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BCK-ALGEBRAS OF EXTENDED POGROUPOID

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ABSTRACT. In this paper we construct a BCK-algebra $(X^\star; *, w)$ from the extended pogroupoid $(X^\star, \cdot)$, and obtain a necessary and sufficient condition for the algebraic system $(X^\star; *, \cdot, w)$ to have a property $(x \cdot y) * z = (x * z) \cdot (y * z)$ for all $x, y, z \in X^\star$.

1. Introduction

BCK-algebras and BCI-algebras were introduced by K. Iséki and Y. Imai in 1966 ([1, 2, 3, 5]), and then many authors have investigated various properties of these algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. On the while, J. Neggers ([6]) introduced the notion of pogroupoid, and J. Neggers and H. S. Kim ([8]) obtained a necessary and sufficient condition that a pogroupoid is to be a semigroup. Moreover, in [9] they investigated a class of algebras whose bases over field K are pogroupoid, and discussed several properties of these algebras as they relate to the structure of their associated pogroupoids and through these to the associated posets also. Given a pogroupoid $(X, \cdot)$ we define an extended pogroupoid $(X^\star, \cdot)$ by adding $w \notin X$ with the condition $w \cdot a = w = a \cdot w$ for any $a \in X \cup \{w\}$. In this paper we construct a BCK-algebra $(X^\star; *, w)$ from the extended pogroupoid $(X^\star, \cdot)$, and obtain a necessary and sufficient condition for the algebraic system $(X^\star; *, \cdot, w)$ to have a property $(x \cdot y) * z = (x * z) \cdot (y * z)$ for all $x, y, z \in X^\star$.

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2. Extended pogroupoid

A groupoid $(X,\cdot)$ is called a *pogroupoid* if (i) $x \cdot y \in \{x, y\}$ (ii) $x \cdot (y \cdot x) = y \cdot x$ (iii) $(x \cdot y) \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in X$ ([6]). J. Neggers [6] defined an associated partially ordered set $(X, \leq)$ by $x \leq y$ iff $y \cdot x = y$. On the one hand, for a given poset $(X, \leq)$ he also defined a binary operation on X by $y \cdot x = y$ if $x \leq y$, $y \cdot x = x$ otherwise, and proved that $(X,\cdot)$ is a pogroupoid. Let $(X,\cdot)$ be a pogroupoid and let $w \notin X$. Define $w \cdot a = a \cdot w = w$ for all $a \in X^\star := X \cup \{w\}$. Then $(X^\star,\cdot)$ is a pogroupoid, called the *extended pogroupoid* of $(X,\cdot)$. Define a partial order $\leq$ on $X^\star$ by $x \leq y$ iff $y \cdot x = y$. Then $(X^\star, \leq)$ is a poset, called the *associated poset with respect to* $(X^\star,\cdot)$.

PROPOSITION 2.1. *Let $(X,\cdot)$ be a pogroupoid and let $(X^\star := X \cup \{w\},\cdot)$ be the extended pogroupoid of X. Then w is the greatest element of the associated poset $(X^\star, \leq)$.*

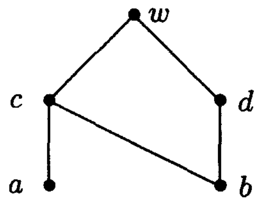
Proof. Since $w \cdot a = w$ for any $a \in X^\star$, we have $a \leq w$. This means that w is the greatest element of $X^\star$. □

EXAMPLE 2.2. Let $X = \{a, b, c, d\}$ be a pogroupoid with the left table below:

$\cdot$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	c	c	d
d	a	d	c	d

$\cdot$	a	b	c	d	w
a	a	b	c	d	w
b	a	b	c	d	w
c	c	c	c	d	w
d	a	d	c	d	w
w	w	w	w	w	w

Then its extended pogroupoid $(X^\star,\cdot)$ is described as in the right table above. The associated poset $(X^\star, \leq)$ has w as the greatest element, and can be represented as follows:



3. BCK-algebra of $(X^{\star}, \cdot)$

Let X be a set with a binary operation '$*$' and a constant 0. Then $(X; *, 0)$ is called a *BCK-algebra* if it satisfies the following conditions: (I) $((x*y)*(x*z))*(z*y) = 0$, (II) $(x*(x*y))*y = 0$, (III) $x*x = 0$, (IV) $x*y = 0$ and $y*x = 0$ imply $x = y$, (V) $0*x = 0$, for all $x, y, z \in X$. We construct a BCK-algebra $(X^{\star}; *, w)$ from the extended pogroupoid $(X^{\star}, \cdot)$ motivated from S. Tanaka ([10]).

THEOREM 3.1. *Let $(X^{\star} := X \cup \{w\}, \cdot)$ be the extended pogroupoid of a pogroupoid $(X, \cdot)$. Define*

$$x * y := \begin{cases} w & \text{if } x \cdot y = x, \\ x & \text{otherwise.} \end{cases}$$

Then $(X^{\star}; *, w)$ is a BCK-algebra.

Proof. (III). For any $x \in X^{\star}, x \cdot x = x$ implies $x * x = w$. (IV). If $x * y = w$ and $y * x = w$, then $x \cdot y = x$ and $y \cdot x = y$, and hence $x = x \cdot y = x \cdot (y \cdot x) = y \cdot x = y$. (V). Since $w \cdot x = w$ for any $x \in X^{\star}, w * x = w$. Note that $x * w = x$, since $x \cdot w = w$ for any $x \in X^{\star}$. (I). We have two cases: (a) $x \cdot y = x$, (b) $x \cdot y = y$. If $x \cdot y = x$, then $x * y = w$. Hence by applying (V) we have $((x*y)*(x*z))*(z*y) = (w*(x*z))*(z*y) = w$. If $x \cdot y = y$, then $x * y = x$. We consider two subcases: (b_1) $x \cdot z = z$, (b_2) $x \cdot z = x$. If $x \cdot z = z$, then $x * z = x$, and hence $((x*y)*(x*z))*(z*y) = (x*x)*(z*y) = w*(x*y) = w$. If $x \cdot z = x$, then $x * z = w$. We claim that $z \cdot y = y$. Assume that $z \cdot y = z$. Then $x \cdot y = (x \cdot z) \cdot y = (x \cdot z) \cdot (z \cdot y) = x \cdot z = x$, a contradiction. Thus we have $z*y = z$. Therefore $((x*y)*(x*z))*(z*y) = (x*w)*z = x*z = w$, which proves (I). Finally we consider (II). If $x \cdot y = x$, then $x * y = w$ and $(x*(x*y))*y = (x*w)*y = x*y = w$. If $x \cdot y = y$, then $x * y = x$

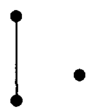
and $(x * (x * y)) * y = (x * x) * y = w * y = w$. This proves (II). Hence $(X^\star; *, w)$ is a BCK-algebra. □

In Theorem 3.1 above we say $(X^\star; *, w)$ the *BCK-algebra associated with the extended pogroupoid* $(X^\star, \cdot)$. H. Yutani [11] proved that a BCK-algebra $(X; *, 0)$ having a binary operation "$\cdot$" on X with $(x * y) * z = x * (y \cdot z)$, $\forall x, y, z \in X$ is with condition (S). With this notion J. Meng [4] proved that implicative commutative semigroups are equivalent to BCK-algebras with condition (S). We consider some relations between the BCK-operation "$*$" and the pogroupoid operation "$\cdot$".

PROPOSITION 3.2. *If $(X^\star; *, w)$ is a BCK-algebra associated with the extended pogroupoid $(X^\star, \cdot)$, then $(y \cdot x) * x = w$, $\forall x, y \in X^\star$.*

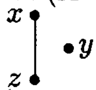
Proof. If $y \cdot x = y$, then $y * x = w$ and $(y \cdot x) * x = y * x = w$. If $y \cdot x = x$, then $(y \cdot x) * x = x * x = w$. This proves the proposition. □

Let $(X, \leq)$ be a poset and $S \subseteq X$. A poset whose underlying set is S and whose poset structure is that inherited from $(X, \leq)$ is called a *full subposet* of X ([7]). Given a poset $(X, \leq)$ it is *Q-free* if there is no full subposet $(P, \leq)$ of $(X, \leq)$ which is order isomorphic to the poset $(Q, \leq)$. If C_n denotes a chain of length n and if $\underline{n}$ denotes an antichain of cardinal number n , while $+$ denotes the disjoint union of posets, then the poset $(C_2 + \underline{1})$ (or $C_2 + C_1$) has Hasse diagram



and may be represented as $\{p \leq q, p \circ r, q \circ r\}$, where $a \circ b$ denotes the relation of not being comparable (i.e., $a \circ b$ iff $a \leq b$ and $b \leq a$ are both false).

THEOREM 3.3. *Let $(X^\star, \leq)$ be a poset associated with the BCK-algebra $(X^\star; *, w)$. Then $(X^\star, \leq)$ is $(C_2 + \underline{1})$-free if and only if $(X^\star; *, \cdot, w)$ has a property: $(x \cdot y) * z = (x * z) \cdot (y * z)$, $\forall x, y, z \in X^\star$.*

Proof. ($\Leftarrow$) Assume that $(X^\star, \leq)$ is not $(C_2 + \underline{1})$-free. Then $(X^\star, \leq)$ has a full subposet, say  This means that $(x \cdot y) * z = y * z = y \neq w = w \cdot y = (x * z) \cdot (y * z)$, a contradiction.
($\Rightarrow$) We consider 3 cases: (a) $y \leq x$ (b) $x \leq y$ (c) $x \circ y$. Case (a). $y \leq x$. Then $x \cdot y = x$ and $x * y = w$. If $x \geq z$, then $x * z = w$, and hence $(x \cdot y) * z = x * z = w$ and $(x * z) \cdot (y * z) = w \cdot (y * z) = w$. If $z \geq x$, then $x * z = x$. Since $z \geq x \geq y$, $y * z = y$. Hence $(x * z) \cdot (y * z) = x \cdot y = x = x * z = (x \cdot y) * z$. If $x \circ z$, then we may consider two subcases: (i) $y \leq z$ and (ii) $y \circ z$. Since $(X^\star, \leq)$ is $(C_2 + \underline{1})$-free we need only to consider the subcase (i) $y \leq z$. It means that $y * z = y$. Hence $(x \cdot y) * z = x * z = x = x \cdot y = (x * z) \cdot (y * z)$.
Case (b). Similar to the Case (a).
Case (c). $x \circ y$. Then $x \cdot y = y$ and $x * y = x$. We consider three subcases: (i) $\{x \circ y, x < z, y < z\}$, (ii) $\{x \circ y, z < x, z < y\}$ and (iii) $\{x \circ y, y \circ z, z \circ x\}$. If (i) holds, then $x \cdot y = y, y * z = y$ and $x * z = x$. Hence $(x \cdot y) * z = y = (x * z) \cdot (y * z)$. If (ii) holds, then $(x \cdot y) * z = y * z = w = (x * z) \cdot (y * z)$. If (iii) holds, then $x \cdot y = y, y * z = y$ and $x * z = x$. Hence $(x \cdot y) * z = y * z = y = x \cdot y = (x * z) \cdot (y * z)$. This proves the theorem. □

EXAMPLE 3.4. In Example 2.2 $(X^\star, \leq)$ has a full subposet $\{c \leq a, c \circ d, a \circ d\}$, and so it is not $(C_2 + \underline{1})$-free. Moreover, we can see that $(c \cdot d) * a = d * a = d$, while $(c * a) \cdot (d * a) = w \cdot d = w$.

REMARK. We may have another candidates for such laws described in Theorem 3.3 as follows:

$$(\delta_1) \quad (x * y) \cdot z = ((x \cdot z) * y) \cdot (y * (y * z)),$$
$$(\delta_2) \quad x \cdot (y * z) = ((x \cdot y) * z) \cdot (x * (x * z)),$$
$$(\delta_3) \quad x * (y \cdot z) = (x * y) * z.$$

It will be interesting to obtain a necessary and sufficient conditions in poset structure or else for such conditions to be hold.

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