique maximal t-ideal.

EXAMPLE 2.1. Let Q (resp. Z) be the ring of rational numbers (resp. integers). Let $Q[|X|] = Q + XQ[|X|]$, let $D_1 = Z_{(2)} + XQ[|X|]$, and let $D_2 = Z_{(3)} + XQ[|X|]$. Then

1. D_1 and D_2 are valuation domains on $Q[|X|][\frac{1}{X}]$ [1, Ex.13, page 203],
2. $D = D_1 \cap D_2 = Z_{(2)} \cap Z_{(3)} + XQ[|X|]$ is a Prüfer domain with maximal ideals $P_1 = (2Z_{(2)}+XQ[|X|]) \cap D$, $P_2 = (3Z_{(3)}+XQ[|X|]) \cap D$ [1, Theorem 22.8],
3. $D_{P_1} = D_1, D_{P_2} = D_2$ [1, Ex.13, page 203].

Let v_i be the valuation on K associated with D_{P_i}, $i = 1, 2$. Then there is no element f of $Q[|X|][\frac{1}{X}]$ such that $v_1(f) = v_1(X)$ and $v_2(f) = 0$.

Example 2.1 shows that the approximation theorem for Krull domains [1, Theorem 44.1] cannot be generalized to a ring of Krull type. Thus [1, Ex. 1, page 553] fails. But we can generalize the approximation theorem for Krull domains to an independent ring of Krull type. Its proof is an easy adaptation of the proof of [1, Theorem 44.1], and is hence omitted.

THEOREM 2.2. (cf. [1, Theorem 44.1]) *Let R be an independent ring of Krull type which is not its quotient field K. Let t-MaxR $= \{P_\lambda\}_{\lambda\in\Lambda}$. For each λ, let v_λ be the valuation on K associated with R_{P_λ} with value group G_λ. If $\{v_1, v_2, \ldots, v_n\}$ is a finite subset of $\{v_\lambda\}_{\lambda\in\Lambda}$ and if $g_i \in G_i$, $i = 1, 2, \ldots, n \in \Lambda$, there exists an element $t \in K$ such that $v_i(t) = g_i$ for $i = 1, 2, \ldots, n$ and $v_\lambda(t) \geq 0$ for $\lambda \in \Lambda - \{1, 2, \ldots, n\}$.*

LEMMA 2.3. (cf. [5, Lemma 5.2]) *If V is a valuation domain with maximal ideal M, the followings are equivalent.*

1. *Each nonzero ideal of V is divisorial.*
2. *M is principal.*
3. *M is divisorial.*

Proof. (1)$\Longleftrightarrow$ (2) [5, Lemma 5.2].
(2) $\Longrightarrow$ (3) Clear.
(3) $\Longrightarrow$ (2) Since $M_v = M$, $M^{-1} \supsetneq V$. For $x \in M^{-1} - V$, $M^{-1} \supseteq V + xV$ and $M = (M^{-1})^{-1} \subseteq (V + xV)^{-1} \subsetneq V$. Since M is the maximal ideal of V, $M = (V + xV)^{-1} = V \cap x^{-1}V$. Since $x \notin V$, $x^{-1} \in M$. So $M = x^{-1}V$ is principal (cf. [5, Theorem 5.1]). □

REMARK. If $G(V)$ is complete, then every divisorial ideal of V is principal. (Proof. This follows from the definition of "complete" and [1, Theorem 34.1]).

LEMMA 2.4. (cf. [7, Corollary 1.8]) *If a maximal t-ideal P of an integral domain R is divisorial, then PR_P is divisorial.*

Proof. Since $P_v = P$, $P^{-1} \supsetneq R$. So we can find an element $\frac{a}{b} \in P^{-1} - R$ where $a, b \in R$. Hence $P = (b) : a = \{r \in R | ra \in (b)\}$ and $PR_P = bR_P : aR_P$. So PR_P is divisorial. □

In the proof of the following result, we use a $*$-operation. The reader may consult section 32 and section 34 of [1] for the definition and properties of $*$-operation.

THEOREM 2.5. (cf. [1, Theorem 44.2]) *Let R is an independent ring of Krull type which is not its quotient field K, let $\{P_\lambda\}_{\lambda \in \Lambda} = t$-Max$R$, and let $F \longrightarrow F_w$ be the w-operation on R induced by the family $\{R_{P_\lambda}\}_{\lambda \in \Lambda}$ of valuation overrings of R. If each $P_\lambda R_{P_\lambda}$ is divisorial, then $F_w = F_v$ for each non-zero fractional ideal F of R.*

Proof. For a fractional ideal F of R, there is a nonzero element d of R such that dF is an integral ideal of R. Since F is divisorial if and only if dF is divisorial, it is sufficient to consider an integral ideal A of R. Since $F \longrightarrow F_w$ is a $*$-operation on R, $A_w \subseteq A_v$. Conversely, for $x \in R - A_w$, $A_w = \cap AR_{P_\lambda} = \cap(AR_{P_\lambda} \cap R) = \cap_{i=1}^n (AR_{P_i} \cap R)$ for some finite subset $\{1, \dots, n\} \subseteq \Lambda$. For convenience, we may assume that $x \notin AR_{P_1}$. Now Lemma 2.3 and Lemma 2.4 yield that each nonzero ideal of R_{P_1} is divisorial. Since AR_{P_1} is a divisorial ideal, there exists an element $y \in R$ such that $AR_{P_1} \subseteq yR_{P_1} \subsetneq xR_{P_1}$. So $v_1(x) < g_1 = v_1(y) \leq v_1(a)$ for each $a \in AR_{P_1}$. Similary, we can find $y_i \in R$ such that $AR_{P_i} \subseteq y_i R_{P_i} \subsetneq R_{P_i}$ for $i = 2, \dots, n$. Let $v_i(y_i) = g_i > 0$ for $i = 2, \dots, n$. By Theorem 2.2, there exists an element $t \in K$ such that $v_i(t) = -g_i$ for $i = 1, 2, \dots, n$ and $v_\lambda(t) \geq 0$ for $\lambda \in \Lambda - \{1, \dots, n\}$. So $v_1(xt) = v_1(x) + v_1(t) = v_1(x) - g_1 < 0$ and hence $xt \notin R$ and $x \notin (t^{-1})$. But for $a \in A$, $v_i(at) = v_i(a) - g_i \geq 0$ for $i = 1, \dots, n$ and $v_\lambda(at) = v_\lambda(a) + v_\lambda(t) \geq 0$ for $\lambda \in \Lambda - \{1, \dots, n\}$. Thus $at \in R$ and $a \in (t^{-1})$ and $A \subseteq (t^{-1})$. Hence $x \notin A_v$ and $A_v = A_w$. □

We next give an example which shows that in Theorem 2.5, the assumption that each $P_\lambda R_{P_\lambda}$ is divisorial is necessary.

EXAMPLE 2.6. Let $\{(V_i, M_i)\}_{i=1}^n$ be a set of valuation domains on the field K such that if $i \neq j$, V_i, V_j are independent and M_1 is not principal. Let $R = \cap_{i=1}^n V_i$, then R is a Prüfer domain with Max$R = \{M_i \cap R = P_i\}$

[1, Theorem 22.8]. Then R is an independent ring of Krull type but not a TV-domain. Since M_1 is not divisorial, P_1 is not divisorial. Hence $(P_1)_v = R$ but $(P_1)_w = \cap_{i=1}^{n} P_1 R_{P_i} = P_1$.

LEMMA 2.7. (cf. [8, Theorem 3.5]) *An integral domain R is a TV-PVMD if and only if R is integrally closed and $I_v = \cap_{P\in\Gamma} I_P$ for every nonzero ideal I of R where Γ is the set of maximal t-ideals of R.*

Proof. ($\Longrightarrow$) Since R is a PVMD, R is integrally closed and $I_t = \cap_{P\in\Gamma} I_P$ ([6, Proposition 0.1], [8, Theorem3.5]). So $I_v = \cap_{P\in\Gamma} I_P$.
($\Longleftarrow$) Since $\cap_{P\in\Gamma} I_P \subseteq I_t$ [3, Proposition 4] and $I_t \subseteq I_v$, $I_v = I_t = \cap_{P\in\Gamma} I_P$. So R is a TV-PVMD [8, Theorem 3.5]. □

PROPOSITION 2.8. (cf. [1, Ex. 3, page 554]) *Let R be a PVMD. If a maximal t-ideal P of R is divisorial, then P is t-invertible. Moreover, if P is a maximal ideal, P is invertible.*

Proof. Let P be a maximal t-ideal of R. Since $(P^{-1})^{-1} = P$, $R \subsetneq P^{-1}$. So there is an element $x \in P^{-1} - R$. So $R \subsetneq R+xR \subseteq P^{-1}$ and $P = P_v \subseteq (R+xR)^{-1} \subsetneq R$. Since P is a maximal t-ideal, $P = (R+xR)^{-1}$. Since R is a PVMD, $(PP^{-1})_t = ((R+Rx)_t(R+Rx)^{-1})_t = ((R+Rx)(R+Rx)^{-1})_t = R$. Thus P is t-invertible and $PP^{-1} \supsetneq P$. Hence if P is maximal, $PP^{-1} = R$. □

The following lemma appears in the proof of [10, Theorem 3.2].

LEMMA 2.9. *If an integral domain R is a TV-PVMD, then each nonzero fractional ideal of $R[X]_{N_v}$ is divisorial where $N_v = \{f \in R[X] | (A_f)_v = R\}$.*

Proof. Since R is a PVMD, every ideal of $R[X]_{N_v}$ is extended from R [8, Theorem3.1]. Thus if A is an ideal of $R[X]_{N_v}$, there is an ideal I of R for which $A = I[X]_{N_v}$. Since $R[X]_{N_v}$ is a Prüfer domain [8, Theorem 3.7], A is a t-ideal of $R[X]_{N_v}$. Thus I is a t-ideal of R. By assumption, I is a divisorial ideal of R and hence A is a divisorial ideal of $R[X]_{N_v}$. □

We give a characterization of a TV-PVMD (cf. [7, Theorem 3.1]).

THEOREM 2.10. *Let R be an integrally closed domain with quotient field K, then the followings are equivalent.*

1. *R is an essential TV-domain.*
2. *R is a v-domain which is a TV-domain.*

3. *R is a TV-PVMD.*
4. *Each nonzero ideal of $R[X]_{N_v}$ is divisorial.*
5. *R is an independent ring of Krull type whose maximal t-ideals are quasi-finite (and so t-invertible).*
6. *There is a family $\{P_\alpha\}_{\alpha\in A}$ of prime ideals of R such that*
   (a) *R_{P_α} is a valuation domain such that $P_\alpha R_{P_\alpha}$ is divisorial.*
   (b) *$R = \cap_{\alpha\in A} R_{P_\alpha}$.*
   (c) *Each pair of $\{R_{P_\alpha}\}_{\alpha\in A}$ are independent.*
   (d) *Each nonzero element of R belongs to only finitely many P_α.*

Proof. (1) $\Longrightarrow$ (2) [9, Lemma 3.1].

(2) $\Longrightarrow$ (3) If A is a finitely generated fractioal ideal of R, $(AA^{-1})_v = (AA^{-1})_t = R$. Hence R is a PVMD.

(3) $\Longrightarrow$ (1) Clear.

(3) $\Longrightarrow$ (4) Lemma 2.9.

(4) $\Longrightarrow$ (5) If Q is a prime ideal of R which is contained in a maximal t-ideal, $Q[X]_{N_v}$ is a proper prime ideal of $R[X]_{N_v}$. Since each prime ideal of $R[X]_{N_v}$ is contained in a unique maximal ideal [5, Theorem 2.4] and $\{P[X]_{N_v} | P \in t\text{-Max}R\} = \text{Max}(R[X]_{N_v})$ [8, Proposition 2.1], Q is contained in a unique maximal t-ideal of R. Since $R[X]_{N_v}$ is a Prüfer domain [5, Theorem 5.1], $R_P = R[X]_{P[X]} \cap K = (R[X]_{N_v})_{P[X]_{N_v}} \cap K$ is a valuation domain for each $P \in t\text{-Max}R$. Since $R[X]_{N_v} = \cap(R[X]_{N_v})_{P[X]_{N_v}} = \cap(R[X]_{P[X]})$, $R = \cap R_P$. [5, Theorem 2.5 and Theorem 5.1] show that R is an independent ring of Krull type.

(5) $\Longrightarrow$ (6) Let $t\text{-Max}R = \{P_\lambda\}_{\lambda\in\Lambda}$. Then $\{P_\lambda\}_{\lambda\in\Lambda}$ satisfies the conditions stated in (6).

(6) $\Longrightarrow$ (3) Since R is a ring of Krull type, R is a PVMD. If M is a maximal t-ideal, $R_M = \cap(R_{P_\alpha})_S = R_{P_\alpha}$ for some P_α where $S = R - M$. So $M = P_\alpha$. For $P \in \{P_\alpha\}_{\alpha\in A}$, $P_t \subseteq P_v = \cap PR_{P_\alpha} = PR_P \cap R = P$ where the first equality follows from Theorem 2.5. Thus P is a t-ideal and $\{P_\alpha\}_{\alpha\in A}$ is the set of maximal t-ideals. By Lemma 2.7, R is a TV-domain. □

REMARK. In [10, Theorem 3.2], B.G.Kang proved the PVMD version of [5, Theorem 5.1] which shows that 3 and 5 of Theorem 2.10 are equivalent.

EXAMPLE 2.11. If a domain V is an n-dimensional discrete valuation domain, then V is a TV-PVMD. But if $n > 1$, V is not a Krull domain.

3. Semigroup Ring over a TV-PVMD

Throughout this section, S denotes a torsion free cancellative semigroup with zero. It is well known that S admits a total order $<$ compatible with the semigroup structure. Thus each element of the semigroup ring $R[X;S]$ can be uniquely expressed as the form $f = a_0X^{s_0} + a_1X^{s_1} + \cdots + a_nX^{s_n}$ where $a_i \in R$ with $a_n \neq 0$ and $s_i \in S$ with $s_0 < s_1 < \cdots < s_n$. Let $deg f = s_n$ and A_f the ideal of R generated by the coefficients of f. Let $N_v = \{f \in R[X;S] | (A_f)_v = R\}$, then if R is integrally closed, using [11, Proposition 5.1.4], we can show that N_v is a multiplicative closed subset of $R[X;S]$ and $\text{Max}(R[X;S]_{N_v}) = \{P[X;S]_{N_v} | P \in t\text{-Max}R\}$.

Using [11, Proposition 5.1.4] instead of [1, Propostition 34.8] in the proof of [8, Proposition 2.2], we have the following results.

LEMMA 3.1. (cf. [8, Proposition 2.2]) *Suppose that R is integrally closed. Let T be a multiplicative closed subset of $R[X;S]$ contained in $N_v = \{f \in R[X;S] | (A_f)_v = R\}$. Let I be a nonzero fractional ideal of R. Then*

1. $(I[X;S]_T)^{-1} = I^{-1}[X;S]_T$.
2. $(I[X;S]_T)_v = I_v[X;S]_T$.
3. $(I[X;S]_T)_t = I_t[X;S]_T$.

COROLLARY 3.2. (cf. [8, Corollary 2.3]) *Let I be a nonzero ideal of R. Then*

1. $(I[X;S])_v = I_v[X;S], (I[X;S])_t = I_t[X;S]$.
2. $(I(X;S))_v = I_v(X;S), (I(X;S))_t = I_t(X;S)$.
3. $(I[X;S]_{N_v})_v = I_v[X;S]_{N_v}, (I[X;S]_{N_v})_t = I_t[X;S]_{N_v}$.

The following theorem generalizes Lemma 2.9 to a semigroup ring $R[X;S]$.

THEOREM 3.3. *Let $N_v = \{f \in R[X;S] | (A_f)_v = R\}$. Then $R[X;S]_{N_v}$ is a TV-PVMD if and only if R is a TV-PVMD.*

Proof. ($\Longrightarrow$) Corollary 3.2.(3) shows that $P[X;S]_{N_v}$ is a maximal t-ideal of $R[X;S]_{N_v}$ for each maximal t-ideal P of R. So $(R[X;S]_{N_v})_{(P[X;S]_{N_v})}$ $R[X;S]_{P[X;S]}$ is a valuation domain. So $R_P = R[X;S]_{P[X;S]} \cap K$ is a valuation domain. Hence R is a PVMD [3, Theorem 5]. If A is a t-ideal of R, $A_v[X;S]_{N_v} = (A[X;S]_{N_v})_v = (A[X;S]_{N_v})_t = A_t[X;S]_{N_v} = A[X;S]_{N_v}$. Thus $A = A_v$ and hence R is a TV-domain.

($\Longleftarrow$) Since R_P is a valuation domain for each $P \in t$-Max$R$,

$$(R[X;S]_{N_v})_{P[X;S]_{N_v}} = R[X;S]_{P[X;S]}$$

is a valuation domain. Since Max$(R[X;S]_{N_v})$=$\{P[X;S]_{N_v} | P \in t\text{-Max}R\}$, $R[X;S]_{N_v}$ is a Prüfer domain. So each nonzero ideal of $R[X;S]_{N_v}$ is a t-ideal. By Corollary 3.2 and the proof of Lemma 2.9, each nonzero ideal of $R[X;S]_{N_v}$ is divisorial since each ideal of $R[X;S]_{N_v}$ is extended from R. Hence $R[X;S]_{N_v}$ is a TV-PVMD. □

COROLLARY 3.4. *With the notation of* Theorem 3.3, $R[X;S]_{N_v}$ *is a PID if and only if R is a Krull domain.*

Proof. ($\Longrightarrow$) Since $R[X;S]_{N_v}$ is a PID, $R[X;S]_{N_v}$ is a Krull domain. So $R = R[X;S]_{N_v} \cap K$ is a Krull domain.

($\Longleftarrow$) Since $\{P[X;S]_{N_v} | P \in t\text{-Max}R\}$ is the set of nonzero prime ideals of $R[X;S]_{N_v}$, we only need to show that for each $P \in t$-Max$R$, $P[X;S]_{N_v}$ is principal. Since $(P[X;S]_{N_v})(P^{-1}[X;S]_{N_v})$ =$(PP^{-1})[X;S]_{N_v} = R[X;S]_{N_v}$, $P[X;S]_{N_v}$ is invertible and so $P[X;S]_{N_v} = (f_1, \dots, f_n)R[X;S]_{N_v}$ for some elements $f_1, \dots, f_n \in P[X;S]$. Let

$$f = f_1 + f_2X^{g_1+g} + \cdots + f_n^{g_1+g_2+\cdots+g_{n-1}+(n-1)g}$$

where $g_i, g \in S$ with $g_i = degf_i$ and $g > 0$. Since $A_{f_i} \subseteq A_f$, $A_{f_i}R_P \subseteq A_fR_P$ for each $P \in t$-Max$R$.
Since R_P is a valuation domain, $(R[X;S]_{N_v})_{P[X;S]_{N_v}} = R[X;S]_{P[X;S]}$ is a valuation domain. Thus $(P[X;S]_{N_v})_{P[X;S]_{N_v}}$ =$fR[X;S]_{P[X;S]}$ is locally principal and hence $P[X;S]_{N_v} = fR[X;S]_{N_v}$ is globally principal. □

EXAMPLE 3.5. Let S be the set of nonnegative rational numbers, and let I be the ideal of the semigroup ring $R[X;S]$ generated by $\{X^g | g \in S, g > 0\}$. Then I is a t-ideal but not a divisorial ideal. So $R[X;S]$ is not a TV-domain.

We know that R is a TV-PVMD if and only if $R[X]$ is a TV-PVMD ([7, Proposition 4.6] and [8, Theorem 3.7]). Example 3.5 shows that this cannot be generalized to a semigroup ring.

Let G be a group, g an element of G. We say that g is of type $(0, 0, 0, \dots)$ if the set $M = \{n \in Z^+ | nx = g \text{ for some } x \in G\}$ is finite, where Z^+ is the set of positive integers. It is known [2, Theorem 15.6] that $R[X;S]$ is a Krull domain if and only if R is a Krull domain, S is a

Krull monoid and every element of a maximal subgroup of S is of type $(0,0,0,\dots)$.

COROLLARY 3.6. (cf, [2, Theorem 15.6]) *Let S be a Krull monoid such that each element of a maximal subgroup of S is of type $(0,0,0,\dots)$. Then the semigroup ring $R[X;S]$ is a TV-PVMD if and only if R is a TV-PVMD.*

Proof. ($\Longrightarrow$) If I is a t-ideal of R, then $I[X;S]$ is a t-ideal of $R[X;S]$. Thus $I[X;S]$ is also a divisorial ideal of $R[X;S]$ and hence I is a divisorial ideal of R. For each $P \in t$-Max$R$, $R[X;S]_{P[X;S]}$ is a valuation domain since $P[X;S]$ is a prime t-ideal of $R[X;S]$. Thus $R_P = R[X;S]_{P[X;S]} \cap K$ is a valuation domain and hence R is a PVMD.

($\Longleftarrow$) It is clear that $R[X;S] = K[X;S] \cap R[X;S]_{N_v}$. Since $K[X;S]$ is a Krull domain [2, Theorem 15.6] and $R[X;S]_{N_v}$ is a TV-PVMD, by Theorem 2.10 $R[X;S]$ is a TV-PVMD. □

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