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A NOTE ON THE UPPER AND LOWER BOUNDS OF SMALL BALL PROBABILITIES FOR GAUSSIAN PROCESSES WITH STATIONARY INCREMENTS

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ABSTRACT. In this paper we obtain sharp upper and lower bounds of small ball probabilities for Gaussian processes with stationary increments, whose results are essential to estabilish Chung type laws of iterated logarithm.

1. Introduction

Recently, the upper and lower bounds of small ball probabilities for Gaussian processes have been studied in several situations by many authors: Shao [7], Kuelbs, Li and Shao [4], Shao and Wang [9], Monrad and Rootzén [6], Talagrand [12], Shao [8], Kuelbs and Li [3] and Li and Shao [5], etc.

Among the above recent results, Shao [7] proved the following fundamental theorem on the upper and lower bounds of small ball probabilities for a Gaussian process:

THEOREM A. *Let $\{X(t),\ 0 \le t \le 1\}$ be a real-valued Gaussian process on the probability space $(\Omega, \mathcal{S}, P)$ with mean zero, stationary increments and $X(0) = 0$. Put $\sigma^2(h) = E\{X(t+h) - X(t)\}^2$, $0 \le t \le t+h \le 1$, where $\sigma^2(h)$ is nondecreasing and concave on $[0,1]$. Then we*

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have

$$P\Big\{\sup_{0\le t\le 1}|X(t)|\le\sigma(x)\Big\}\le 2\exp\Big(-0.17/x\Big),$$

$$P\Big\{\sup_{0\le t\le 1}|X(t)|\le\sigma(x)+6e\int_0^\infty\sigma\big(xe^{-y^2}\big)\,dy\Big\}\ge\exp\Big(-2/x\Big)$$

for every $x\in(0,1)$.

It is well known that such a kind of small ball probabilities is the key step in establishing a Chung type law of the iterated logarithm. The main aim of this paper is to improve the above Theorem A to the wider class of small ball probabilities in various situations, and to obtain sharp upper and lower bounds of them.

2. Results

Let $\{X(t),\ 0\le t\le 1\}$ be an almost surely continuous Gaussian process on the probability space $(\Omega,\mathcal{S},P)$ with mean zero, stationary increments and $X(0)=0$. Put $\sigma^2(h)=E\{X(t+h)-X(t)\}^2$, $0\le t\le t+h\le 1$, where $\sigma^2(\cdot)$ is a nondecreasing function on $[0,1]$. Throughout this paper we always assume that $X(\cdot)$ and $\sigma(\cdot)$ are as in the above statements. First we shall consider upper bounds of small ball probabilities for the Gaussian process $X(\cdot)$. For proving our results, we need the following lemmas:

LEMMA 1. [7] *Assume that* $\sigma^2(h)$ *is concave on* $[0,1]$. *Then we have*

$$P\Big\{\max_{1\le i\le n}|X(t_i)|\le x\Big\}\le\prod_{i=1}^{n}\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{2}x/(\sigma(t_i-t_{i-1}))}e^{-t^2/2}\,dt$$

for every $0=t_0<t_1<t_2<\cdots<t_n\le 1$ *and for every* $x>0$.

LEMMA 2. [11] *Let* $\{\xi_i,\ i=1,2,\cdots,n\}$ *and* $\{\eta_i,\ i=1,2,\cdots,n\}$ *be sequences of jointly standardized normal random variables with* covariance$(\xi_i,\ \xi_j)\le$ covariance (η_i,η_j), $i\ne j$. *Then for any real numbers* $u_1,u_2,\cdots,u_n$,

$$P\{\xi_j\le u_j,\ j=1,2,\cdots,n\}\le P\{\eta_j\le u_j,\ j=1,2,\cdots,n\}.$$

THEOREM 1. *Assume that $\sigma^2(h)$ is concave on $[0,1]$. Then the following inequalities hold:*

$$\text{(i)} \qquad P\Big\{\sup_{0<t\le h}|X(t)| \le \sigma(x)\Big\} \le \exp\Big(-0.17\Big[\frac{h}{x}\Big]\Big),$$

$$\text{(i)}' \qquad P\Big\{\sup_{0<t\le h}|X(t)| > \sigma(x)\Big\} \ge 0.17\Big[\frac{h}{x}\Big]\exp\Big(-0.17\Big[\frac{h}{x}\Big]\Big)$$

for every $0 < x \le h$ with $h \le 1$.

(ii)

$$P\Big\{\sup_{0\le s\le 1-h}\sup_{0<t\le h}(X(t+s)-X(s)) \le \sigma(x)\Big\} \le \exp\Big(-0.17\ \Big[\frac{1-h}{x}\Big]\Big)$$

for every $0 < x \le \min\{h, 1-h\}$, *where* $[\cdot]$ *denotes the integer part.*

Proof. (i) For any $0 < x \le h$ with $h \le 1$, it follows from Lemma 1 that

$$\begin{aligned} P\Big\{\sup_{0<t\le h}|X(t)| \le \sigma(x)\Big\} &\le P\Big\{\max_{1\le i\le [h/x]}|X(ix)| \le \sigma(x)\Big\} \\ &\le \prod_{i=1}^{[h/x]}\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{2}\sigma(x)/\sigma(x)} e^{-t^2/2}\,dt = \big(2\Phi(\sqrt{2})-1\big)^{[h/x]} \\ &\le \exp\Big(-0.17\ \Big[\frac{h}{x}\Big]\Big), \end{aligned}$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-z^2/2}\,dz$. The inequality (i)$'$ immediately follows from the relation

$$1 - e^{-z} \ge ze^{-z}, \quad z \ge 0.$$

(ii) For any $0 < x \le \min\{h, 1-h\}$, put $U_i = X((i+1)x) - X(ix)$, $i = 1, \cdots, [(1-h)/x]$. It follows from the relation $ab = \big(a^2+b^2-(a-b)^2\big)/2$

that, for $l = |i-j| \geq 1$,

$$\begin{aligned} covariance(U_i, U_j) &= E(U_iU_j) \\ &= E\{X((i+1)x)X((j+1)x)\} - E\{X((i+1)x)X(jx)\} \\ &\qquad - E\{X(ix)X((j+1)x)\} + E\{X(ix)X(jx)\} \\ &= \frac{1}{2}\Big\{(\sigma^2((l+1)x) - \sigma^2(lx)) - (\sigma^2(lx) - \sigma^2((l-1)x))\Big\} \\ &\leq 0, \end{aligned} \tag{1}$$

because $\sigma^2(h)$ is concave. In order to apply Lemma 2, set $\xi_i = U_i/\sigma(x)$ in Lemma 2 and let η_i be independent standard normal random variables. From (1), $covariance(\xi_i, \xi_j) \leq 0 = covariance(\eta_i, \eta_j)$, $i \neq j$. Now applying Lemma 2, we have

$$\begin{aligned} P\Big\{&\sup_{0\leq s\leq 1-h}\sup_{0<t\leq h}(X(t+s)-X(s)) \leq \sigma(x)\Big\} \\ &\leq P\Big\{\max_{1\leq i\leq[(1-h)/x]} U_i \leq \sigma(x)\Big\} = P\Big\{\max_{1\leq i\leq[(1-h)/x]} \xi_i \leq 1\Big\} \\ &\leq P\Big\{\max_{1\leq i\leq[(1-h)/x]} \eta_i \leq 1\Big\} = P\Big\{\eta_1 \leq 1, \cdots, \eta_{[(1-h)/x]} \leq 1\Big\} \\ &= \Big(\Phi(1)\Big)^{[(1-h)/x]} \leq \exp\Big(-0.17\left[\frac{1-h}{x}\right]\Big). \end{aligned}$$

□

Let $\{X(t),\ 0 \leq t \leq 1\}$ be a fractional Brownian motion of order 2β with $0 < \beta < 1$ on the probability space $(\Omega, \mathcal{S}, P)$, that is, let $\{X(t),\ 0 \leq t \leq 1\}$ be an almost surely continuous, real-valued Gaussian process on $(\Omega, \mathcal{S}, P)$ with mean zero, $X(0) = 0$ and stationary increments $\sigma^2(h) = E\{X(t+h) - X(t)\}^2 = h^{2\beta}$, $0 \leq t \leq t+h \leq 1$, where $0 < \beta < 1$. Then the concave condition in Theorems A and 1 is satisfied only for $0 < \beta \leq 1/2$. But the condition (2) in the following Theorem 2 also contains the case that $\sigma^2(h)$ may not be concave. More precisely, for some $\beta < \alpha$ in (2) with $1/2 < \beta < 1$, the condition (2) is stisfied, and $\sigma^2(h) = h^{2\beta}(1/2 < \beta < 1)$ is convex on $[0, 1]$. It is obvious that (3) is also satisfied if $\sigma^2(h) = h^{2\beta}$ for $1/2 \leq \beta < 1$.

THEOREM 2. *Let $\sigma^2(x)/x^{2\alpha}$ be a quasi-decreasing function for some α $(0<\alpha<1)$: concretely, suppose that there exists a constant C_0 with $0<C_0<3/4^{\alpha}$ such that*

$$\frac{\sigma^2(x)}{x^{2\alpha}} \le C_0 \frac{\sigma^2(y)}{y^{2\alpha}} \qquad \text{for all} \quad x>y>0. \tag{2}$$

Assume also that

$$\begin{aligned} 6\sigma^2(mx) + \sigma^2((m+2)x) + \sigma^2((m-2)x) \\ \ge 4\sigma^2((m+1)x) + 4\sigma^2((m-1)x) \end{aligned} \tag{3}$$

for $x>0$ and $2 \le m \le (1/x)-2$. Then the following inequalities hold:

$$\text{(i) } P\Big\{\sup_{0\le s\le 1-h}\sup_{0<t\le h}(X(t+s)-X(s)) \le \sigma(x)\Big\} \le \exp\Big(-\Big[\frac{1-h}{2x}\Big]\Big\{1-\Phi(2/\sqrt{3-C_0 4^{\alpha}})\Big\}\Big)$$

for every $0<x\le \min\{h, 1-h\}$.

$$\text{(ii) } P\Big\{\sup_{0\le t\le 1-h}\sup_{0<s\le h}(X(t+s)-X(s)) \le \sigma(x)\Big\} \le \exp\Big(-\Big[\frac{h}{2x}\Big]\Big\{1-\Phi(2/\sqrt{3-C_0 4^{\alpha}})\Big\}\Big)$$

for every $0<x\le \min\{h, 1-h\}$.

Proof. The proof of (ii) is similar to that of (i). So we only prove (i): Put $\xi_i = X(ix) - X((i-1)x)$ for $i = 1, 2, \cdots, [(1-h)/x]$, and $\eta_j = \xi_{2j} - \xi_{2j-1}$ for $j = 1, 2, \cdots, [(1-h)/2x]$. Then we have, for $i \ne j$,

$$E\eta_i\eta_j = E\left(\xi_{2i}\xi_{2j} - \xi_{2i}\xi_{2j-1} - \xi_{2i-1}\xi_{2j} + \xi_{2i-1}\xi_{2j-1}\right).$$

Using the relation $ab = (a^2+b^2-(a-b)^2)/2$, we have

$$\begin{aligned} E\xi_{2i}\xi_{2j} &= E\Big\{X(2ix)X(2jx) - X(2ix)X((2j-1)x) \\ &\qquad - X((2i-1)x)X(2jx) + X((2i-1)x)X((2j-1)x)\Big\} \\ &= \frac{1}{2}\Big\{-\sigma^2(2|i-j|x) + \sigma^2((2|i-j|+1)x) \\ &\qquad + \sigma^2(|2(i-j)-1|x) - \sigma^2(2|i-j|x)\Big\}, \end{aligned}$$

$$
\begin{aligned}
E\xi_{2i}\xi_{2j-1} &= \frac{1}{2}\Big\{-\sigma^2((2|i-j|+1)x) + \sigma^2(2|i-j+1|x) \\
&\qquad + \sigma^2(2|i-j|x) - \sigma^2((2|i-j|+1)x)\Big\}, \\
E\xi_{2i-1}\xi_{2j} &= \frac{1}{2}\Big\{-\sigma^2((2|i-j|-1)x) + \sigma^2(2|i-j|x) \\
&\qquad + \sigma^2(2|i-j-1|x) - \sigma^2((2|i-j|-1)x)\Big\}, \\
E\xi_{2i-1}\xi_{2j-1} &= \frac{1}{2}\Big\{-\sigma^2(2|i-j|x) + \sigma^2((2|i-j|+1)x) \\
&\qquad + \sigma^2((2|i-j|-1)x) - \sigma^2(2|i-j|x)\Big\}.
\end{aligned}
$$

Set $m = 2|i-j| \geq 2$. It follows from the condition (3) that

$$
\begin{aligned}
E\eta_i\eta_j &= -\frac{1}{2}\Big\{6\sigma^2(mx) + \sigma^2((m+2)x) + \sigma^2((m-2)x) \\
&\qquad - \Big(4\sigma^2((m+1)x) + 4\sigma^2((m-1)x)\Big)\Big\} \\
&\leq 0, \qquad 2 \leq m \leq (1/x) - 2.
\end{aligned}
$$

Using the condition (2), we have

$$
\begin{aligned}
E\eta_j^2 &= E\xi_{2j}^2 - 2E(\xi_{2j}\xi_{2j-1}) + E\xi_{2j-1}^2 \\
&= 2\sigma^2(x) - 2E\big\{X(2jx)X((2j-1)x) - X(2jx)X((2j-2)x) \\
&\qquad - X((2j-1)x)X((2j-1)x) + X((2j-1)x)X((2j-2)x)\big\} \\
&= 4\sigma^2(x) - \sigma^2(2x) - \sigma^2(0) \\
&\geq \big(3 - C_0 4^\alpha\big)\sigma^2(x), \quad 0 < C_0 < 3/4^\alpha,
\end{aligned}
$$

and

$$
E\eta_j^2 \leq 4\sigma^2(x) \quad \text{for } j = 1, 2, \cdots, [(1-h)/2x].
$$

It follows from Lemma 2 that

$$\begin{aligned}
P\Big\{\sup_{0\le s\le 1-h}\ \sup_{0<t\le h}(X(t+s)-X(s))\le\sigma(x)\Big\}
&\le P\Big\{\max_{1\le i\le[(1-h)/x]}(X(ix)-X((i-1)x))\le\sigma(x)\Big\}\\
&\le P\Big\{\max_{1\le j\le[(1-h)/2x]}(\xi_{2j}-\xi_{2j-1})\le 2\sigma(x)\Big\}\\
&\le P\Big\{\max_{1\le j\le[(1-h)/2x]}\frac{\eta_j}{\sqrt{E\eta_j{}^2}}\le\frac{2}{\sqrt{3-C_0 4^{\alpha}}}\Big\}\\
&\le\Big\{\Phi(2/\sqrt{3-C_0 4^{\alpha}})\Big\}^{[(1-h)/2x]}\\
&\le\exp\Big(-\Big[\frac{1-h}{2x}\Big]\Big\{1-\Phi(2/\sqrt{3-C_0 4^{\alpha}})\Big\}\Big). \qquad\square
\end{aligned}$$

From now on we study lower bounds of small ball probabilities. For this purpose, we start with three lemmas that will be needed for the proof of Theorem 3:

LEMMA 3. ([2], [10]) *Let T be a parameter set and $\{Y(t),\ t\in T\}$ be a Gaussian process with mean zero and finite variances. Then*

$$P\Big\{\sup_{t\in A}\frac{|Y(t)|}{x(t)}\le 1, |Y(t_0)|\le x(t_0)\Big\}\ \ge P\big\{|Y(t_0)|\le x(t_0)\big\}P\Big\{\sup_{t\in A}\frac{|Y(t)|}{x(t)}\le 1\Big\}$$

for every $A\subset T$, $x(t)>0$, $t_0\in T$ and $x(t_0)>0$.

LEMMA 4. [7] *Let $\{\xi_i,\ 1\le i\le n\}$ be jointly normal random variables with mean zero and finite variances. Then, for every $x>0$,*

$$P\Big\{\max_{1\le i\le n}\Big|\sum_{j=1}^{i}\xi_j\Big|\le x\Big\}\ge\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\int_0^{2x/\rho_i}e^{-y^2/2}\,dy,$$

where $\rho_i^2=\sum_{j=1}^{n}|E\xi_i\xi_j|$.

The following lemma is easily obtained by a slight modification of the proof of Lemma 2.3 in Shao [7]:

LEMMA 5. *Let $\{Y(t),\ t \geq 0\}$ be a Gaussian process with mean zero and finite variance. Assume that there exists a nondecreasing function $u(h)$ on $[0,1]$ such that*

$$E\{Y(t+h)-Y(t)\}^2 \leq u^2(h) \quad \text{for all } \ 0 \leq t \leq t+h \leq 1.$$

Then we have

$$P\Big\{\sup_{0\leq t\leq h} |Y(t)| \leq x + 2e\int_0^\infty u\Big(\frac{ee^{-y^2}}{R}\Big)\,dy\Big\} \geq e^{-R}P\Big\{\max_{0\leq i\leq [Rh]}\Big|Y\Big(\frac{i}{R}\Big)\Big| \leq x\Big\}$$

for every $R \geq 1$ and $x > 0$.

THEOREM 3. *For $h > 0$, let $\sigma(h)$ be nondecreasing on $[0,1]$. Then we have*

$$P\Big\{\sup_{0\leq t\leq h}|X(t)| \leq \sigma(x) + 6e\int_0^\infty \sigma(xe^{-y^2})\,dy\Big\} \geq \exp\Big(-\frac{1}{x}\Big)\Big\{\Phi\Big(2\sqrt{\frac{x}{h}}\Big) - \frac{1}{2}\Big\}^{h/x} \tag{4}$$

for every $0 < x \leq h$ with $h \leq 1$. If, in addition, $\sigma^2(h)$ is concave, then the lower bound of (4) is equal to or greater than $\exp(-(1+0.87h)/x)$.

Proof. Take $R = 1/x$ in Lemma 5. Then we have

$$P\Big\{\sup_{0\leq t\leq h}|X(t)| \leq \sigma(x) + 2e\int_0^\infty \sigma(xe\,e^{-y^2})\,dy\Big\} \geq e^{-1/x}P\Big\{\max_{1\leq i\leq [h/x]}|X(ix)| \leq \sigma(x)\Big\}. \tag{5}$$

Applying lemma 4, we get

$$P\Big\{\max_{1\leq i\leq [h/x]}|X(ix)| \leq \sigma(x)\Big\} \geq \prod_{i=1}^{[h/x]} \frac{1}{\sqrt{2\pi}}\int_0^{2\sigma(x)/\rho_i} e^{-y^2/2}\,dy\,,$$

where

$$\rho_i^2 = \sum_{j=1}^{[h/x]} |E(X(ix)-X((i-1)x))(X(jx)-X((j-1)x))|, \quad 1 \le i \le \left[\frac{h}{x}\right]$$

Using the relation $ab \le (a^2+b^2)/2$, we have

$$\begin{aligned}\rho_i^2 &\le \frac{1}{2}\sum_{j=1}^{[h/x]} \left(E\{X(ix)-X((i-1)x)\}^2 + E\{X(jx)-X((j-1)x)\}^2\right) \\ &= \left[\frac{h}{x}\right]\sigma^2(x).\end{aligned}$$

Hence

$$\begin{aligned} & P\Big\{ \max_{1\le i\le [h/x]} |X(ix)| \le \sigma(x)\Big\} \\ (6) \qquad & \ge \prod_{i=1}^{[h/x]} \frac{1}{\sqrt{2\pi}} \int_0^{2\sigma(x)/\sqrt{[h/x]}\sigma(x)} e^{-y^2/2}\,dy \\ & = \Big\{\Phi\big(2\sqrt{[x/h]}\big) - \frac{1}{2}\Big\}^{[h/x]} \ge \Big\{\Phi\Big(2\sqrt{\frac{x}{h}}\,\Big) - \frac{1}{2}\Big\}^{h/x}. \end{aligned}$$

A combination of the above inequalities (5) and (6) yields

$$(7) \qquad \begin{aligned} & P\Big\{\sup_{0\le t\le h} |X(t)| \le \sigma(x) + 2e\int_0^\infty \sigma(xe\cdot e^{-y^2})\,dy\Big\} \\ & \qquad \ge \exp\Big(-\frac{1}{x}\Big)\Big\{\Phi\Big(2\sqrt{\frac{x}{h}}\,\Big) - \frac{1}{2}\Big\}^{h/x}. \end{aligned}$$

Noting that, for every $0 \le h \le 1/3$,

$$\begin{aligned}\sigma(3h) &= \{EX^2(3h)\}^{1/2} \\ &= \{EX^2(h)\}^{1/2} + \{E(X(2h)-X(h))^2\}^{1/2} \\ &\qquad + \{E(X(3h)-X(2h))^2\}^{1/2} \\ &= 3\sigma(h)\end{aligned}$$

by the Minkowski inequality, we have

$$\sigma(x \cdot e \cdot e^{-y^2}) \le \sigma(3xe^{-y^2}) \le 3\sigma(xe^{-y^2}) \qquad \text{for } y \ge 0. \tag{8}$$

Now the inequality (4) immediately follows from (7) and (8). If $\sigma^2(h)$ is concave, then the result is straightforward from [7]. □

From Theorem 3, we obtain the following corollary:

COROLLARY. *Assume that $\sigma(x)/x^\alpha$ is nondecreasing on $(0,1)$ for some $\alpha > 0$. Then the inequality*

$$P\Big\{\sup_{0\le t\le h} |X(t)| \le C_\alpha \sigma(x)\Big\} \ge \exp\Big(-\frac{1}{x}\Big)\Big\{\Phi\Big(2\sqrt{\frac{x}{h}}\,\Big) - \frac{1}{2}\Big\}^{h/x}$$

holds for every $0 < x \le h$ with $h \le 1$, where $C_\alpha = 1 + 3e\sqrt{\pi/\alpha}$.

Proof. If $\sigma(x)/x^\alpha$ is nondecreasing on $(0,1)$ for some $\alpha > 0$, then

$$\int_0^\infty \sigma(xe^{-y^2})\,dy \le \sigma(x)\int_0^\infty e^{-\alpha y^2}\,dy = \frac{\sigma(x)}{2}\sqrt{\frac{\pi}{\alpha}}\,.$$

Hence, it follows from (4) that

$$\begin{aligned} &P\Big\{\sup_{0\le t\le h} |X(t)| \le C_\alpha \sigma(x)\Big\} \\ &\qquad = P\Big\{\sup_{0\le t\le h} |X(t)| \le \sigma(x) + 6e\frac{\sigma(x)}{2}\sqrt{\frac{\pi}{\alpha}}\Big\} \\ &\qquad \ge P\Big\{\sup_{0\le t\le h} |X(t)| \le \sigma(x) + +6e\int_0^\infty \sigma(xe^{-y^2})\,dy\Big\} \\ &\qquad \ge \exp\Big(-\frac{1}{x}\Big)\Big\{\Phi\Big(2\sqrt{\frac{x}{h}}\,\Big) - \frac{1}{2}\Big\}^{h/x} \end{aligned}$$

for every $0 < x \le h$ with $h \le 1$. □

Next we shall estimate upper bounds of another type of large deviation probabilities, whose results are also essential to obtain the results related to the iterated logarithm.

Let $\mathbb{D} = \{\mathbf{t} : \mathbf{t} = (t_1, \cdots, t_N),\ a_i \le t_i \le b_i,\ i = 1, 2, \cdots, N\}$ be a real N-dimensional parameter space. We assume that the space $\mathbb{D}$ has the usual Euclidean norm $\|\cdot\|$ such that

$$\|\mathbf{t} - \mathbf{s}\|^2 = \sum_{i=1}^{N} (t_i - s_i)_2.$$

Let $\{X(\mathbf{t}),\ \mathbf{t} \in \mathbb{D}\}$ be a real-valued separable Gaussian process with $EX(\mathbf{t}) = 0$. Suppose that

$$0 < \sup_{\mathbf{t}\in\mathbf{D}} EX(\mathbf{t})^2 =: \Gamma^2 < \infty, \quad \Gamma > 0,$$

and

$$E\{X(\mathbf{t}) - X(\mathbf{s})\}^2 \le \varphi^2(\|\mathbf{t} - \mathbf{s}\|),$$

where $\varphi(\cdot)$ is a nondecreasing continuous function such that

$$\int_0^\infty \varphi(e^{-y^2})\, dy < \infty. \tag{9}$$

LEMMA 6. [1] *Let $\{X(\mathbf{t}),\ \mathbf{t} \in \mathbb{D}\}$ be given as in the above statements. Then, for $\lambda > 0, z \ge 1$ and $\mathcal{A} > 2\sqrt{N \ln 2}$, we have*

$$P\Big\{\sup_{\mathbf{t}\in\mathbf{D}} X(\mathbf{t}) \ge z\{\Gamma + (2\sqrt{2} + 2)\mathcal{A} \int_0^\infty \varphi(\sqrt{N}\lambda 2^{-y^2})\, dy\}\Big\}$$
$$\le (4^N + \psi) \prod_{i=1}^{N} \Big(\frac{b_i - a_i}{\lambda} \vee 1\Big) e^{-z^2/2},$$

where $m \vee n = \max\{m, n\}$, and

$$\psi = \sum_{n=1}^{\infty} \exp\Big\{\frac{1}{2} - 2^n\Big(\frac{\mathcal{A}^2}{2} - 2N \ln 2\Big)\Big\} < \infty.$$

THEOREM 4. *Let $0 < \alpha < 1$. For a fixed $K > 0$, let $\sigma(Kx) \le K^\alpha \sigma(x)$ for every $x > 0$. Then for any small $\epsilon > 0$ there exists a positive constant $C = C(\epsilon)$ depending only on ϵ such that, for every $x > 0$,*

$$\text{(i)} \quad P\Big\{\sup_{0<t\le h} |X(t)| > \sigma(x)\Big\} \le C \exp\Big(-\Big(\frac{1}{2+\epsilon}\Big)\frac{\sigma^2(x)}{\sigma^2(h)}\Big),$$

$$\text{(ii)} \quad P\Big\{\sup_{0\le s\le 1-h}\ \sup_{0<t\le h} |X(t+s) - X(s)| > \sigma(x)\Big\}$$

$$\le C\Big(\frac{1-h}{h} \vee 1\Big) \exp\Big(-\Big(\frac{1}{2+\epsilon}\Big)\frac{\sigma^2(x)}{\sigma^2(h)}\Big).$$

Proof. We prove only (ii). The proof of (i) follows the same method as that of (ii). Let $\mathbb{D} = \{(t,s) : 0 < t \le h, 0 \le s \le 1-h\}$ be a two-dimensional space. Set

$$Y(t,s) = \frac{X(t+s) - X(s)}{\sigma(h)}, \quad (t,s) \in \mathbb{D},$$

and

$$\varphi(z) = \frac{2\sigma(\sqrt{2}z)}{\sigma(h)}, \quad z > 0,$$

where $\varphi(\cdot)$ is a nondecreasing continuous function satisfying (9). It is clear that

$$EY(t,s) = 0, \quad \Gamma = \sup_{(t,s)\in\mathbb{D}} EY(t,s)^2 = 1 \quad \text{and}$$

$$\begin{aligned} E\{Y(t_1,s_1) - Y(t_2,s_2)\}^2 &\le 2\frac{\sigma^2(|t_1-t_2| + |s_1-s_2|) + \sigma^2(|s_1-s_2|)}{\sigma^2(h)} \\ &\le \frac{4}{\sigma^2(h)}\sigma^2\big(\sqrt{2}\sqrt{(t_1-t_2)^2 + (s_1-s_2)^2}\,\big) \\ &= \varphi^2\big(\sqrt{(t_1-t_2)^2 + (s_1-s_2)^2}\,\big). \end{aligned}$$

For any small $\epsilon > 0$, there exists a small $c = c(\epsilon) > 0$ such that

$$(10) \qquad (2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi\big(\sqrt{2}\, ch\, 2^{-y^2}\big)\, dy \le \frac{\epsilon}{8},$$

where $\mathcal{A} > 2\sqrt{2\ln 2}$. Indeed, it follows from the assumption that for any small $\epsilon > 0$ there exists a small $c = c(\epsilon) > 0$ such that

$$\int_0^\infty \varphi(\sqrt{2}\; ch\; 2^{-y^2})\, dy = \int_0^\infty \frac{2\sigma(2ch2^{-y^2})}{\sigma(h)}\, dy \le \int_0^\infty 2(2c2^{-y^2})^\alpha\, dy$$
$$= (2c)^\alpha \frac{\sqrt{\pi}}{\sqrt{\alpha \ln 2}} \le \frac{\epsilon}{8}\,.$$

Let $\sigma(x)/\sigma(h) = z(1 + (\epsilon/8)),\ z \ge 1$. From (10) and Lemma 6, we have

$$P\Big\{\sup_{(t,s)\in\mathbb{D}} |Y(t,s)| > \frac{\sigma(x)}{\sigma(h)}\Big\}$$
$$\le 2P\Big\{\sup_{(t,s)\in\mathbb{D}} Y(t,s) > z\{1 + (2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi(\sqrt{2}\; ch\; 2^{-y^2})\, dy\}\Big\}$$
$$\le K\Big(\frac{h}{ch} \vee 1\Big)\Big(\frac{1-h}{ch} \vee 1\Big)\, e^{-z^2/2}$$
$$\le C\Big(\frac{1-h}{h} \vee 1\Big) \exp\Big(-\Big(\frac{1}{2+\epsilon}\Big)\frac{\sigma^2(x)}{\sigma^2(h)}\Big)\,,$$

where K is a constant and $C = C(\epsilon)$. In the case of $x \le h$, the result (ii) is obvious, since the right hand side of (ii) is larger than one for C big enough. □

3. An application

In this section we shall establish some results related to the iterated logarithm from the theorems in section 2.

Let $\{X(t),\ 0 \le t \le 1\}$ be a fractional Brownian motion of order 2β with $0 < \beta < 1$, i.e., $\{X(t),\ 0 \le t \le 1\}$ is a real-valued Gaussian process with mean zero, stationary increments, $X(0) = 0$ and $\sigma^2(t) := EX^2(t) = t^{2\beta}$ for some $0 < \beta < 1$. When $\beta = 1/2$, $\{X(t),\ 0 \le t \le 1\}$ is a standard Wiener process.

THEOREM 5. *Let $\{X(t),\ 0 \le t \le 1\}$ be a fractional Brownian motion of order 2β with $0 < \beta < 1$. Then we have*

$$\text{(i)} \qquad \liminf_{h\to 0} \sup_{0\le s\le 1-h} \sup_{0<t\le h} \frac{X(t+s)-X(s)}{\sigma(0.16h/\log|\log h|)} \ge 1 \qquad \text{a.s.}$$

for some $0 < \beta \le 1/2$, and

$$\text{(ii)} \qquad \liminf_{h\to 0} \sup_{0\le s\le 1-h} \sup_{0<t\le h} \frac{X(t+s)-X(s)}{\sigma\big(Ah/2\log|\log h|\big)} \ge 1 \qquad \text{a.s.}$$

for some $1/2 < \beta < 1$, where $A = 1-\Phi(2/\sqrt{3-C_0 4^\beta})$ and $0 < C_0 < 3/4^\beta$.

Proof. (i) From Theorem 1 (ii), we have, for any $0 < \epsilon < 1$,

$$P\Big\{\sup_{0\le s\le 1-h} \sup_{0<t\le h} \big(X(t+s)-X(s)\big) \le \sigma(x)(1-\epsilon)\Big\} \le 2\exp\big(-0.17h/x\big), \qquad 0 < x \le h,$$

where $h > 0$ is small enough. Choose

$$x = \frac{0.16h}{\log|\log h|} \quad \text{and} \quad h = e^{-n}, \quad n \in \mathbb{N},$$

where $\mathbb{N}$ is a set of positive integers. Then we have

$$\sum_n P\Big\{\sup_{0\le s\le 1-e^{-n}} \sup_{0<t\le e^{-n}} \big(X(t+s)-X(s)\big) \le \sigma\Big(\frac{0.16e^{-n}}{\log n}\Big)(1-\epsilon)\Big\} \le 2\sum_n \exp\Big(-\frac{0.17}{0.16}\log n\Big) < \infty.$$

So, the Borel-Cantelli lemma implies that

$$\liminf_{n\to\infty} \sup_{0\le s\le 1-e^{-n}} \sup_{0<t\le e^{-n}} \frac{X(t+s)-X(s)}{\sigma\big(0.16e^{-n}/\log n\big)} \ge 1-\epsilon \qquad \text{a.s.}$$

This gives the result (i).

(ii) In Theorem 2, let $\sigma^2(t) = t^{2\beta}$ for some β with $1/2 < \beta < \alpha < 1$. Then the conditions (2) and (3) of Theorem 2 are satisfied. It follows from Theorem 2 (i) that, for any $0 < \epsilon < 1$,

$$P\Big\{\sup_{0\le s\le 1-h}\sup_{0<t\le h}\big(X(t+s)-X(s)\big) \le \sigma(x)(1-\epsilon)\Big\}$$
$$\le C\exp\Big(-\frac{h}{2x}\Big\{1-\Phi(2/\sqrt{3-C_0 4^{\beta}})\Big\}\Big),$$

where C is a constant and $h > 0$ is small enough. Choose

$$x = \frac{\{1-\Phi(2/\sqrt{3-C_0 4^{\alpha}})\}h}{2\log|\log h|} \quad \text{and} \quad h = e^{-n}, \quad n \in \mathbb{N}.$$

Then we have

$$\sum_n P\Big\{\sup_{0\le s\le 1-e^{-n}}\sup_{0<t\le e^{-n}}(X(t+s)-X(s))$$
$$\le \sigma\Big(\frac{\{1-\Phi(2/\sqrt{3-C_0 4^{\alpha}})\}e^{-n}}{2\log n}\Big)(1-\epsilon)\Big\}$$
$$\le C\sum_n \exp\Big(-\frac{1-\Phi(2/\sqrt{3-C_0 4^{\beta}})}{1-\Phi(2/\sqrt{3-C_0 4^{\alpha}})}\log n\Big) < \infty.$$

So, the result (ii) follows from the Borel-Cantelli lemma. □

THOEREM 6. *Let $\{X(t),\ 0 \le t \le 1\}$ be a fractional Brownian motion of order 2β with $0 < \beta < 1$. Then we have*

$$\text{(i)} \qquad \limsup_{h\to 0}\sup_{0<t\le h}\frac{|X(t)|}{\sigma\big(h(2\log|\log h|)^{1/(2\beta)}\big)} \le 1 \qquad \text{a.s.,}$$

$$\text{(ii)} \ \limsup_{h\to 0}\sup_{0\le s\le 1-h}\sup_{0<t\le h}\frac{|X(t+s)-X(s)|}{\sigma\big(h\{2(\log(1/h)+\log|\log h|)\}^{1/(2\beta)}\big)} \le 1 \quad \text{a.s.}$$

Proof. (i) From Theorem 4 (i), it follows that for any small $\epsilon > 0$ there exists a positive constant $C = C(\epsilon)$ such that, for every $x > 0$,

$$P\Big\{\sup_{0<t\le h}|X(t)| > \sigma(x)\Big\} \le C\exp\Big(-\frac{1}{2+\epsilon}\Big(\frac{x}{h}\Big)^{2\beta}\Big).$$

Set $x = h(2(1+\epsilon)\log|\log h|)^{1/(2\beta)}$ and $h = e^{-n}$, $n \in \mathbb{N}$. Then we have

$$\sum_n P\Big\{\sup_{0<t\le e^{-n}}\big|X(t)\big| > \sigma\big(e^{-n}(2(1+\epsilon)\log n)^{1/(2\beta)}\big)\Big\}$$
$$\le C\sum_n \exp\Big(-\frac{1}{2+\epsilon}((2+2\epsilon)\log n)\Big) < \infty.$$

By the Borel-Cantelli lemma, we have

$$\limsup_{n\to\infty}\sup_{0<t\le e^{-n}}\frac{|X(t)|}{\sigma\big(e^{-n}(2\log n)^{1/(2\beta)}\big)} \le 1 \qquad \text{a.s.}$$

This yields (i).

The proof of (ii) is similar to that of (i): From Theorem 4 (ii), it follows that for any small $\epsilon > 0$ there exists a positive constant $C = C(\epsilon)$ such that, for every $x > 0$,

$$P\Big\{\sup_{0\le s\le 1-h}\sup_{0<t\le h}\big|X(t+s) - X(s)\big| > \sigma(x)\Big\}$$
$$\le C\frac{1}{h}\exp\Big(-\frac{1}{2+\epsilon}\Big(\frac{x}{h}\Big)^{2\beta}\Big).$$

Set $x = h\,\{2(1+\epsilon)(\log(1/h) + \log|\log h|)\}^{1/(2\beta)}$ and $h = e^{-n}$, $n \in \mathbb{N}$. Then we have

$$\sum_n P\Big\{\sup_{0\le s\le 1-e^{-n}}\sup_{0<t\le e^{-n}}\big|X(t+s) - X(s)\big|$$
$$> \sigma\big(e^{-n}\{2(1+\epsilon)(n+\log n)\}^{1/(2\beta)}\big)\Big\}$$
$$\le C\sum_n e^n\exp\Big(-\frac{1}{2+\epsilon}\{(2+2\epsilon)(n+\log n)\}\Big) < \infty.$$

So, the result (ii) immediately follows from the Borel-Cantelli lemma.□

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