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ALEXANDER POLYNOMIAL FOR LINK CROSSINGS

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ABSTRACT. We define a crossing of a link without referring to a specific projection of the link and describe a construction of a non-normalized Alexander polynomial associated to collections of such crossings of oriented links under an equivalence relation, called homology relation. The polynomial is computed from a special Seifert surface of the link. We prove that the polynomial is well-defined for the homology equivalence classes, investigate its relationship with the combinatorially defined Alexander polynomials and study some of its properties.

1. Link projections (or diagrams) contain crossings as double points of projection maps. Under the Reidemeister moves, crossings can be created or eliminated and their relative positions can be altered. It is possible to define a crossing so that it remains unaffected up to an equivalence relation by the Reidemeister moves. The definition we give here is a generalization of conventional crossings. Assume that all links are oriented.

Let $B = D^1 \times D^1$ be the subset of the $x-y$ plane in Figure 1, where $D^1 = [-1, 1]$. Orient $\partial B = (\partial D^1 \times D^1) \cup (D^1 \times \partial D^1)$ as in the figure. Let $W = D^1 \times \{0\}$, and orient W in the positive direction of the x axis.

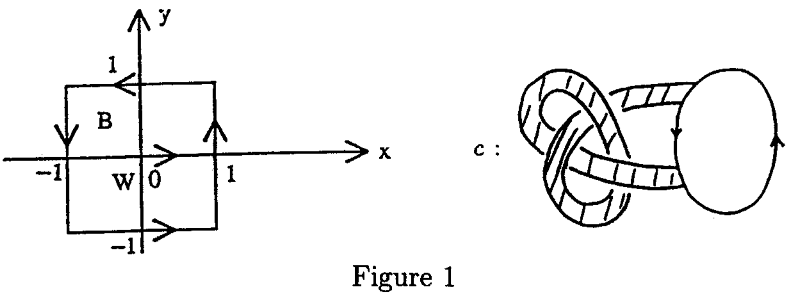

Figure 1

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DEFINITION 1. Given a link L, a crossing c of L is defined to be embedding $c : B \to \mathbf{R}^3$ such that $c(B) \cap L = c(\partial D^1 \times D^1)$ and $c \mid \partial D^1 \times I$ preserves the orientation. Two crossings c and c' of L are equivalent there exists a diffeomorphism f of $(R)^3$ such that $c' = fc$ and $f(L) = L$ preserving the orientation of L. Given a crossing c, we call $c(B)$ the band of the crossing.

In Figure 1, an example of a crossing of the trivial knot is given. A conventional crossing in a link projection is naturally a crossing under Definition 1 as in Figure 2. Conversely, a crossing in the definition is easily seen to be equivalent to a conventional crossing in a projection of the link, where it can be positive or negative. So a sign can not be assigned to a crossing. Furthermore, the change of a crossing is not defined for crossings under the definition although splicing at a crossing is. Given a crossing c of link L, define $L_0(c)$, the link obtained by splicing L at c, by $L_0(c) = (L - c(B)) \cup c(D^1 \times \partial D^1)$.

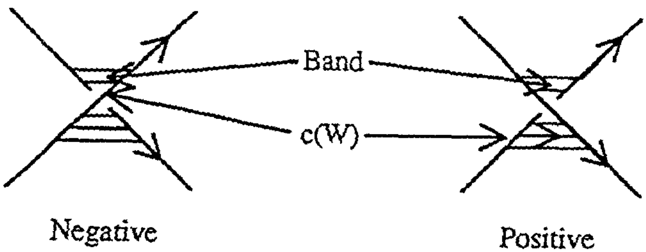


Figure 2

DEFINITION 2. A collection of crossings of a link is defined to be a finite set of disjoint crossings of the link. Two collections are equivalent if there exists a diffeomorphism of $\mathbf{R}^3$ carrying one collection onto the other such that each corresponding pair of crossings is equivalent under the diffeomorphism.

To avoid confusion, we make the following definition of a Seifert surface of a link.

DEFINITION 3. A Seifert surface of a link L is defined to be a connected, orientable 2-surface F in $\mathbf{R}^3$ such that $\partial F = L$ and F can be oriented compatibly with L.

A canonical Seifert surface of any link projection is a Seifert surface in the sense of the definition if it is connected. The shaded surface F in Figure 3 is not a Seifert surface of the 2 component link L although F is orientable.

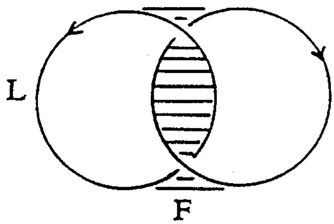


Figure 3

Let C be a collection of crossings of a link L. Then it is easy to see that there exists a Seifert surface F of L that contains the bands of the crossing in C as a submanifold [1]. We call F a Seifert surface of C. If c is a crossing and F a Seifert surface of c, then $c_*(w) \in H_1(F, \partial F;\ \mathbf{Z})$, where w is the canonical generator of $H_1(W, \partial W;\ \mathbf{Z}) \cong \mathbf{Z}$ corresponding to the orientation of W in Figure 1. We say that two crossings are homologous if they determine the same element in $H_1(F, \partial F;\ \mathbf{Z})$ for some Seifert surface F of the two crossings. Given collections C and D of crossings, we say that they are homologous if there exists a Seifert surface F of C and D, and a one-to-one correspondence between C and D such that corresponding crossings are homologous in $H_1(F, \partial F;\ \mathbf{Z})$.

DEFINITION 4. The equivalence relation in Definition 2 and the above homologous relation generate an equivalence relation on the set of collections of crossings of a link. We say that two collections are homology equivalent if they are equivalent under this extended equivalence. It may be possible that two collections are homology equivalent without being homologous.

The paper should be regarded as a study of non-normalized Alexander polynomial of links computed from Seifert surfaces. A problem with this old approach is that the polynomial is only well-defined up to a factor of $\pm$(a power of t). We show (Theorem 1 of Section 3) that the non-normalized polynomials constructed from Seifert surfaces in Section 2 are well-defined for homology equivalence classes of collections of crossings.

For a collection C of a link L, we denote this polynomial by $\lambda(C)$ and call it the Alexander polynomial for the collection.

Given a link L, let $\tilde{\Delta} L$ be the Alexander polynomial of L defined combinatorially using the skein relation, $\tilde{\Delta} L_{+} - \tilde{\Delta} L_{-} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta} L_{0} = 0$.

We show (Theorem 2 of Section 3) that $\lambda(C) = (-1)^{|C|} t^{\frac{k-1-|C|}{2}} \tilde{\Delta} L_0(C)$, where k is the number of components in L, $|C|$ the number of crossings in C and $L_0(C)$ the link obtained by splicing L at the crossings in C . The identity implies that $\tilde{\Delta} L_0(C)$ is an invariant of the homology equivalence class of C, and it answers partially the question about what properties of crossings imply that the Alexander polynomials of the spliced links at distinct crossings are the same. In Section 5, we give an example of two homology equivalent collections C and D of crossings in a link, thus $\tilde{\Delta} L_0(C) = \tilde{\Delta} L_0(D)$, but $L_0(C)$ is not equivalent to $L_0(D)$ as links.

If $C = \emptyset$, we denote $\lambda(C)$ by ΔL and call it the signed Alexander polynomial of L. Under this notation the above relation becomes $\Delta L = t^{\frac{k-1}{2}} \tilde{\Delta} L$. This raises the question whether or not there exist links with the same Alexander polynomial but with different signed Alexander polynomial. We answer the question positively in Remark 3 of Section 3.3.

The skein relation of the Alexander polynomial for a single crossing is generalized (Theorem 4 of Section 4) to that of the signed Alexander polynomial for a collection of crossings: Given a collection C of crossings in a link projection L, let $L'(C)$ be the link obtained from L by changing all the crossings in C and $\varepsilon(C)$ the product of the signs of the crossings in C. Then

$$\Delta L'(C) = \sum_{S \in P(C)} \varepsilon(S)(1-t)^{|S|}\lambda(S),$$

where $P(C)$ is the power set of C.

In Section 5, for two knots 10_{18} and 10_{24} [3] we compute the Alexander and other polynomials derived from the construction given in Section 3. In Section 6, we study how the Alexander polynomial changes when twists are introduced to the band of a crossing. Using a result of [1], we apply this to show that given any crossing of a knot, a proper number of twists on the band always causes the index of the knot to change if the crossing with the additional twists is changed. The second application is that Alexander polynomial can be used as an invariant for the equivalence classes of arcs (or tunnels) of a link if it is regarded as an element of a quotient ring of $\mathbf{Z}[t, t^{-1}]$.

The collection of homology equivalence classes of crossings of a link is an invariant of the link and can be useful in the study of links. The collection could be quite small if we ignore knotted crossings like the one given for the trivial knot in Figure 1, and the Alexander polynomials of the crossings in the collection may be used to distinguish different links. We will take up this in another paper. Finally, the author would like to thank the referee of this paper for carefully checking the computations and finding numerous mistakes.

2. Let C be a collection of crossings of a link L and F a Seifert surface of C. Choose an orientation of F and a positive normal direction of F in $\mathbf{R}^3$. Let $\alpha = (a_1, a_2, \cdots, a_n)$ be an ordered basis for $H_1(F; \mathbf{Z})$. Then there exists a basis $\alpha^* = (a_1^*, a_2^*, \cdots, a_n^*)$ of $H_1(S^3 - F; \mathbf{Z})$ dual to α under the linking pairing. Let $h_* : H_1(F; \mathbf{Z}) \to H_1(S^3 - F; \mathbf{Z})$ be the homomorphism induced by h, where h is an embedding obtained by pushing F into $S^3 - F$ in the positive normal direction of F. Let $M_\alpha = (m_{ij})$ be the $n \times n$ matrix representing h_* with respect to the basis α (and α^*). Then m_{ij} is equal to the linking number of $h(v)$ with u, where u and v are the i^{th} and j^{th} elements of α, respectively. M_α is a Seifert matrix of L [3]. Let $V_\alpha = tM_\alpha^T - M_\alpha$, where M_α^T denotes the transpose of M_α. If L is a knot, then $\det(V_\alpha) \neq 0$.

Now consider the unimodular bilinear pairing, $H_1(F, \partial F;\ \mathbf{Z}) \otimes H_1(F; \mathbf{Z}) \to \mathbf{Z}$, defined by $x \otimes y = u(y)$, where $u \in H^1(F; \mathbf{Z})$ is the Poincare dual of x with respect to the orientation of F. Let $\overline{\alpha} = (\overline{a}_1, \overline{a}_2, \cdots, \overline{a}_n)$ be the basis of $H_1(F, \partial F;\ \mathbf{Z})$ dual to α under this pairing.

For any $x \in H_1(F, \partial F;\ \mathbf{Z})$, let $X_\alpha(x)$ be the row coordinate vector of x with respect to $\overline{\alpha}$ so that $x = X_\alpha(x)\overline{\alpha}^T$. The i^{th} coordinate of $X_\alpha(x)$ is equal to the intersection number of x with the i^{th} element of α.

Given a crossing c in a collection C, let $X_\alpha(c)$ denote $X_\alpha(c_*(w))$. To save notation we use c for $X_\alpha(c)$ if there is no danger of confusion. Finally, for any n-dimensional vectors $X_1, X_2, \cdots, X_r$ and an $n \times n$ matrix V, let

$$V(X_1, X_2, ..., X_r) = \begin{bmatrix} & & & X_r \\ & 0 & & \vdots \\ & & & X_1 \\ X_r^T & \cdots & X_1^T & V \end{bmatrix}.$$

DEFINITION 5. For a collection $C = (c_1, c_2, \cdots, c_r)$ and a Seifert surface F of C, define

$$\lambda(C) = t^{-g} \det(V_\alpha(c_1, c_2, \cdots, c_r)),$$

where g is the genus of F. If V_α is non-singular, then for any pair (c, d), define $\langle c, d\rangle = X_\alpha(c)V_\alpha^{-1}X_\alpha(d)^T$. The matrix V_α^{-1} has been used in [4] for the study of non-singular Seifert matrices.

REMARK 1. Both $\det(V_\alpha(\ ))$ and the pairing $< , >$ do not depend on the choice of an orientation of F and a basis for $H_1(F, \partial F;\ \mathbf{Z})$: First, the reverse of orientation changes $X_\alpha(c)$ to $-X_\alpha(c)$. So the quantities do not change. Secondly, if β is another basis for $H_1(F, \partial F;\ \mathbf{Z})$, then there exists a unimodular matrix A such that $\beta = \alpha A$, where α and β are regarded as row vectors. This implies that $M_\beta = A^T M_\alpha A$, $V_\beta = A^T V_\alpha A$, and $X_\beta(c) = X_\alpha(c)A$. With respect to β,

$$\begin{aligned}\lambda(C) &= t^{-g} \det(V_\beta(X_\beta(c_1), \cdots, X_\beta(c_r))) \\ &= t^{-g} \det(A^T V_\alpha A(X_\alpha(c_1)A, \cdots, X_\alpha(c_r)A)) \\ &= t^{-g} \det(I \oplus A^T) \det(V_\alpha(c_1, c_2, \cdots, c_r)) \det(I \oplus A) \\ &= t^{-g} \det(V_\alpha(c_1, c_2, \cdots, c_r)).\end{aligned}$$

$$\begin{aligned} As for < , >,\ & X_\beta(c)V_\beta^{-1}X_\beta(d)^T \\ &= X_\alpha(c)AA^{-1}V_\alpha^{-1}(A^T)^{-1}A^T X_\alpha(d)^T \\ &= X_\alpha(c)V_\alpha^{-1}X_\alpha(d)^T. \end{aligned}$$

REMARK 2. In the definition of $\lambda(C)$, the order of crossings in $V_\alpha(c_1, c_2, ..., c_r)$ or a replacement of a crossing with one that has the opposite orientation does not matter. We show later (Theorem 5 of Section 4) that $\lambda(C)$ can be described in terms of $< , >$ when V_α is non-singular. It is curious to note that $< c , d >$ is not an invariant of a pair. If c is a crossing of a knot, then $< c , c > (-1)$ is shown to be an invariant of the homology equivalence class of c in [1]. This rational number valued invariant of a crossing completely determines the change in the index of the knot when the crossing is changed. The construction of the Alexander polynomial for collections of crossings of links grew out of that of the rational invariant of crossings of knots [1]. But the definition of the rational invariant does not generalize to crossings of links since the matrix V_α may be singular for links.

3. We now show that the construction in Section 2 gives a well-defined polynomial.

THEOREM 1. *The function λ is well-defined for homology equivalence classes of collections of crossings of links.*

Proof. We need to show that λ is invariant under a change of orientation and positive normal direction of a chosen Seifert surface, under a change of basis for the first homology group of the Seifert surface, under a different choice of Seifert surfaces, and under the homology equivalence. □

In Remark 1, we observed that λ is invariant under a change of orientation of a Seifert surface and a change of basis of the first homology group of the Seifert surface.

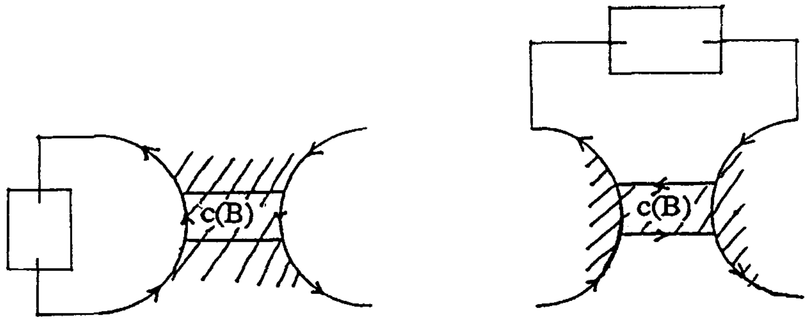
3.1. We next consider a change of positive normal direction of a Seifert surface. The reverse of a positive normal direction changes M_α to M_α^T. With the new normal direction,

$$\begin{aligned}\lambda(C) &= t^{-g}\det((tM_\alpha - M_\alpha^T)(c_1, c_2, ..., c_r)) \\ &= t^{-g}\det(((tM_\alpha - M_\alpha^T)(c_1, c_2, ..., c_r))^T) \\ &= t^{-g}\det((tM_\alpha^T - M_\alpha)(c_1, c_2, ..., c_r)).\end{aligned}$$

Therefore, λ is invariant under a change of positive normal direction.

3.2. For this part of the proof we assume that the collection C has a single crossing c. Let F and F' be Seifert surfaces of the crossing c. We first show that after an isotopy there exists a cobordism between F and F' away from $c(B)$. There are two cases to consider (see Figure 4):

Case 1. Splicing L at c reduces the number of components by one.
Case 2. Splicing L at c increases the number of components by one.



Case 1 Case 2

Figure 4

Let $G = Closure(F - c(B))$ and $G' = Closure(F' - c(B))$. In Case 1, let $K = L$, and in Case 2, $K = \partial G = \partial G'$. Here we only verify the existence of the desired cobordism for Case 2. A similar argument works for Case 1 if G and G' are replaced with F and F', respectively. Observe that it suffices to show that there exists an isotopy of $\mathbf{R}^3$ that keeps the points of $K \cup c(B)$ fixed and put G in general position with G' such that $G \cup G'$ bounds a 3-manifold in $\mathbf{R}^3$.

The orientation of L induces an orientation of K, and the surfaces, G and G', admit orientations compatible with that of K. Let K_0 be a component of K. Then G and G' are oriented cobordisms between K_0 and the rest of the components of K which can be regarded as an element of $H_1(S^3 - K_0; \mathbf{Z})$. This implies that if N is a thin solid torus neighborhood of K_0, then $\partial N \cap G$ and $\partial N \cap G'$ are two isotopic circles in ∂N. Using the fact that K_0 intersects $c(B)$ in no more than one arc, find an isotopy of $\mathbf{R}^3$ that fixes the points of $K \cup c(B)$ and puts G in general position with G' such that $G \cap G'$ consists of K and circles embedded in the interior of G and G'. After the isotopy, G and G' are cobordant in a general sense, i.e., there exists a compact 3-manifold H in $\mathbf{R}^3$ such that $\partial H = (H \cap G) \cup (H \cap G')$ and $(H \cap G) \cap (H \cap G') = G \cap G'$. Since we can choose H such that $H \cap c(B) \subset G \cap G'$, H is a cobordism between F and F' relative to $c(B)$.

By the Morse function theory [2], F' is obtained from F by attaching, away from $c(B)$, 0, 1, 2 and 3-dimensional handles, and then taking a proper subset of the boundary. Without loss of generality, assume that the handles are attached in the order of dimension. By working with each component of H, eliminate all 0-handles with an equal number of 1-handles, and similarly, eliminate all 3-handles regarding them as complementary 0-handles. Therefore, we may assume that H has a handle decomposition consisted of a sequence of 1 and 2-handles. Any surface, obtained as the result of successive handle attachings in this sequence, is connected since F and F' are. Furthermore, we may assume that each 1-handle is attached on one side of the surface. Therefore, to show λ is invariant under a different choice of Seifert surface, it suffices to verify the assertion, assuming that F and F' are connected and F' is obtained from F by attaching a 1-handle on one side of F since a 2-handle is complementary to a 1-handle.

Suppose that F' is obtained from F by attaching a 1-handle H as in Figure 5, where $H \cong D^1 \times D^2$, $H \cap F = \partial D^1 \times D^2 \subset F - c(B)$ and $H \cap F' = D^1 \times \partial D^2$. Let a and b be the elements of $H_1(F'; \mathbf{Z})$ represented by the oriented circles as in the figure.

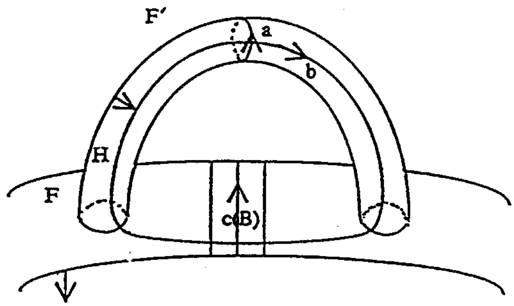


Figure 5

Let α be a basis of $H_1(F; \mathbf{Z})$. Then $\beta = (a, b) \cup \alpha$ is a basis for $H_1(F'; \mathbf{Z})$. If the positive normal directions of F and F' are chosen consistently as in the figure, then

$$M_\beta = \begin{bmatrix} 0 & -1 & 0 \\ 0 & x & x \\ 0 & x & M_\alpha \end{bmatrix} \text{ and } V_\beta = \begin{bmatrix} 0 & 1 & 0 \\ -t & x & x \\ 0 & x & V_\alpha \end{bmatrix}, \quad \text{where 0 denotes}$$

a zero row or column vector of a proper dimension, and x denotes a row or column vector of a proper dimension with unspecified entries.

Since the coordinates of c in $H_1(F', \partial F'; \mathbf{Z})$ with respect to $\overline{\beta}$ is $X_\beta(c) = (0, x, X_\alpha(c))$, if $\lambda(c)$ is computed from F' and β, then $\lambda(c) = t^{-(g+1)}\det(V_\beta((0, x, X_\alpha(c)))) = t^{-g}\det(V_\alpha(X_\alpha(c)))$. Therefore, λ is invariant.

3.3 Suppose that crossing c is equivalent to crossing d through a diffeomorphism f of $\mathbf{R}^3$. Let F be a Seifert surface of c and α a basis of $H_1(F; \mathbf{Z})$. Then $F' = f(F)$ is a Seifert surface of d and $\beta = f_*(\alpha)$ is a basis of $H_1(F'; \mathbf{Z})$. Choose also orientations and positive normal directions of F and F' such that f preserves them. Since both data give the identical λ-value for c and d, λ is invariant. Finally, if two collections are homologous, they clearly have the same λ-value by definition. This completes the proof of Theorem 1 when the collection has a single crossing. To begin the proof of general case, we first make the following definition.

DEFINITION 6. For a link L, define the signed Alexander polynomial of L by $\Delta L = \lambda(\emptyset) = t^{-g}\det(tM^T - M)$, where M is a Seifert matrix of L associated to a Seifert surface of genus g. Here we consider the determinant of an empty matrix to be 1.

REMARK 3. It follows from the proof of Theorem 1 that ΔL is an invariant of link L. We show (Theorem 2) that if L has k components, then $t^{-(k-1)/2}\Delta L = \widetilde{\Delta} L$, the Alexander polynomial of L defined combinatorially using the skein relation.

This implies that if two links with the same number of components have the same Alexander polynomial, then they must have the same signed Alexander polynomial. The assertion is no longer true if we consider links with different number of components. Two links $L_1$and L_2 are given in Figure 6. Link L_1 has 2 components and L_2 4 components. They have the same Alexander polynomial $t^{-\frac{3}{2}} - 3t^{-\frac{1}{2}} + 3t^{\frac{1}{2}} - t^{\frac{3}{2}}$ but $\Delta L_1 = t^{-1} - 3 + 3t - t^2$ and

$$\Delta L_2 = 1 - 3t + 3t^2 - t^3.$$

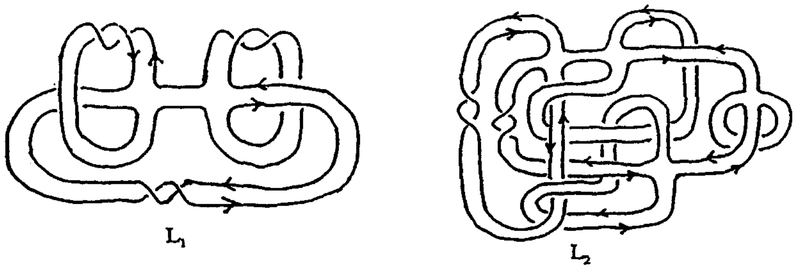


Figure 6

3.4 Here we study the effect of crossing changes on ΔL and $\lambda(C)$, from which the proof of the general case of the theorem follows. We first make it clear what it means by changing a collection of crossings. Given any collection C of crossings of a link L, there exists a projection of L such that each crossing in the collection is equivalent to one of the two standard forms in Figure 2. To change the collection C, put the crossings in C into standard forms in a projection of L and then change the crossings. This operation is not well-defined since the new link obtained by changing the collection depends on the standard forms which the crossings are put into. In the rest of this section assume that all crossings are in standard forms in a link projection.

DEFINITION 7. For a crossing c, define the sign ε of c by $\varepsilon(c) = 1$ if c is a positive crossing, and $\varepsilon(c) = -1$ otherwise. For a collection C of crossings, let $\varepsilon(C)$ be the product of signs of crossings in C.

Given a crossing c of link L, denote by $L'(c)$ the new link obtained by changing c, and c' the crossing in $L'(c)$ corresponding to c. Note that c and c' have the opposite signs. If C is a collection of crossings, then $L'(C)$ denotes the link obtained by changing every crossing in C. For a collection, $(c_1, c_2, \cdots, c_r, d_1, d_2, \cdots, d_s)$ of crossings, let $(c_1, c_2, \cdots, c_r; d'_1, d'_2, \cdots, d'_s)$ or $(c_1, c_2, \cdots, c_r; L'(d_1, d_2, \cdots, d_s))$ be the collection $(c_1, c_2, \cdots, c_r)$ in the link $L'(d_1, d_2, \cdots, d_s)$.

LEMMA 1. *Under the above notation, for any collection* $(c_1, c_2, \cdots, c_r, c)$ *of crossings of a link* L, *we have:*

(1) $\Delta L'(c) = \Delta L + \varepsilon(c)(1-t)\lambda(c)$.

(2) $\lambda(c_1, c_2, \cdots, c_r, c') = \lambda(c_1, c_2, \cdots, c_r, c)$.

(3) $\lambda(c_1, c_2, \cdots, c_r; c') = \lambda(c_1, c_2, \cdots, c_r) + \varepsilon(c)(1-t)\lambda(c_1, c_2, \cdots, c_r, c)$.

(4) $\Delta L_0(c) = -t\lambda(c)$ *if* L_0 *has one more component than* L, *and* $\Delta L_0(c) = -\lambda(c)$ *if* L_0 *has one less component than* L.

Proof of Lemma 1. Let F be a canonical Seifert surface of L as in Figure 7 that is also a Seifert surface of c. Let F' be the surface obtained from F by attaching two 1-handles (or bands) to F. Then F' is a Seifert surface of the crossing c' of L'. Choose positive normal directions and orientations of F and F' as in the figure. Let a and b be the elements of $H_1(F'; \mathbf{Z})$ represented by the oriented circles as in Figure 7. Then for any basis α of $H_1(F; \mathbf{Z})$, $\beta = (a, b) \cup \alpha$ is a basis for $H_1(F'; \mathbf{Z})$. Put $\varepsilon = \varepsilon(c)$. □

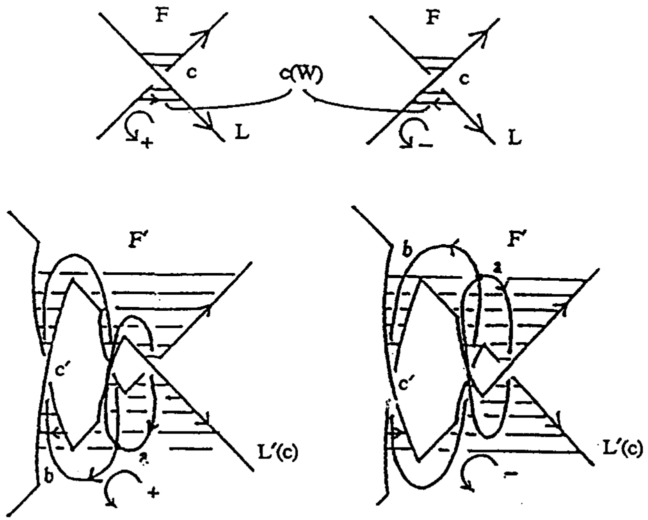


Figure 7

The Seifert matrix associated to F' and β is

$$M_\beta = \begin{bmatrix} 0 & -1 & X \\ 0 & \varepsilon & 0 \\ 0 & 0 & M_\alpha \end{bmatrix} \text{ and } V_\beta = \begin{bmatrix} 0 & 1 & -X \\ -t & \varepsilon(t-1) & 0 \\ tx & 0 & V \end{bmatrix},$$

where $X = X_\alpha(c)$ is the coordinate of $c_*(w)$ with respect to $\overline{\alpha}$, $x = X^T$, and $V = tM_\alpha^T - M_\alpha$.

$$\begin{aligned}\Delta L'(c) &= t^{-(g+1)}\det(V_\beta)\\ &= t^{-(g+1)}(t\det(V) - \varepsilon(t-1)t\det(V(X)))\\ &= t^{-g}\det(V) + \varepsilon(1-t)t^{-g}\det(V(X)) = \Delta L + \varepsilon(c)(1-t)\lambda(c).\end{aligned}$$

This completes the proof of (1).

To prove (2), observe that $X_\beta(c') = (0,-1,0)$ and $X_\beta(c_i) = (0,0,X_\alpha(c_i))$. By definition,

$$\begin{aligned}&\lambda(c_1, c_2, \cdots, c_r, c')\\ &= t^{-(g+1)}\det(V_\beta((0,-1,0),(0,0,X_\alpha(c_1)),(0,0,X_\alpha(c_2)),\cdots,(0,0,X_\alpha(c_r))).\end{aligned}$$

Successive expansions of the above determinant in proper rows and columns show that the expression is equal to

$$t^{-g}\det(V(X, X_\alpha(c_1), X_\alpha(c_2), \cdots, X_\alpha(c_r)).$$

Therefore, $\lambda(c_1, c_2, \cdots, c_r, c') = \lambda(c_1, c_2, \cdots, c_r, c)$.

Similarly,

$$\begin{aligned}&\lambda(c_1, c_2, \cdots, c_r; c')\\ &= t^{-(g+1)}\det(V_\beta((0,0,X_\alpha(c_1)),(0,0,X_\alpha(c_2)),\cdots,(0,0,X_\alpha(c_r))))\\ &= t^{-(g+1)}t\det(V(X_\alpha(c_1),X_\alpha(c_2),\cdots,X_\alpha(c_r)))\\ &\quad - t^{-(g+1)}\varepsilon(t-1)t\det(V(X,X_\alpha(c_1),X_\alpha(c_2),\cdots,X_\alpha(c_r)))\\ &= \lambda(c_1,c_2,\cdots,c_r) + \varepsilon(c)(1-t)\lambda(c_1,c_2,\cdots,c_r,c).\end{aligned}$$

This proves (3).

For the last assertion of the lemma, let F_0 be the Seifert surface of $L_0(c)$ obtained from surface F' in Figure 7 by deleting the band of c'. Then F_0 is obtained from F by attaching a single band. If the band is attached to the same component of L, then L_0 has one more component than L. This does not affect the genus of the Seifert surface. Therefore, F_0 and F have the same genus. Since (a,α) is a basis for $H_1(F_0;\mathbf{Z})$, $\Delta L_0 = t^{-g}\det\begin{bmatrix} 0 & -X \\ tx & V\end{bmatrix} = -tt^{-g}\det(V(X)) = -t\lambda(c)$. If the band is attached to two distinct components of L, then L_0 has one less component than L and the genus of F_0 is one larger than that of F. So $\Delta L_0 = t^{-g-1}\det\begin{bmatrix} 0 & -X \\ tx & V \end{bmatrix} = -t^{-g}\det(V(X)) = -\lambda(c)$. This completes the proof of Lemma 1.

Back to the proof of Theorem 1. Given a collection C of $(r+1)$ crossings, the identity (3) of Lemma 1 expresses $\lambda(C)$ in terms of $\lambda-$values of collections containing r crossings. Therefore, λ is well-defined for collections with two crossings since it was proven in the first part that λ is well-defined for collections with one crossing. By induction, it is well-defined for arbitrary collections. This completes the proof of Theorem 1. □

THEOREM 2. *For any link L, $\widetilde{\Delta}L = t^{-(k-1)/2}\Delta L$, where k is the number of components of L.*

Proof. Let L_+, L_- and L_0 be the usual three link projections that differ only at the neighborhood of a crossing c. Without loss of generality, assume that c is a negative crossing, and apply (1) of Lemma 1 to get $\Delta L_+ - \Delta\ L_- + (1-t)\lambda(c) = 0$. Multiply $t^{-(k-1)/2}$ to the identity. If L_0 has one more component than L, then from (4) of Lemma 1,

$$t^{-(k-1)/2}\Delta L_+ - t^{-(k-1)/2}\Delta L_- + (t^{1/2} - t^{-1/2})t^{-k/2}\Delta L_0 = 0.$$

Similarly, if L_0 has one less component than L, then

$$t^{-(k-1)/2}\Delta L_+ - t^{-(k-1)/2}\Delta L_- + (t^{1/2} - t^{-1/2})t^{-(k-2)/2}\Delta L_0 = 0.$$

Therefore, $t^{-(k-1)/2}\Delta L$ satisfies the defining skein relation for $\widetilde{\Delta}L$, which implies that the two polynomials are the same. □

REMARK 4. The proof of the above theorem shows that if we use a Seifert matrix M to define the (normalized) Alexander polynomial, then we must use $\det(tM^T - M)$ rather than $\det(M - tM^T)$. Otherwise, the polynomial does not satisfy the skein relation after normalization.

THEOREM 3. *For any collection C of crossings of L,*

$$\lambda(C) = (-1)^{|C|}t^{\frac{k-1-|C|}{2}}\widetilde{\Delta}L_0(C),$$

where k is the number of components in L and $|C|$ the number of crossings in C.

Proof. We induct on the number of crossings in C. If C is empty, the assertion is that of Theorem 2. Suppose that the theorem holds for any collection with r or less crossings. Let $C = (c_1, c_2, \cdots, c_r, c)$ and $D = (c_1, c_2, \cdots, c_r)$. Then by (3) of Lemma 1,

$$\varepsilon(c)(1-t)\lambda(C) = \lambda(D; c') - \lambda(D)$$

By the induction hypothesis, Lemma 1 and Theorem 2, if ℓ is the number of components in $L_0(D)$, then

$$\begin{aligned}
\lambda(D;c') &= (-1)^r t^{\frac{k-r-1}{2}} \tilde{\Delta}(L'(c))_0(D) \\
&= (-1)^r t^{\frac{k-r-1}{2}} t^{-\frac{\ell-1}{2}} \Delta(L'(c))_0(D) \\
&= (-1)^r t^{\frac{k-r-\ell}{2}} \left[\Delta L_0(D) + \varepsilon(c)(1-t)\lambda(c; L_0(D))\right] \\
&= (-1)^r t^{\frac{k-r-\ell}{2}} \left[t^{\frac{\ell-1}{2}} \tilde{\Delta} L_0(D) + \varepsilon(c)(1-t)(-t^{\frac{\ell-2}{2}})\tilde{\Delta} L_0(C))\right] \\
&= (-1)^r t^{\frac{k-1-r}{2}} \tilde{\Delta} L_0(D) + (-1)^{r+1}\varepsilon(c)(1-t)(t^{\frac{k-r-2}{2}})\tilde{\Delta} L_0(C)
\end{aligned}$$

On the other hand,

$$\lambda(D) = (-1)^r t^{\frac{k-1-r}{2}} \tilde{\Delta} L_0(D)$$

Therefore,

$$\lambda(C) = (-1)^{r+1} t^{\frac{k-1-r-1}{2}} \tilde{\Delta} L_0(C) = (-1)^{|C|} t^{\frac{k-1-|C|}{2}} \tilde{\Delta} L_0(C)$$ □

4. We derive other identities involving λ.

LEMMA 2. *Let $(c_1, c_2, \cdots, c_r, d_1, d_2, \cdots, d_s)$ be a collection of crossings of link L. Then*

$$\begin{aligned}
&\lambda(c_1, c_2, \cdots, c_r; d_1', d_2', \cdots, d_s') \\
&= \sum_{S \in P(\{d_1, d_2, \cdots, d_s\})} \varepsilon(S)(1-t)^{|S|}\lambda(c_1, c_2, \cdots, c_r, S).
\end{aligned}$$

Proof. Use induction on s. If $s = 1$, the lemma is (3) of Lemma 1. Assume that the lemma is true for $(d_1, d_2, \cdots, d_{s-1})$. Then by Lemma 1,

$$\begin{aligned}
&\lambda(c_1, c_2, \cdots, c_r; d'_1, d'_2, \cdots, d'_s) \\
&= \lambda(c_1, c_2, \cdots, c_r; d'_1, d'_2, \cdots, d'_{s-1}) \\
&\quad + \varepsilon(d_s)(1-t)\lambda(c_1, c_2, \cdots, c_r, d_s; d'_1, d'_2, \cdots, d'_{s-1}) \\
&= \sum_{T \in P(\{d_1, d_2, \cdots, d_{s-1}\})} \varepsilon(T)(1-t)^{|T|}\lambda(c_1, c_2, \cdots, c_r, T) \\
&\quad + \varepsilon(d_s)(1-t) \sum_{U \in P(\{d_1, d_2, \cdots, d_{s-1}\})} \varepsilon(U)(1-t)^{|U|}\lambda(c_1, c_2, \cdots, c_r, d_s, U).
\end{aligned}$$

The lemma follows from the fact that $P(\{d_1, d_2, \cdots, d_s\})$ is the disjoint union of $P(\{d_1, d_2, \cdots, d_{s-1}\})$ and $\{U \cup \{d_s\} : U \in P(\{d_1, d_2, \cdots, d_{s-1}\})\}$. □

THEOREM 4. *For any collection C of crossings of link L,*

$$\Delta L'(C) = \sum_{S \in P(C)} \varepsilon(S)(1-t)^{|S|}\lambda(S).$$

Proof. We prove the theorem by an induction on $|C|$. If $|C| = 1$, then the theorem is (1) of Lemma 1. Assume that it is true for any C, $|C| \leq r-1$. Suppose that $C = (c_1, c_2, \cdots, c_r)$. By Lemma 1, 2 and the induction hypothesis,

$$\begin{aligned}
&\Delta L'(c_1, c_2, \cdots, c_r) \\
&= \Delta L'(c_1, c_2, \cdots, c_{r-1}) + \varepsilon(c_r)(1-t)\lambda(c_r; c'_1, c'_2, \cdots, c'_{r-1}) \\
&= \sum_{T \in P(\{c_1, c_2, \cdots, c_{r-1}\})} \varepsilon(T)(1-t)^{|T|}\lambda(T) \\
&\quad + \varepsilon(c_r)(1-t) \sum_{U \in P(\{c_1, c_2, \cdots, c_{r-1}\})} \varepsilon(U)(1-t)^{|U|}\lambda(c_r, U) \\
&= \sum_{S \in P(C)} \varepsilon(S)(1-t)^{|S|}\lambda(S).
\end{aligned}$$

This completes the proof. □

Suppose that $\Delta L \neq 0$, equivalently, $\widetilde{\Delta} L \neq 0$. In this case, recall that $\langle c, d\rangle$ is defined for a pair of crossings of L in Definition 5.

DEFINITION 8. Given a collection $C = (c_1, c_2, \cdots, c_r)$, define the matrix of C, $\langle C, C\rangle$, to be $t^{-g}(c_{ij})$, where $c_{ij} = \langle c_i, c_j\rangle$.

THEOREM 5. *Let C be a collection of crossings of a link L. If $\Delta L \neq 0$, then $\lambda(C) = (-1)^{|C|}\Delta L \det(\langle C, C\rangle)$.*

Proof. Put $V = V_\alpha$ and $X_i = X_\alpha(c_i)$. By definition,

$$\lambda(C) = t^{-g}\det(V(X_1, X_2, \cdots, X_r)).$$

Multiply the determinant of $\begin{bmatrix} 1 & & 0 & -X_r V^{-1} \\ & \ddots & & \vdots \\ 0 & & 1 & -X_1 V^{-1} \\ 0 & \cdots & 0 & V^{-1} \end{bmatrix}$ to the both sides of the preceding identity. Then

$\det(V^{-1})\lambda(C) = (-1)^r t^{-g}\det(\langle C, C\rangle)$.

So

$\lambda(C) = (-1)^r t^{-g}\det(V)\det(\langle C, C\rangle) = (-1)^{|C|}\Delta L \det(\langle C, C\rangle)$,

which completes the proof. □

In the above theorem, if $C = (c, d)$, then

$$\begin{aligned} \lambda(c, d) &= \Delta L(\langle c, c\rangle \langle d, d\rangle - \langle c, d\rangle \langle d, c\rangle) \\ &= \Delta L(-(\Delta L)^{-1}\lambda(c)(-1)(\Delta L)^{-1}\lambda(d) - \langle c, d\rangle \langle d, c\rangle) \\ &= (\Delta L)^{-1}\lambda(c)\lambda(d) - \Delta L \langle c, d\rangle \langle d, c\rangle . \end{aligned}$$

Therefore, we have:

THEOREM 6. *If $\Delta L \neq 0$, then for any pair (c, d) of crosssings of link L, $(\Delta L)^2 \langle c, d\rangle \langle d, c\rangle$ is an invariant of the pair under the homology equivalence. We denote this polynomial by $\lambda_0(c, d)$.*

5. We compute some of the polynomials constructed in previous sections. Knots 10_{18} and 10_{24} [3] are given in Figure 8 together with some chosen crossings and their Seifert surfaces. Both knots have $-4t^{-2} + 14t^{-1} - 19 + 14t - 4t^2$ as their Alexander polynomial. Choose $\alpha =$

(a_1, b_1, a_2, b_2) as bases for the first homology groups of the canonical Seifert surfaces as in the figure. Then

$$M_\alpha(10_{18}) = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 2 \end{bmatrix} \text{ and } M_\alpha(10_{24}) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

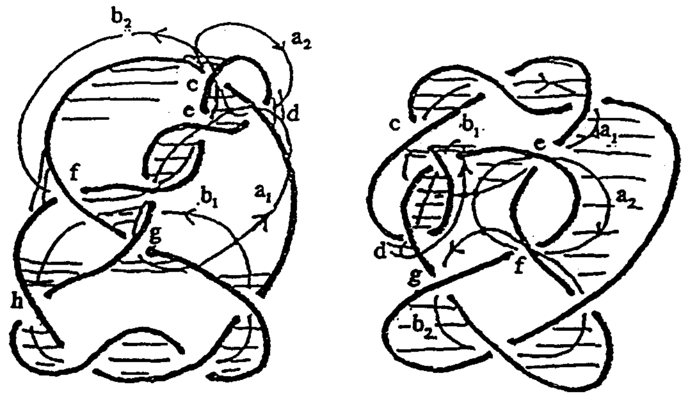


Figure 8

The coordinates of the crossings are:

10_{18}	$X_\alpha(\ )$
c	$(0,0,1,1)$
d	$(1,0,1,0)$
e	$(1,0,0,-1)$
f	$(0,0,0,1)$
g	$(1,0,0,0)$
h	$(0,1,0,0)$

10_{24}	$X_\alpha(\ )$
c	$(1,0,0,0)$
d	$(0,1,0,0)$
e	$(1,0,-1,0)$
f	$(0,0,1,1)$
g	$(0,0,0,1)$

The Alexander polynomials of the crossings are:

10_{18}	$\lambda(\ )$
c	$-4t^{-2}+12t^{-1}-12+4t$
d	$-4t^{-2}+12t^{-1}-12+4t$
e	$-2t^{-2}+7t^{-1}-7+2t$
f	$-4t^{-2}+11t^{-1}-11+4t$
g	$-4t^{-2}+10t^{-1}-10+4t$
h	$2t^{-2}-4t^{-1}+4-2t$

10_{24}	$\lambda(\ )$
c	$-2t^{-2}+5t^{-1}-5+2t$
d	$4t^{-2}-8t^{-1}+8-4t$
e	$-4t^{-2}+13t^{-1}-13+4t$
f	$-4t^{-2}+13t^{-1}-13+4t$
g	$-2t^{-2}+6t^{-1}-6+2t$

The Alexander polynomials for some pairs are:

10_{18}	$\lambda(c,\ )$
d	$-2t^{-2}+5t^{-1}-2$
e	$-2t^{-2}+5t^{-1}-2$
f	$-4t^{-2}+9t^{-1}-4$
g	$-4t^{-2}+8t^{-1}-4=-4(t^{-1}-1)^2$
h	$2t^{-2}-3t^{-1}+2$

10_{24}	$\lambda(c,\ )$
d	$2t^{-2}-3t^{-1}+2$
e	$-2(t^{-1}-1)^2$
f	$-2(t^{-1}-1)^2$
g	$-(t^{-1}-1)^2$

The value of λ_0 for some of the pairs are:

10_{18}	$\lambda_0(c,\ )$
d	$(2t^{-2}-7t^{-1}+10-4t)(4t^{-2}-10t^{-1}+7-2t)$
e	$-t^{-1}(2t^{-1}-5+2t)^2$
f	$t^{-2}(2-3t)(3-2t)$
g	$4(t^{-1}-1)^2$
h	$-t^{-1}$

10_{24}	$\lambda_0(c,\ )$
d	$-t^{-1}(2t^{-1}-3+2t)^2$
e	$(-t^{-1}+3-2t)(2t^{-2}-3t^{-1}+1)$
f	$(-2t^{-1}+3-t)(t^{-2}-3t^{-1}+2)$
g	$(t^{-1}-1)^2$

The second set of tables show that no two crossings in knot 10_{18} or 10_{24} are homology equivalent, in particular, no two are equivalent under diffeomorphisms, including orientation-reversing ones, of $\mathbf{R}^3$ (Property 1 of Section 6) except possibly for c and d in 10_{18} or e and f in 10_{24}.

We next give an example of two homologous collections such that the spliced links at the collections are not equivalent as links. Let $C=(c_1,c_2)$ and $D=(d_1,d_2)$ be the collections of crossings in knot $8_{21}[3]$ as in Figure

9. C and D are homologous, thus $\widetilde{\Delta}L_0(C) = \widetilde{\Delta}L_0(D)$ $(= 0)$ but $L_0(C)$ is not equivalent to $L_0(D)$ as links.

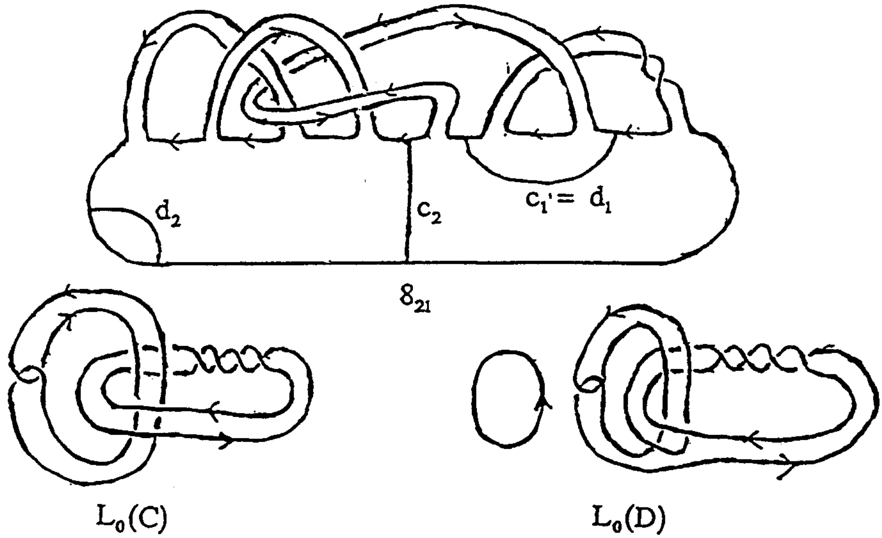


Figure 9

6. We discuss more properties of Alexander polynomials of crossings. One of them deals with the change in $\lambda(c)$ when twists are introduced to c.

Suppose that C is a collection crossings of link L. Let rL and rC be the mirror images of L and C, respectively. If M is Seifert matrix of L, then $-M$ is a Seifert matrix of rL. It follows:

PROPERTY 1. *If C is a collection of crossings of link L with k components. Then $\lambda(C) = (-1)^{k-1+|C|}\lambda(rC)$, where $|C|$ is the number of crossing in C.*

DEFINITION 9. Given a crossing c of link L and an integer n, define the crossing c_n to be the one obtained from c by giving its band n right handed full-twists as in Figure 10. Notice that $c \mid W = c_n \mid W$.

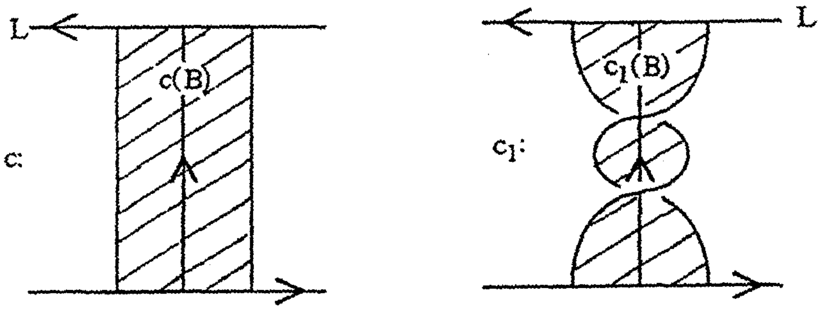


Figure 10

Property 2. *For any pair* (c, d) *of link* L *and integer* n, $\lambda(c_n) = n(t^{-1} - 1)\Delta L + \lambda(c)$, $\lambda(c_n, d) = n(t^{-1} - 1)\lambda(d) + \lambda(c, d)$ *and*

$$\lambda_0(c_n, d) = \lambda_0(c, d).$$

Proof. Let F be a Seifert surface of (c, d). Perform a 0-surgery on $c(B) \subset F$ as in Figure 11. Let F' be the result of the surgery.

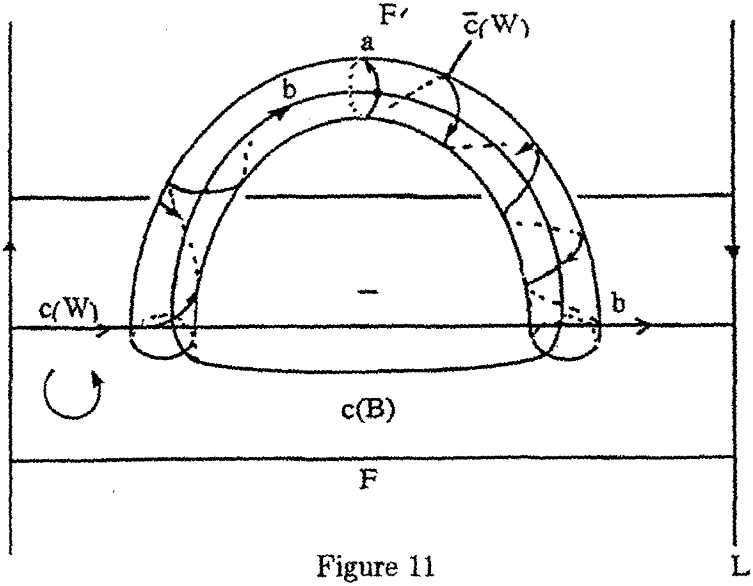


Figure 11

Define an embedded arc $\bar{c}(W)$ as in the figure such that the arc has the same the end points as $c(W)$ and winds around the surgery tube n times in the right handed direction. We consider $\bar{c}$ as a crossing by taking a thin tubular neighborhood of $\bar{c}(W)$ in F' as its band. It is clear that $\bar{c}$ is equivalent to c_n as crossings of L. Let a and b be the elements of $H_1(F';\mathbf{Z})$ represented by the oriented circles as in the figure. Let α be a basis of $H_1(F;\mathbf{Z})$. Then $\beta = (a,b)\cup\alpha$ is a basis of $H_1(F';\mathbf{Z})$. We may assume that b does not intersect any elements of α. By choosing positive normal directions and orientations of F and F' consistently as in the figure, we obtain

$$M_\beta = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_\alpha \end{bmatrix}, \quad V_\beta = \begin{bmatrix} 0 & 1 & 0 \\ -t & 0 & 0 \\ 0 & 0 & V_\alpha \end{bmatrix}, \quad \text{and}$$

$$V_\beta^{-1} = \begin{bmatrix} 0 & -t^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & V_\alpha^{-1} \end{bmatrix}.$$

Now $X_\beta(\bar{c}) = (1, n, X_\alpha(c))$ and $X_\beta(d) = (0,0,X_\alpha(d))$.

$$\begin{aligned}\lambda(c_n) &= t^{-(g+1)}\det(V_\beta((1,n,X_\alpha(c))))\\ &= t^{-(g+1)}(-tn\det(V_\alpha) + n\det(V_\alpha) + t\det(V_\alpha(X_\alpha(c))))\\ &= n(t^{-1}-1)\Delta L + \lambda(c).\end{aligned}$$

$$\begin{aligned}\lambda(c_n, d) &= t^{-(g+1)}\det(V_\beta((1,n,X_\alpha(c)),(0,0,X_\alpha)))\\ &= t^{-(g+1)}(-tn\det(V_\alpha(X_\alpha(d)))\\ &\quad + n\det(V_\alpha(X_\alpha(d))) + t\det(V_\alpha(X_\alpha(c),X_\alpha(d)))\\ &= n(t^{-1}-1)\lambda(d) + \lambda(c,d).\end{aligned}$$

$$\begin{aligned}\lambda_0(c_n, d) &= (\Delta L)^2(1,n,X_\alpha(c))V_\beta^{-1}(0,0,X_\alpha(d))^T\\ &\quad (0,0,X_\alpha(d))V_\beta^{-1}(1,n,X_\alpha(c))^T\\ &= (\Delta L)^2 X_\alpha(c)V_\alpha^{-1}X_\alpha(d)^T X_\alpha(d)V_\alpha^{-1}X_\alpha(c)^T\\ &= \lambda_0(c,d).\end{aligned}$$

□

REMARK 5. It follows from Lemma 1 that for any link L, if $\Delta L \neq 0$, then $L'_+(c_n)$(or $L'_-(c_n)$), $n \in \mathbf{Z}$, are distinct links, where $L'_+(c_n)$ is the link obtained from L by changing c_n after c_n is put into a negative crossing (or a positive crossing). A picture of $L'_+(c_n)$ is given in Figure 12.

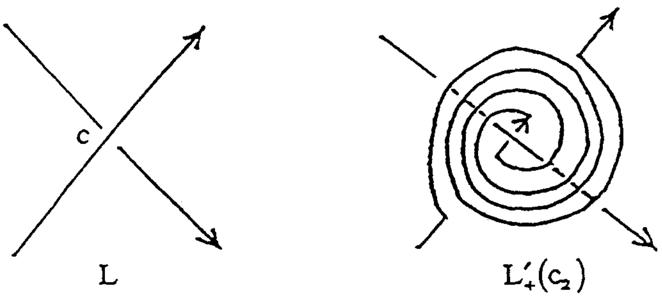


Figure 12

If c is a crossing of a knot projection K, it is shown in Theorem 2 of [1] that

$$\overline{\lambda}(c) = \langle c,\ c\rangle(-1) = -\lambda(c)(-1)/\Delta K(-1)$$

determines completely the change in the index of the knot when c is changed. We used $\lambda(c)$ for $\overline{\lambda}(c)$ in [1] and called it the rational invariant of c. The above property implies that $\overline{\lambda}(c_n) = \overline{\lambda}(c) + 2n$. Hence by Theorem 2 of [1], we have

PROPERTY 3. *Let c be a crossing in a knot projection K. If c is a positive crossing, then the index of the knot increases by 2 if we change c_n provided that $n < -\frac{1}{4} - \frac{1}{2}\overline{\lambda}(c)$ and it remains the same otherwise. If c is a negative crossing, then the index of the knot decreases by 2 if we change c_n provided that $n > \frac{1}{4} - \frac{1}{2}\overline{\lambda}(c)$ and it remains the same otherwise.*

The above property can be used to produce knots with a given index from any knot projection by introducing a proper number of twists to each member of a collection of crossings.

DEFINITION 10. An arc u of a link L is defined to be an embedding u: $[-1, 1] \to \mathbf{R}^3$ such that $u([-1, 1]) \cap L = u(\{-1, 1\})$.

A pair of arcs of a link is defined the same way as a pair of crossings is defined, and the homology equivalence of arcs and pairs of arcs is defined similarly.

Given a pair (u, v) of arcs of link L, there exists a pair (c, d) of crossings of L such that $c \mid W = u$ and $d \mid W = v$. Notice that c and d are not unique.

DEFINITION 11. Under the above notation, define $\lambda(u) = \lambda(c) \in \mathbf{Z}[t, t^{-1}]/(\Delta L)$ and $\lambda_0(u, v) = \lambda_0(c, d)$, where (ΔL) is the ideal generated by ΔL.

It follows from Property 2:

PROPERTY 4. *The functions λ and λ_0 in the above definition are invariants of the homology equivalence classes of arcs and pairs of arcs of a link.*

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