Bull. Korean Math. Soc. **35** (1998), No. 2, pp. 227–234

STABILITY OF A CLASS OF pTH-ORDER NONLINEAR AUTOREGRESSIVE PROCESSES

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ABSTRACT. Criteria are derived for the existence of a unique invariant probability distribution of a class of nonlinear pth-order autoregressive processes, which reformulate those of Tweedie's. It will be shown that the criteria in this paper are easily applicable to the linear or piecewise linear case so that some of the earlier results are immediate consequences of our main results.

1. Introduction

Consider a *nonlinear pth-order autoregressive process* $\{X_n : n \geq 0\}$ defined by

$$X_{n+1} = h(X_{n-p+1}, \cdots, X_n) + \eta_{n+1} \quad (n \geq p-1), \tag{1.1}$$

where h is real-valued Borel measurable function on $\mathbb{R}^p$, $\{\eta_n : n \geq 1\}$ is a sequence of i.i.d real-valued random variables, and $\{X_0, X_1, \cdots, X_{p-1}\}$ are arbitrarily prescribable real-valued random variables independent of $\{\eta_n : n \geq 1\}$. We are mainly interested in the case $p > 1$. By vectorization of (1.1), we have the associated Markov process $\{Y_n : n \geq 0\}$ on $\mathbb{R}^p$ defined by an arbitrarily specified random vector Y_0 with values in $\mathbb{R}^p$, independent of $\{\eta_n : n \geq 1\}$, and by

$$Y_{n+1} := f(Y_n) + \varepsilon_{n+1} \quad (n \geq 0), \tag{1.2}$$

Received June 18, 1997.
1991 Mathematics Subject Classification: Primary 60G10, 60J05.
Key words and phrases: Markov process, invariant probability, irreducibility, Harris ergodic.
Research supported in part by KOSEF Grant 951-0103-021-2.

where $f : \mathbb{R}^p \to \mathbb{R}^p$ and ε_n are defined by

$$(1.3) \qquad f(y) := \begin{pmatrix} y_2 \\ \vdots \\ y_p \\ h(y) \end{pmatrix}, \quad \varepsilon_n := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \eta_n \end{pmatrix} \quad (n \geq 1).$$

Here $y = (y_1, y_2, \cdots, y_p)' \in \mathbb{R}^p$. It is convenient to read off the asymptotic properties of the stochastic process $\{X_n : n \geq 0\}$ from those of $\{Y_n : n \geq 0\}$. Let $\mathcal{B}^p$ denote the Borel sigma field on $\mathbb{R}^p$, and $p(x, dy)$ (or $p(x, A)$, $A \in \mathcal{B}^p$) the transition probability of Y_n. More generally, $p^{(n)}(x, dy)$ denotes the n-step transition probability of $Y_n (n \geq 1)$, so that $p(x, dy) \equiv p^{(1)}(x, dy)$. Let λ_p denote *Lebesgue measure* on $(\mathbb{R}^p, \mathcal{B}^p)$. A probability measure π on $(\mathbb{R}^p, \mathcal{B}^p)$ is said to be *invariant* for $\{Y_n : n \geq 0\}$, or for $p(x, dy)$ if

$$(1.4) \qquad \int_{\mathbb{R}^p} p(x, A)\pi(dx) = \pi(A), \qquad A \in \mathcal{B}^p.$$

The Markov process is *ϕ-irreducible* with respect to a nontrivial measure ϕ on $\mathbb{R}^p$, if for every $A \in \mathcal{B}^p$ with $\phi(A) > 0$ one has

$$\sum_{n \geq 1} p^{(n)}(x, A) > 0, \qquad x \in \mathbb{R}^p.$$

A set $B \in \mathcal{B}^p$ is said to be *small* (with respect to ϕ) if $\phi(B) > 0$, and for every $A \in \mathcal{B}^p$ with $\phi(A) > 0$ there exists $j \geq 1$ such that

$$(1.5) \qquad \inf_{x \in B} \sum_{n=1}^{j} p^{(n)}(x, A) > 0.$$

A ϕ-irreducible aperiodic Markov process with transition probability $p(x, dy)$ is said to be (*Harris*) *ergodic* if there exists a probability measure π such that

$$(1.6) \quad ||p^{(n)}(x, dy) - \pi(dy)|| \longrightarrow 0 \quad \text{as} \quad n \to \infty, \qquad x \in \mathbb{R}^p.$$

Here $|| \cdot ||$ denotes the variation norm on the Banach space of finite signed measure on $(\mathbb{R}^p, \mathcal{B}^p)$.

If (1.6) holds then π is necessarily the unique invariant probability for $p(x, dy)$. The following theorem of Tweedie in [5] provides one set of sufficient conditions for (Harris)ergodicity.

THEOREM 1.1. *Suppose that the process $\{Y_n : n \geq 0\}$ is ϕ-irreducible and B is a small set. (a) If there exists a nonnegative Borel measurable function $g(\cdot)$ on $\mathbb{R}^p$ and a positive constant θ such that*

$$(1.7) \qquad (i) \qquad \sup_{x \in B} \int g(y)p(x, dy) < \infty,$$

and

$$(ii) \quad \int g(y)p(x, dy) \leq g(x) - \theta, \qquad x \in B^c,$$

then the process is (Harris) ergodic.

The function $g(\cdot)$ above is often referred to as a (stochastic) Lyapunov function because of the analogous role similar functions play in the theory of asymptotic stability for dynamical systems.

Under some mild conditions on h (accordingly on f), the process $\{Y_n : n \geq 0\}$ in (1.2) can be represented as

$$(1.8) \qquad Y_{n+1} = A(Y_n)Y_n + \varepsilon_{n+1},$$

where, for given $Y_n = y$, $A(Y_n) = A(y)$ is a $p \times p(y$ dependent) matrix. In particular, in the familiar autoregressive situation where $h(y)$ is linear or piecewise linear, and the corresponding $p \times p$ matrix $A(y)$ is constant, sufficient conditions for ergodicity (or for geometric ergodicity) have been found before (see [1], [2] and [4]), most of them requiring the knowledge of $h(y)$ for all y. It will be shown that the criteria in this paper are easily applicable to the linear or piecewise linear case so that some of the results in [2] and [4] are immediate consequences of our main results, and also shown that these can be extended to the non-linear case as well.

Here is a brief overview of the paper.

In section 2 some criteria are provided, which reformulate the criteria in Theorem 1.1 so that they fit our purpose better to improve and extend those results obtained earlier in [2], [4] and others.

In section 3 two examples are given, in which some aspects of our main results in application are observed.

2. The main result

Consider the AR(p) model

$$X_{n+1} = h(X_{n-p+1}, \cdots, X_n) + \eta_{n+1}(n \geq p-1), \tag{2.1}$$

where h is a real-valued Borel measurable function on $\mathbb{R}^p$, $\{\eta_n : n \geq 1\}$ is a sequence of i.i.d random variables, and the initial $\{X_0, X_1, \cdots, X_{p-1}\}$ are real-valued random variables independent of $\{\eta_n : n \geq 1\}$.

Write $Y_n := \begin{pmatrix} X_{n-p+1} \\ \vdots \\ X_n \end{pmatrix}$. Then (2.1) may be expressed as

$$Y_{n+1} = f(Y_n) + \varepsilon_{n+1}, \tag{2.2}$$

where $f : \mathbb{R}^p \to \mathbb{R}^p$ and ε_n are defined by $f(y) := \begin{pmatrix} y_2 \\ \vdots \\ y_p \\ h(y) \end{pmatrix}$,

$\varepsilon_n := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \eta_n \end{pmatrix}$. Here $y = (y_1, y_2, \cdots, y_p)' \in \mathbb{R}^p$.

We make the following assumptions:

(A_1) Assume that h is a continuous function which is bounded on compacts and has bounded first order derivatives outside some ball.

(A_2) Assume that η_n has a density $\varphi(\cdot)$ w.r.t. Lebesgue measure λ_1 on $\mathbb{R}^1$ such that φ is positive a.e. (λ_1).

Under (A_1) there exists $R > 0$ and a h_0 on $\mathbb{R}^p$ having bounded first order derivatives on $\mathbb{R}^p$ such that

$$h(x) = h_0(x) \quad \text{for} \quad |x| \geq R \tag{2.3}$$

Now let $\mathbf{D}f$ denote the Jacobian matrix of f:

$$Df(y) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ D_1h(y) & \cdots & \cdots & \cdots & \cdots & \cdots & D_ph(y) \end{bmatrix} \quad (|y| > R),$$

where $D_ih(y) := \frac{\partial h(y)}{\partial y_i}$.

Let f_0 denote the map obtained by replacing h by h_0 in f. Then the Jacobian $\mathbf{D}f_0$ is

$$Df_0(y) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ D_1h_0(y) & \cdots & \cdots & \cdots & \cdots & D_ph_0(y) \end{bmatrix}, \quad y \in \mathbb{R}^p.$$

Define the $(p \times p)$ matrix-valued function

$$\begin{aligned} A(y) \;&:=\; \int_0^1 \mathbf{D}f_0(sy)ds \\ &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \int_0^1 D_1h_0(sy)ds & \cdots & \cdots & \cdots & \cdots & \int_0^1 D_ph_0(sy)ds \end{bmatrix} \end{aligned}$$

Then one may express f_0 as

(2.4) $$f_0(y) = A(y)y \quad (y \in \mathbb{R}^p),$$

and (2.2), with f replaced by f_0, becomes

(2.5) $$Y_{n+1} = A(Y_n)Y_n + \varepsilon_{n+1} \quad (|Y_n| > R)$$

Let $p^{(m)}$ denote the m-step transition probability of $\{Y_n\}$ and let z' and $A'(z)$ the transposes of $(p \times 1)$ vector z and $(p \times p)$ matrix-valued function $A(z)$, respectively.

THEOREM 2.1. *Suppose* $E\eta_n = 0$, $0 \leq \sigma^2 := E\eta_n^2 < \infty$. *If* $(A_1), (A_2)$ *hold and there exists a symmetric, positive definite matrix* $B = ((b_{ij}))$ *such that*

$$\liminf_{|z|\to\infty} z'(B - A'(z)BA(z))z > b_{pp}\sigma^2, \tag{2.6}$$

then $\{Y_n\}$ *has a unique invariant probability* π *and* $||p^{(n)}(z, dy) - \pi(dy)|| \to 0$ *as* $n \to \infty$, *for every* $z \in \mathbb{R}^p$. *Here* $|z|$ *is an arbitrary norm on* $\mathbb{R}^p$.

Proof. Define $g(y) := y'By$. Then

$$\begin{aligned} Tg(z) &:= E[g(Y_{n+1})|Y_n = z] = E[Y'_{n+1}BY_{n+1}|Y_n = z] \\ &= E[z'A'(z)BA(z)z + 2\varepsilon'_{n+1}BA(z)z + \varepsilon'_{n+1}B\varepsilon_{n+1}] \\ &= z'A'(z)BA(z)z + b_{pp}\sigma^2 \qquad (|z| > R) \end{aligned}$$

Therefore, by (2.6), there exists $R' > R$ such that

$$Tg(z) - g(z) \equiv z'[A'(z)BA(z) - B]z + b_{pp}\sigma^2 < -\theta \quad (|z| > R')$$

for some $\theta > 0$.

Also, since f is bounded on $(|z| \leq R')$,

$$\begin{aligned} Tg(z) &= E[Y'_{n+1}BY_{n+1}|Y_n = z] \\ &= E[(f(z) + \varepsilon_1)'B(f(z) + \varepsilon_1)] \\ &= f'(z)Bf(z) + b_{pp}\sigma^2 \leq M < \infty \qquad (|z| \leq R'), \end{aligned}$$

where $M := b_{pp}\sigma^2 + \sup_{|z|\leq R'} f'(z)Bf(z)$.

Note that the set $(|z| \leq R')$ is a small set (see [1], Lemma 1).

Thus conditions (1.7) (i), (ii) of Theorem 1.1 are verified for the 1-step transition probability. Since the Markov process $\{Y_n : n \geq 0\}$ is aperiodic, it now follows that it is (Harris) ergodic. □

REMARK 2.1. A well known sufficient condition for the global asymptotic stability of the origin for the deterministic system

$$U_{n+1} = A(U_n)U_n$$

is the existence of a positive definite symmetric matrix B such that

$\Gamma(z) := A'(z)BA(z) - B$ is negative definite for all $z \in \mathbb{R}^p$. If one further assumes that, for the eigenvalue $\alpha(z)$ of $\Gamma(z)$ having the smallest magnitude, $|z|^2|\alpha(z)| \to \infty$ as $|z| \to \infty$ (or, $\overline{\lim}_{|z|\to\infty}|z|^2|\alpha(z)| > b_{pp}\sigma^2$), then (2.6) holds.

REMARK. For the verification of the criteria in Theorem 2.1, it suffices to require the knowledge of $A(z)$ only at 'infinity' except that h is bounded on compacts.

The following proposition is a basic result due to Lasalle in [3].

PROPOSITION 2.2. *Suppose $A(z)$ ($z \in \mathbb{R}^p$) is a $p \times p$ matrix-valued function on $\mathbb{R}^p$ such that $A(z)$ is constant and its spectral radius is less than 1. Then there exists a unique positive definite (symmetric) $(p \times p)$ matrix B such that $B - A'(z)BA(z)$ is equal to the $p \times p$ identity matrix I.*

Proof. See [3], Theorem 9.11, p.17. □

REMARK 2.3 As references for earlier work on pth-order piecewise or nonlinear autoregressive models ($p > 1$), see [1] and [4].

Now suppose $A(z)$ ($z \in \mathbb{R}^p$) is a $p \times p$ matrix-valued function on $\mathbb{R}^p$ such that its spectral radius $r(z)$ is bounded away from 1 at infinity (i.e., $\limsup_{|z|\to\infty} r(z) < 1$). Then we have the following lemma.

LEMMA 2.3. *Assume that $A(z) \to A$ in matrix norm as $|z| \to \infty$, where the spectral radius of a constant A is less than 1. Then there exists a (symmetric) positive definite matrix $\tilde{B}$ (constant, i.e., independent of z) such that $C(z) := \tilde{B} - A'(z)\tilde{B}A(z)$ is positive definite at infinity i.e., the smallest eigenvalue of $C(z)$ is bounded away from 0 near infinity.*

Proof. Let $\tilde{B}$ be the (symmetric) positive definite matrix B corresponding to A in proposition 2.2. Then, with $C(z) = B - A'(z)BA(z)$,

$$\begin{aligned} \inf_{|x|=1} x'C(z)x &= \inf_{|x|=1} [x'Ix + x'(C(z) - I)x] \\ &\geq \inf_{|x|=1} [x'Ix - ||C(z) - I||] \\ &> 0 \quad \text{on } (|z| > R) \end{aligned}$$

for some sufficiently large $R > 0$, where $||C||$ is the norm of $p \times p$ matrix C. □

The following theorem is a consequence of Theorem 2.1 and Lemma 2.3.

THEOREM 2. 4. *Suppose* $E\eta_n = 0, \quad 0 \leq \sigma^2 := E\eta_n^2 < \infty$. *Assume that* $(A_1), (A_2)$ *hold and that* $A(z) \to A$ *in matrix norm as* $|z| \to \infty$, *where the spectral radius of* A *is less than* 1. *Then the process* $\{Y_n\}$ *is (Harris) ergodic.*

REMARK 2.4. With a little additional work one can derive geometric ergodicity in Theorm 2.1 and also in Theorem 2.4 under some mild additional conditions. Details concerning this and some others will appear elsewhere.

3. Some examples

We assume that $\{Y_n\}$ satisfies (2.5).

EXAMPLE 3.1. Assume that h is bounded on compacts. Suppose that h is linear outside a bounded set: for some constants $R > 0$, b_i $(1 \leq i \leq p)$, $h(y) = \sum_1^p b_i y_i$ on the set $\{y \in \mathbb{R}^p : |y| > R\}$, where $y = (y_1, \cdots, y_p)'$, and that the spectral radius of the matrix A of the transformation $(y_1, \cdots, y_p) \to (y_2, \cdots, y_p, \sum_1^p b_i y_i)$ is less than 1. Then the process $\{Y_n\}$ is ergodic.

EXAMPLE 3.2. In particular, the spectral radius of the matrix A refered to in Example 3.1 is less than one if $\sum_{i=1}^p |b_i| < 1$. For, by Rouche's theorem, the function $f(z) = z^p$ and the characteristic polynomial $g(z) = -\sum_{i=1}^p b_i z^{i-1}$ of A have the same number of zero's inside the unit circle.

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