On Development of Lower Order Aggregated Model for the Linear Large-Scale Model

Beyong Woo Yoo*

■ Abstract ■

The aggregation of linear large-scale dynamic systems is examined in this paper and a "two-step" approach is proposed. In this procedure, the aggregated system consists of two subsystems.

The first subsystem represents aggregation through the retainment of dominant eigenvalues of the original system, leading to a first approximation of the desired output of the original system. The purpose of augmenting it with a second subsystem is to provide an estimation of the error on the first approximation, thus permitting a second correction to the output approximation and resulting in an output approximation of greater accuracy.

Optimization techniques are discussed for the determination of unknown parameters in the aggregated system. These techniques use minimization principles of certain suitable performance indices and are developed for both single input-single output and multiple input-multiple output system. Numerical examples illustrating these proceuures are given and the results are compared with those obtained using existing methods. Finally, a pharmacokinetics problem is studied from the aggregation point of view.

I. Introduction

In dynamical studies of large scale systems, the concept of aggregation is well known to economists, but less well known to control engineers(Aoki, 1968).

Aggregation techniques enable one to represent an original large-scale system dynamically by the reduced-order systems so that the desired output of the original system can be approximated by that associated with the reduced-order system(Mitra,1967, 1969, Chidambare & Schainker,1971, Saidahmed,1990, Shen, 1990, Hinrichsen, 1990, Kolla,1991, Su, Gajic & Shen,1992, Chiu, 1993, Castelan,1996).

These aggregation techniques are becoming increasingly attractive in the study of urban, environmental, engineering, and other related

* School of Business Administration, Ajou University

social systems governing large dimensionality. Aoki(1968) has shown the usefulness of aggregation matrices for designing suboptimal controllers. As yet, no general method exists for calculating the aggregation matrix for a given large-scale system and an associated reduced model. Several aggregation methods exist for linear time-invariant systems(Kleinman & Athans, 1968, Mitra,1969, Chidambara & Schainker,1971, Saidahmed,1990, Su, Gajic & Shen, 1992, Castelan,1996).

Methods of model reduction proposed by Davison(1966), Chidambara(1971) and Mitra(1969) are the projection methods. Fellow, Sinha and Wismath(1970) have established a simple formula relating the aggregation matrices for a class of reduced-order models derived by projection methods for large-scale systems. Castelan, Gomes and Silva Jr.(1996) have shown that it is possible to solve a class of constrained control problems of linear largescale systems by using a reduced-order system obtained by the projection of the trajectories of the original system onto a subspace associated with the undesirable open -loop eigenvalues. But they addressed the important issues of numerical stability and complexity of the computations. Chiu(1995) tested the model reduction by the low-frequency approximation balancing method for unstable systems. He also investigated that the reduced models retain the dominated control gains of the original model which offers a good approximation at low frequencies. In particular, this reduced models ensure the minimality and the number of the unstable poles as the original model.

One aggrgation method derived by Chidambara & Schainker(1971) is to consider a reduced -order system which retains the most dominant eigenvalues of the original large-scale system. However, these procedures generally require the reduction of the state transition matrix into canonical form. This task becomes cumbersome if some of the eigenvalues are complex or repeated. Alternate methods include those discussed by Anderson(1967) and Fellows, Sinha and Wismath(1970) who select the coefficients of the aggregation model such that the corresponding responses of the original and aggregate systems are approximately matched. The determination of these coefficients becomes time consuming if the order of the reduced system is not sufficiently small.

On the other hand, Meier & Luenberger(1966) has proposed aggregation with observers of multivariable systems and Wilson(1970) developed a model aggregation procedure for the single input-single output linear systems, which was further extended to the multiple input- multiple output case by minimizing the performance index in order to synthesize the aggregation model. However, since these aggregation procedures require not only the knowledge of the eigenvalues and eigenvector-matrix associated with the original transition matrix, but also the solution of nonlinear Riccati matrix differential or algebraric equations(Kleinman,1966, Shen and Gajic,1990), these procedures become quite cumbersome when the dimension of the original system is large, particularly when the original transition matrix contains repeated or partially zero eigenvalues(Su, Gajic and Shen,1992).

In this paper, a "two-step" approach for reducing the order of a linear time-invariant system

is proposed as an extension of Beyong's paper (1975). In this approach, the aggregated system consists of two subsystems. The first subsystem represents aggregation through the retainment of dominant eigenvalues of the original system, leading to a first approximation of the desired output of the original system. The purpose of augmenting it with a second subsystem is to provide an estimation of the error in the first approximation, thus permitting a second correction to the output approximation and resulting in an output approximation of greater accuracy.

A nonlinear programming technique is used to minimize the sum of the squares of the elements of an error matrix. The optimum sets of parameters obtained in such way are used to determine the transformation matrix and the output matrix of the aggregation model(Fellows, Sinha and Wismath,1970, Saidahmed,1990, Wiberg and Dewolf, 1993). Finally the sub-optimal control for the large-scale system is determined from the optimal control of an augmented reduced order system where the augmenting equations give an estimate of the error in the outputs of the original and the aggregate systems(Beyong, 1975).

In what follows, an aggregation algorithm is developed, and is demonstrated through numerical examples and an application of the aggregation technique is also made to the study of a pharmacokinetics problem(Teorell, 1937, Phang, Finerman, Singh, Rosenberg, and Berman, 1971).

Finally, the utility of this technique is investigated and compared with the outputs obtained by using the existing techniques. It turns out that the aggrgation technique developed in this paper gives not only more accurate outputs in comparison with those of the existing techniques, but is also more effective, because none of existing ones contain the second subsystem based upon the error terms which are governed by the elements of the generalized phasevariable D matrix, and also permits a second correction to the first output approximation, thus the aggregation technique provides better approximations for the outputs.

Ⅱ. The Problem Formulation

Consider an original large-scale linear time-invariant system S_0 described by the differential equation

$$S_0 : \dot{x} = Ax + Bu \qquad (1)$$
$$y = Hx$$

where x is the $n\times 1$ state vector of the system, u the $r\times 1$ control input vector, y the $\ell \times 1$ output vector, A is the $n\times n$ system-transition matrix, B is the $n\times r$ control matrix, and H is the $\ell \times n$ output matrix.

It is desired to approximate the system output y by a reduced-order aggregated system.

Suppose the first lower-order aggregated system S_a is governed by the following differential equation :

$$S_a : \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \qquad (2)$$
$$\tilde{y} = \tilde{H}\tilde{x}$$

where $\tilde{x}$ is $m\times 1$ the aggregated state vector, $\tilde{y}$ is the $\ell \times 1$ ($\ell \leq m$) output vector, $\tilde{A}$, $\tilde{B}$, and $\tilde{H}$ are $m\times m$, $m\times r$, $\ell \times m$ undetermined constant matrices.

Let the aggregation error e be defined by

$$e = y - \tilde{y} = Hx - \tilde{H}\tilde{x} \tag{3}$$

Following Aoki(1968), one postulates the existence of a linear relationship between $\tilde{x}$ and x given by

$$\tilde{x} = Cx \tag{4}$$

where C is termed as the "aggregation matrix".

Simple calculations by using the equations (1), (2) and (4) give

$$\begin{aligned} \tilde{A}C &= CA \\ \tilde{B} &= CB \end{aligned} \tag{5}$$

Now the proposed two-step aggregation procedure is given as follows :

Define the first augmented aggregated system $\tilde{S}_a$ which added the inherited error term e in S_a such as

$$\begin{aligned} \tilde{S}_a : \dot{\tilde{x}} &= F_{11}\tilde{x} + F_{12}e + G_1 u \\ \tilde{y} &= \tilde{H}\tilde{x} \end{aligned} \tag{6}$$

where $F_{11} = \tilde{A}$:(m×m), F_{12}:(m× ℓ), $G_1 = \tilde{B}$: (m×r) matrices and e :(ℓ ×1) error vector.

On the other hand, from eqs. (3), (4) and (6), the matrix equation (7) is derived as follows:

$$CA = DC + F_{12}H \tag{7}$$

where $D = F_{11} - F_{12}\tilde{H}$

The second step consists of formulating an error-subsystem S_e, whose output is e.

It is assumed that e satisfies the following differential equation:

$$S_e : \dot{e} = F_{21}\tilde{x} + F_{22}e + G_2 u \tag{8}$$

where F_{21} : ℓ ×m, F_{22} : ℓ × ℓ, and G_2 : ℓ ×r matrices.

By combining the two subsystems $\tilde{S}_a$ and S_e, the proposed Modified Aggregated System (MAS) model, S_a^* can be constructed such as

$$\begin{aligned} S_a^* : \dot{z} &= Fz + Gu \\ \tilde{y}^* &= H^* z \end{aligned} \tag{9}$$

where

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},$$

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \text{ and } H^* = [\tilde{H}, 0] \tag{10}$$

where $z^T = [\tilde{x}, e]^T$: 1×(m+l) state vector, $\tilde{y}^*$: (ℓ ×1) output vector,

F : (m+ ℓ)×(m+ ℓ) matrix, G : (m+ ℓ)×r matrix, H^* : ℓ ×(m+ ℓ) matrix, and 0 : (ℓ × ℓ) matrix where elements are zero and (m+ ℓ)<< n.

On the other hand, from eqs.(3), (4) and (6), one can derive the error differntial equation as follows:

$$\dot{e} = -\tilde{H}F_{12}e + HAx - \tilde{H}F_{11}\tilde{x} + (HB - \tilde{H}G_1)u \tag{11}$$

In order to eliminate the unwanted term HAx in eq.(11), consider the relationships among A, H and C, i.e.

$$HA = HAH^T(HH^T)^{-1}H, \text{ or } HA = HAC^T(CC^T)^{-1}C \tag{12}$$

where $(HH^T)^{-1}$ and $(CC^T)^{-1}$ are assumed to exist.

Now substituting eqs. (3), (4) and (12) into eq. (11), and rearranging the eq.

(11), one has the following two similar error differential equations:

$$\dot{e} = (HAH^T(HH^T)^{-1} - \hat{H}F_{12})e + (HAH^T(HH^T)^{-1}\hat{H} - \hat{H}F_{11})\tilde{x} + (HB - \hat{H}G_1)u \quad (13)$$

$$\dot{e} = -\hat{H}F_{12}e + (HAC^T(CC^T)^{-1} - \hat{H}F_{11})\tilde{x} + (HB - \hat{H}G_1)u \quad (14)$$

Comparing the eq. (13) with eq. (14),

$$F_{21} = HAH^T(HH^T)^{-1}\hat{H} - \hat{H}F_{11} \fallingdotseq HAC^T(CC^T)^{-1} - \hat{H}F_{11} \quad (15)$$

$$F_{22} = HAH^T(HH^T)^{-1} - \hat{H}F_{12} \fallingdotseq -\hat{H}F_{12} \quad (16)$$

$$G_2 = HB - \hat{H}G_1 \quad (17)$$

On the other hand, if small aggregation error were assumed, i.e $e \approx 0$, then from eqs. (3) and (4),

$$H - \hat{H}C \approx 0 \quad \text{i.e.} \quad H \approx \hat{H}C \quad (18)$$

and from eqs. (7) and (18), one has

$$CA = DC + F_{12}H \approx F_{11}C - F_{12}\hat{H}C + F_{12}H \approx F_{11}C \quad (19)$$

Sustituting the eq.(4) into eq.(11), the error differential equation (11) can be derived as follows :

$$\dot{e} = -\hat{H}F_{12}e + (HA - \hat{H}F_{11}C)x + (HB - \hat{H}G_1)u \quad (20)$$

From eqs. (18) and (20), one can derive,

$$HA - \hat{H}F_{11}C \approx \hat{H}CA - \hat{H}F_{11}C = \hat{H}(CA - F_{11}C) \approx 0, \quad (21)$$

thus the error differential equation (20) is given as

$$\dot{e} = -\hat{H}F_{12}e + (HB - \hat{H}G_1)u \quad (22)$$

Comparing the eq.(22) with eq.(8), one has $F_{21} = O$, $F_{22} = -\hat{H}F_{12}$ and $G_2 = HB - \hat{H}G_1$ thus in this case, the MAS, in the eq. (9), S_a^* is given as

$$\dot{z} = \begin{bmatrix} \dot{\tilde{x}} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ O & F_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ e \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u,$$

$$\tilde{y}^* = H^* z \quad (23)$$

Figure 1 shows the block diagram for the proposed Modified Agregated System model. The main objective of this paper is to determine the matrices F, G and H^* in an optimal way.

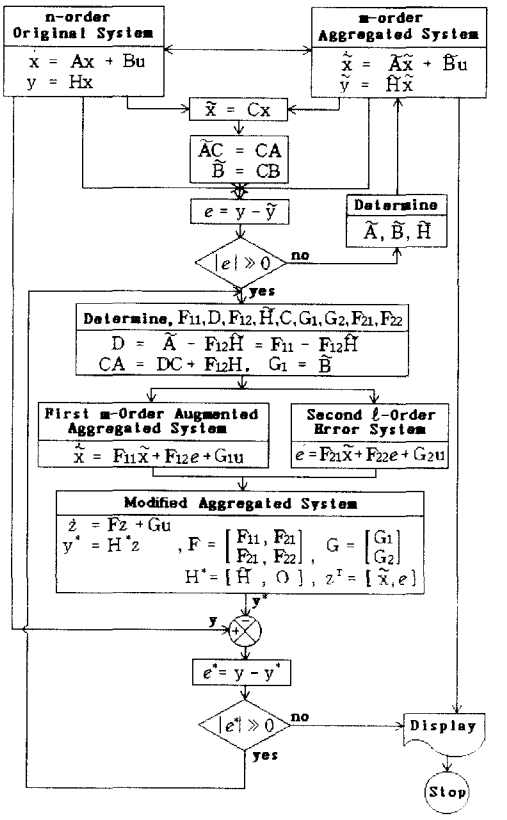

Figure 1. Block Diagram of the Modified Aggregated System Model

Ⅲ. Determination of Matrix F_{11}, D, F_{12}, $\tilde{H}$ & C

3.1 Determination of Matrix $\tilde{A} = F_{11}$

The state $\tilde{x}$ of the first lower-order model eq.(2) is an aggregated state of the original eq.(1). Therefore the model eq.(6) can be shown as a generalized aggregated model. It is possible to select the eigenvalues for the lower-order model eq.(4), if the eigenvalues of A contain a few eigenvalues that are far from the imaginary axis and are in the left half of the complex plan, then these eigenvalues have a negligible effect on the transient response. And therfore the aggregated matrix $\tilde{A}(=F_{11})$ can be chosen to contain the predominant eigenvalues of A like regular phase variable canonical form as in eqs.(24)~(25). Otherwise it becomes necessary to find some other method for specifying the eigenvalues of F_{11}.

$$F_{11} = \tilde{A} = (\tilde{a}_{ij}) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\tilde{a}_1 & -\tilde{a}_2 & -\tilde{a}_3 & \cdots & -\tilde{a}_{m-1} & -\tilde{a}_m \end{bmatrix} \quad (24)$$

and the characteristic polynomial equation of F_{11} is given by

$$\phi_{\tilde{A}}(\lambda) = \lambda^m + \tilde{a}_m\lambda^{m-1} + \tilde{a}_{m-1}\lambda^{m-2} + \cdots + \tilde{a}_2\lambda + \tilde{a}_1 = \prod_{i=1}^{m}(\lambda - \mu_i) \quad (25)$$

where $\mu_i < 0$, $\forall_i$(stability condition) are eigenvalues of

3.2 Selection of D

The essence of modified aggregation method lies in choosing a matrix D so that y^* is the best approximates for y while also satisfying equation (7). An ideal method for selecting D matrix is to adjust its elements in an iterative process until the difference between y and y^* is a minimum.

The number of iterations required to minimize this difference can be reduced by selecting D such that only a few of its elements need to be adjusted.

This objective is realized by writing D matrix in the 「Modified Phase Variable Form」 as shown in the eq.(26)

$$D = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 & & 0 & 0 \\ -d_{11} & -d_{12} & -d_{13} & \cdots & -d_{1m-\ell} & -d_{1m-\ell+1} & \cdots & 0 & 0 \\ -d_{21} & -d_{22} & -d_{23} & \cdots & -d_{2m-\ell} & -d_{2m-\ell+1} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ -d_{\ell 1} & -d_{\ell 2} & -d_{\ell 3} & \cdots & -d_{\ell m-\ell} & -d_{\ell m-\ell+1} & \cdots & -d_{\ell m-1} & 1 \\ -d_{\ell+1\,1} & -d_{\ell+1\,2} & -d_{\ell+1\,3} & \cdots & -d_{\ell+1\,m-\ell} & -d_{\ell+1\,m-\ell+1} & \cdots & -d_{\ell+1\,m-1} & -d_{\ell+1\,m} \end{bmatrix} \quad (26)$$

3.3 Determination of Matrix F_{12}

Theoretically the choice of f_i (i=1, 2, ···, ℓ) of the matrix F_{12} in the eq.(27) is arbitrary, as it does not affect the matrix $\tilde{H}$. However, the elements of $\tilde{H}$ are directly proportional to f_i while those of C are inversely proportional to f_i.

A proper choice of f_i can prevent the elements of $\tilde{H}$ and C from taking values too large or too small. Although there are several choices for f_i in the matrix F_{12} ;

$$F_{12}=(f_{ij})=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ f_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_2 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & f_{\ell-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & f_\ell \end{bmatrix} \quad (27)$$

where the nonzero elements f_i of F_{12} are given by the average values of the coefficients of the row vectors which contain the elements $\hat{d}_{ij}$ of the D matrix such as

$$f_i = \frac{1}{i}\sum_{j=1}^{i} \hat{d}_{kj}, \quad i=(m-\ell+1),$$

$$(m-\ell+2), \cdots, m, \quad k=i-(m-\ell) \quad (28)$$

For the case $\ell=1$, i.e. f_1 can be taken as the average of the coefficients of the characteristic polynomial function of D matrix.

It is also emphasized at this point that F_{12} is not yet known since D is still unknown.

3.4 Determination of $\hat{H}$.

Let

$$\hat{H}=\begin{bmatrix} \hat{h}_{11} & \hat{h}_{12} & \cdots & \hat{h}_{1m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \hat{h}_{\ell 1} & \hat{h}_{\ell 2} & \cdots & \hat{h}_{\ell m} \end{bmatrix} \quad (29)$$

and substitute the eqs. (24), (27) and (29) into the eq. (7), then by rearranging the matrix constraint equation $F_{12}\hat{H}=F_{11}-D$, the elements $\hat{h}_{ij}$ of the matrix $\hat{H}$ can be determined as follows:

$$\hat{h}_{ij}=\begin{cases} \dfrac{\hat{d}_{ij}}{f_i}, & i=1,2,\cdots,(\ell-1), \\ & j=1,2,\cdots,m \\ \dfrac{\hat{d}_{ij}-\hat{a}_j}{f_i}, & i=\ell,\ j=1,2,\cdots,m \end{cases} \quad (30)$$

3.5 Determination of Aggregation Matrix C.

From the eqs. (7) and (10),

$$CA = DC + F_{12}H \quad (31)$$

Let

$$C^T = [c_1^T, c_2^T, \cdots\cdots, c_m^T] \quad (32)$$
$$H^T = [h_1^T, h_2^T, \cdots\cdots, h_i^T]$$

where

$c_j^T = [c_{j1}, c_{j2}, \cdots\cdots, c_{jn}]^T$
: $(n\times 1)$ j^{th} column vector of the matrix c^T for j = 1, 2, ····, m.

$h_j^T = [h_{j1}, h_{j2}, \cdots\cdots, h_{jn}]^T$
: $(n\times 1)$ j^{th} column vector of the matrix H^T for j = 1, 2, ····, l.

then for a given ℓ, C_i can be uniquely determined as the function of elements of D matrix as follows :

$$C_iA=\begin{cases} C_{i+1}, & i=1,2,\cdots\cdots,(m-\ell) \\ -\sum_{j=1}^{i}\hat{d}_{kj}C_j + C_{i+1} + f_k h_k, & i=(m-\ell+1),\cdots,(m-1), \\ & k=i-(m-\ell) \\ -\sum_{j=1}^{i}\hat{d}_{kj}C_j + f_k h_k, & i=m,\ k=\ell \end{cases} \quad (33)$$

where $\hat{d}_{kj}$ and f_k are given as the eqs. (26) and (27).

By using the recursive relation, the eq. (33) can be rewritten as follows :

$$C_i = \begin{cases} C_{m-i+1}(A^{-1})^{(m-\ell+1)-i} \\ \quad , i = 1, 2, 3, \cdots, (m-\ell+1) \\ C_{i-1}[A + \sum_{j=1}^{m-\ell+1} \tilde{d}_{1j}(A^{-1})^{(m-\ell+1)-j}] - f_1 h_1 \\ \quad , i = (m-\ell+2) \\ C_{i-1}A + \sum_{j=1}^{i-1} \tilde{d}_{kj} C_j - f_k h_k \\ \quad , i = (m-\ell+3), \cdots, (m-1) \\ \quad k = i-(m-\ell+1) \\ \begin{cases} C_{i-1}A + \sum_{j=1}^{i-1} \tilde{d}_{k-1,j} C_j - f_{k-1} h_{k-1} \\ \quad , i = m, \quad k = i-(m-\ell) \\ [f_k h_k - \sum_{j=1}^{m-1} \tilde{d}_{kj} C_j][A + \tilde{d}_{km} I]^{-1} \end{cases} \end{cases} \quad (34)$$

From the above discussion, One can see that all C_i vectors can be determined as the function of the $C_{m-\ell+1}$ vector which is the function of the elements $\tilde{d}_{ij}$ of the D matrix.

Hence, the problem to determine the aggregation matrix C is strictly reduced to how to find the $C_{m\ \ell+1}$ vector.

Because of complexity for determining the C_{m-l+1} vector, one can use two or three simple cases when m = 1, 2, 3, for a fixed ℓ = 1 as illustrated examples 1 and 2.

The main purpose of the proposed MAS is to determine the best set of values for $\tilde{d}_{ij}$ in the eq. (26) where the aggregation error should be minimized in optimal tolerance.

A suitable criterion for optimum selection of D matrix would be one which minimized the sum of the squares of the elements of the aggregation error matrix E = H - $\hat{H}$C such that

$$\underset{\tilde{d}_{ij}}{\text{Min}} \{ J = J(\tilde{d}) = \sum_{i=1}^{\ell} \sum_{j=1}^{n} E_{ij}^2 \} \quad (35)$$

where E = (E_{ij}) is $\ell \times$m matrix, and $\hat{H}$ is given as

$$\hat{H} = (F_{12}^T F_{12})^{-1} F_{12}^T (F_{11} - D) \quad (36)$$

where E, $\hat{H}$, F_{12} and C are represented in terms of $\tilde{d}_{ij}$ in eq. (26) with the matrix equation (7),

By using Vec( ) notation(Neudecker, 1969), the eq. (35) is given as

$$\underset{\tilde{d}_{ij}}{\text{Min}} \{ J = J(\tilde{d}) = [\text{Vec}(H - \hat{H}C)^T \cdot [\text{Vec}(H - \hat{H}C)] \} \quad (37)$$

subject to the matrix constraint equation $F_{12}\hat{H} = F_{11} - D$ in eq. (7).

To solve this problem, powerful nonlinear programming techniques using properties of quadratic programming techniques associated with Pattern Search, Steepest Descent, Conjugate Direction, and Conjugate Gradient Method (Curry, 1954, Fletcher & Powell, 1963, Powell, 1964, Afimiwala, 1974) can be used to select the optimum values for $\tilde{d}_{ij}$.

Provided J is minimized to approximately zero, then it implies H - $\hat{H}$C $\approx$ 0 as shown in eq. (18).

Furthermore, this implies that the error differential equation is given as the eq. (22), and $\tilde{y}^* \approx y$. Consequently, this means that $\tilde{y}^*$ is the approximated value of the system output y of the original exact problem by a reduced order aggregated system.

Ⅳ. Numerical Example

Example 1. (Simple Linear Dynamic Model)

To illustrate the numerical example, this section will be devoted to the following example of a time-invariant fourth order process controlled by a reduced order linear dynamic model as defined by the eq. (6).

Let the original linear dynamic system model S_0 in the eq.(1) be given as

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$= \begin{bmatrix} -1. & 0. & 0. & 0. \\ 0. & -3. & 0. & 0. \\ 0. & 0. & -5. & 0. \\ 0. & 0. & 0. & -10. \end{bmatrix} x(t) + \begin{bmatrix} 0.0278 \\ 0.9286 \\ -3.1500 \\ 3.1937 \end{bmatrix} u(t) \tag{38}$$

$$y(t) = Hx(t) = [\,1\ 1\ 1\ 1\,]\,x(t)$$

then if one picks $F_{11}(=\tilde{A})$ to retain the predominant two and three eigenvalue of A matrix and chooses the matrix D as one of the modified phase variable forms in the eq. (26) for $\ell = 1, 2$, and also computes the elements f_i of the F_{12} as an average value of the row coefficients $\hat{d}_{ij}$ of D as given in the eq. (28), then one can determine the elements $\hat{d}_{ij}$ of D by finding the optimal solution for $J(\hat{d})$ due to the sum of squares of the aggregated error matrix in the eq. (35) or the eq. (37), and also from this D matrix, one can easily determine C and $\tilde{H}$ by using the eqs. (34), (36), and the rest of the others is easily calculated by using C , D, $\tilde{H}$ and so forth.

On the other hand, let $F_{12} = 0$ in eq. (6), then, since the matrix A given by eq. (38) is diagonal, one can find the direct aggregated model by simply choosing F_{11}, G_1 and $\tilde{H}$ corresponding to the dimension of the aggregated state vector $\tilde{x}(t)$.

From the calculated results, the desired second and third order aggregated models S_a in the eq.(2) can be determined respectively as follows:

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0.0 & 1.0 \\ -3.0 & -4.0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} -0.017596 \\ 0.577238 \end{bmatrix} u(t)$$

$$\tilde{y}_2(t) = [\,-0.044113 \quad 1.94118\,]\,\tilde{x}(t) \tag{39}$$

and

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ -15.0 & -23.0 & -9.0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0.010339 \\ -0.121022 \\ 3.536070 \end{bmatrix} u(t)$$

$$\tilde{y}_3(t) = [\,1.65551 \quad -0.438639 \quad 0.452788\,]\,\tilde{x}(t) \tag{40}$$

On the other hand, for the order 2 and 3 cases, the proposed MAS models S_a^* in the eqs.(9) and (10) are determined respectively as follows: where $Z[t] = [\,\tilde{x}(t), e(t)\,]^T$

$$\dot{Z}(t) = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ -3.0 & -4.0 & 68.00610 \\ 0.0 & 0.0 & -132.0120 \end{bmatrix} Z(t) + \begin{bmatrix} -0.017596 \\ 0.577238 \\ -0.121200 \end{bmatrix} u(t)$$

$$\tilde{y}_2^*(t) = [\,-0.044113 \quad 1.94118 \quad 0.0\,]\,Z(t) \tag{41}$$

and

$$\dot{Z}(t) = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ -15.0 & -23.0 & -9.0 & 35.3333 \\ 0.0 & 0.0 & 0.0 & -15.9985 \end{bmatrix} Z(t) + \begin{bmatrix} 0.016339 \\ -0.121022 \\ 3.536070 \\ -0.671194 \end{bmatrix} u(t)$$

$$\tilde{y}^*{}_3(t) = [\,1.65551 \quad -0.438639 \quad 0.452788 \quad 0.0\,]\,Z(t) \tag{42}$$

The outputs associated with the aggregated system models $\tilde{y}(t)=(\tilde{y}_i(t),\ i=2,3)$ and $\tilde{y}^*(t)=(\tilde{y}_i^*(t), i=2,3)$ vs. exact system model y(t) are tabulated in Table 1 and 2 in Appendix for the cases of a unit impulse or a unit step input respectively.

Figures 2, 3, 4, 5 and figures 6, 7, 8, 9 show the plots of $\tilde{y}_i(t)$, $\tilde{y}_i^*(t)$ and $y_2(t)$ vs the exact output y(t) when $u(t)$ is a unit impulse input or a unit step input where x(0) = 0, $\tilde{x}(0)$ = 0 and z(0) are given as the initial conditions.

It is observed from these figures that if the aggregated system contains more than two predominant modes, then the output $\tilde{y}^*(t)$ seems to be a better approximation to the original output y(t) and also $\tilde{y}_i^*(t) \approx y(t)$ and $\tilde{y}_i(t) \approx y(t)$ $(i=2,3)$ as shown in figures 5 and 9. It also turns out that figures 2, 3, 4 and 5 for the unit impulse input and figures 6, 7, 8 and 9 for the unit step input show that the modified aggregated model gives better output approximation by virture of the fact that a second correction through incorporating the error vector makes it possible to include more modes in the original system. As shown in the figures 4 and 8, author's modified aggregration results $\tilde{y}_2(t)$ and $\tilde{y}_2^*(t)$ are better approximations compared with $y_2(t)$ from other researches(Mitra(1969), Wilston(1970), Shen(1990), Saidahmed(1990), and Wiberg(1993), etc.) for

Case A : Unit Impulse Input

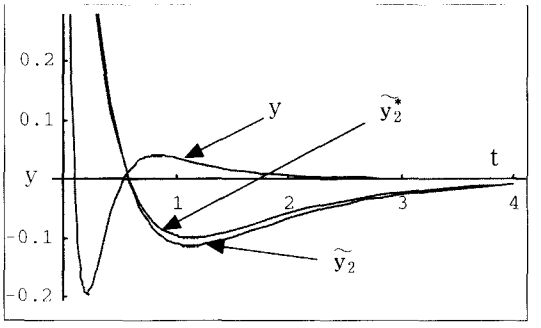

Figure 2. Aggregated system outputs $\tilde{y}_2$ $\tilde{y}_2^*$, vs. exact system output y

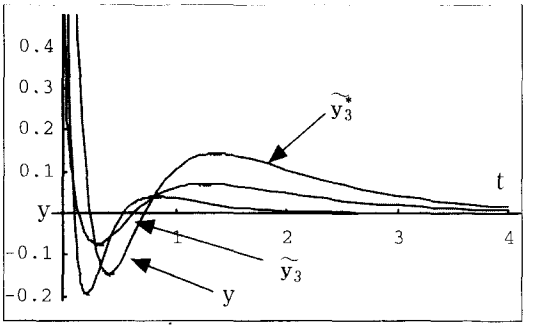

Figure 3. Aggregated system outputs $\tilde{y}_3$, $\tilde{y}_3^*$ vs. exact system output y

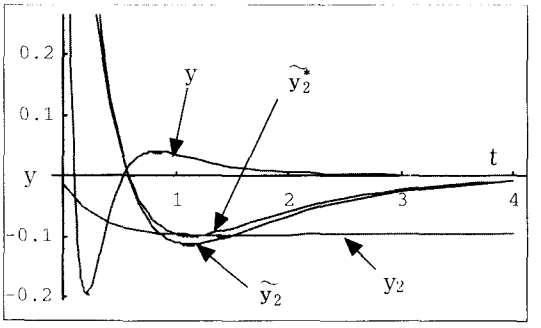

Figure 4. Aggregated system outputs $\tilde{y}_2$ $\tilde{y}_2^*$, y_2 vs. exact system output y

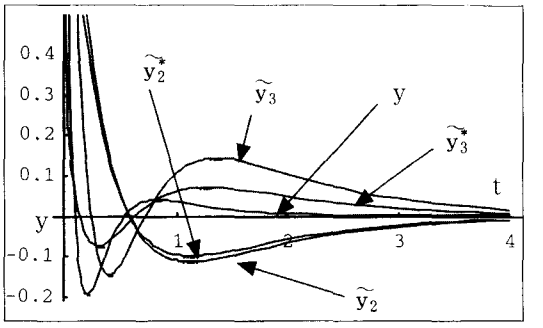

Figure 5. Aggregated system outputs $\tilde{y}_2$, $\tilde{y}_2^*$ $\tilde{y}_3$, $\tilde{y}_3^*$ vs. exact system output y

Case B : Unit Step Input

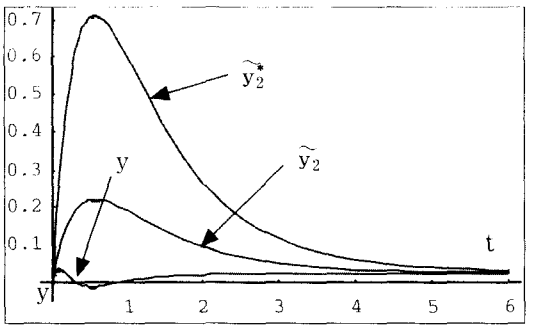

Figure 6. Aggregated system outputs $\tilde{y}_2$, $\tilde{y}_2^*$ vs. exact system output y

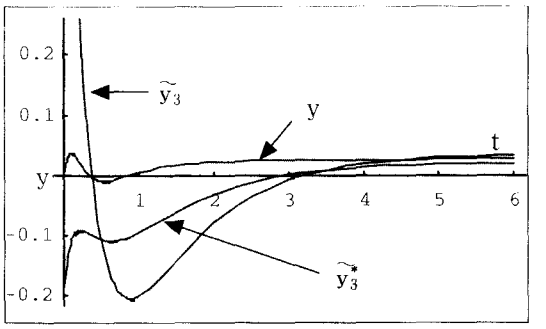

Figure 7. Aggregated system outputs $\tilde{y}_3$, $\tilde{y}_3^*$ vs. exact system output y

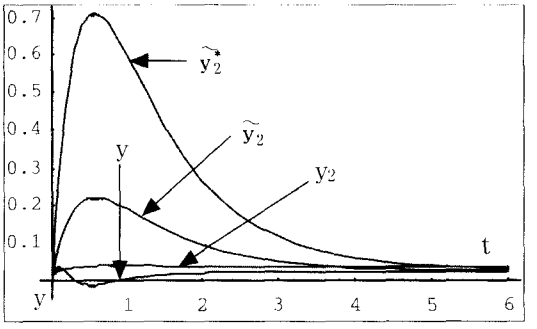

Figure 8. Aggregated system outputs $\tilde{y}_2$, $\tilde{y}_2^*$, y_2 vs. exact system output y

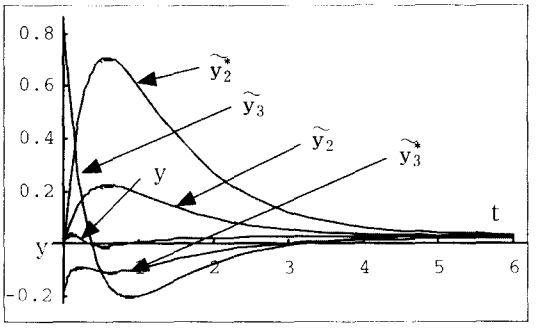

Figure 9. Aggregated system outputs $\tilde{y}_2$, $\tilde{y}_2^*$, $\tilde{y}_3$, $\tilde{y}_3^*$ vs. exact system output y

both unit impulse and unit step inputs.

Example 2. (An the Aggregation for Pharmacokinetic Model)

As an illustration, consider a modified seventh order compartmental model of collagen synthesis in fetal rat calvaria studied by Phang and his coworkers(1971). This model is shown schematically in Figure 10. In this figure 10, a circle (compartment) denotes the location of a biochemical state of substrate. A triangle represents a data sampling sit and can be simply measured as a linear combination of two(or more) compartments of interest. The arrows represent rate constants denoting the transitional probabilities estimated from Gaussian Distributions with means and standard deviations and the dotted lines represent constant coefficients which are taken to be one.

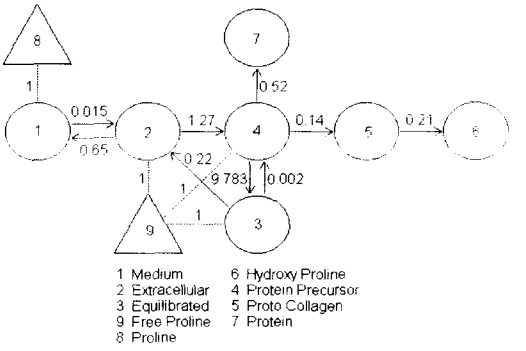

Figure 10. Compartmental Model of Collagen Synthesis(from J.M. Phang et al, 1971)

To develop the mathematical model, let

$x_i(t)$ = biochemical state variables of the compartment i,

$y_i(t)$ = output variables expressed as linear combinations of measured state variables.

a_{ij} = rate constants from the compartment j to the compartment i in Mu moles of proline, and a_{jj} is the sum of the connected rates a_{ij} from the fixed compartment j to the compartment i ($i \neq j$).

then, the desired 7×7 dynamic compartmental model for the system has the form:

$$\begin{bmatrix}\dot{X}_1(t)\\ \dot{X}_2(t)\\ \dot{X}_3(t)\\ \dot{X}_4(t)\\ \dot{X}_5(t)\\ \dot{X}_6(t)\\ \dot{X}_7(t)\end{bmatrix} = \begin{bmatrix} -0.015 & 0.65 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.015 & -1.92 & 0.22 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & -0.222 & 0.0 & 9.783 & 0.0 & 0.0\\ 0.0 & 1.27 & 0.002 & -10.443 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.14 & -0.21 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.21 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.52 & 0.0 & 0.0 & 0.0\end{bmatrix} \begin{bmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\\ X_5(t)\\ X_6(t)\\ X_7(t)\end{bmatrix} \tag{43}$$

$$\begin{bmatrix}Y_8(t)\\ Y_9(t)\end{bmatrix} = \begin{bmatrix}1.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\\ 0.\ 1.\ 1.\ 0.\ 1.\ 0.\ 0.\end{bmatrix} \begin{bmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\\ X_5(t)\\ X_6(t)\\ X_7(t)\end{bmatrix} \tag{44}$$

or, for the vector-matrix notation,

$$\dot{X}(t) = A_0X(t), \quad X(0) = X_0 \tag{45}$$
$$Y(t) = H_0X(t)$$

where A_0 and H_0 are 7×7 and 2×7 matrices, respectively, and

$$X^T(t) = [X_1(t), X_2(t), X_3(t), x_4(t), X_5(t), X_6(t), X_7(t)]$$

and $Y^T = [Y_8(t), Y_9(t)]$.

Our purpose is to perform aggregation with respect to the system above so that the output $Y(t)$ can be approximately obtained from a reduced order system.

Aggregation

Since the matrix A_0 in eq. (45) is of rank 5 by inspection, it can first be factored into a matrix product as follows. Thus

$$A_0 = P_1 \cdot P_2 \tag{46}$$

where P_1 and P_2 are 7×5 and 5×7 matrices with

$$P_1 = \begin{bmatrix} -0.015 & 0.65 & 0.0 & 0.0 & 0.0\\ 0.015 & -1.92 & 0.22 & 0.0 & 0.0\\ 0.0 & 0.0 & -0.222 & 9.783 & 0.0\\ 0.0 & 1.27 & 0.002 & -10.443 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.14 & -0.21\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.21\\ 0.0 & 0.0 & 0.0 & 0.52 & 0.0\end{bmatrix}$$

and $$P_2^T = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0\end{bmatrix} \tag{47}$$

Regarding P_2 as an aggregation matrix which relates the original state vector $X(t)$ to the reduced state vector $x(t)$ in the form

$$x(t) = P_2X(t) \tag{48}$$

The reduced system model is then governed by the following dynamic differential equations,

$$\dot{x}(t) = A\,x(t)$$
$$y(t) = H\,x(t) \tag{49}$$

where A is an 5×5 nonsingular aggregated matrix given by

$$A = P_2 \cdot P_1 = \begin{bmatrix} -0.015 & 0.65 & 0.0 & 0.0 & 0.0 \\ 0.015 & -1.92 & 0.22 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.222 & 9.783 & 0.0 \\ 0.0 & 1.27 & 0.002 & -10.443 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.14 & -0.21 \end{bmatrix} \tag{50}$$

and its eigenvalues are -0.0015, -0.0862, -0.21, -2.099 and -10.4133, and $y(t)$ is a 2×1 output vector and H is given by

$$H = \begin{bmatrix} 1. & 0. & 0. & 0. & 0. \\ 0. & 1. & 1. & 1. & 0. \end{bmatrix} \tag{51}$$

Let us here consider the reduced system characterized by eq. (49) as the original systems.

In order to determine unknown matrices F_{11}, D, F_{12}, $\hat{H}$ and C for the MAS model, the matrix F_{11} is first chosen in a phase variable form as in eq. (24) which retains the predominant eigenvalue of A and then, by using the proposed analytical algorithm, the optimum fundamental D matrix is obtained as follows:

$$D = \begin{bmatrix} -0.755170 & 1.0 \\ 2.400063 & -2.429330 \end{bmatrix} \tag{52}$$

where the minimum value of the performance index $J(\hat{d})$ given by eq. (35) is 0.069456.

The remaining matrices F_{12}, $\hat{H}$, C and their related matrices can now be found by the eqs. (28), (30) and (34) and (38).

Hence, using this approach, the direct 2nd order aggregated model S_a can be represented by

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0.0 & 1.0 \\ -1.27701\times10^{-4} & -8.76744\times10^{-2} \end{bmatrix} \tilde{x}(t)$$
$$\tilde{y}(t) = \begin{bmatrix} 2.0 & 0.0 \\ -0.993991 & 0.96975 \end{bmatrix} \tilde{x}(t) \tag{53}$$

where the relationship between $\tilde{x}(t)$ and $x(t)$ is given by

$$\tilde{x}(t) = \begin{bmatrix} 0.108294 & 0.041265 & 0.017184 & 0.015725 & 0.0 \\ 0.110925 & 0.042295 & 0.018272 & 0.015770 & 0.0 \end{bmatrix} x(t) = Cx(t) \tag{54}$$

Next, the proposed MAS pharmacokinetics model S_a^* can also be determined as follows:

$$\dot{Z}(t) = \begin{bmatrix} 0.0 & 1.0 & 0.377585 & 0.0 \\ -1.27701\times10^{-4} & -8.76744\times10^{-2} & 0.0 & 2.41747 \\ -0.245365 & -1.78989 & -0.77017 & 0.216667 \\ 0.464167 & 0.655556 & 0.390316 & -2.77832 \end{bmatrix} Z(t) \tag{55}$$

$$\tilde{y}^*(t) = \begin{bmatrix} 2.0 & 0.0 & 0.0 & 0.0 \\ -0.993991 & 0.96975 & 0.0 & 0.0 \end{bmatrix} Z(t)$$

where $Z(t) = [\,\tilde{x}_1(t),\ \tilde{x}_2(t),\ e_1(t),\ e_2(t)\,]^T$ and the initial given state vector:

$$Z(0) = Z_0 = [\,\tilde{x}_1(0),\ \tilde{x}_2(0),\ e_1(0),\ e_2(0)\,]^T = [\,1,\ 1,\ 1,\ 1\,] \tag{56}$$

Figure 9 and 10 are plotted with Table 3 in Appendix. Figure 9 shows a comparison of the exact output $y(t)$ in the eq. (49) with the direct aggregated the output $\tilde{y}(t)$ as given in eq.(53) determined by the aggregated state response

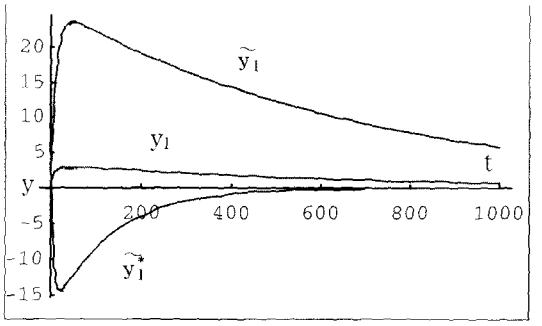

Figure 11. Aggregated Outputs asociated with y_1, $\tilde{y}_1$, & $\tilde{y}_1^*$

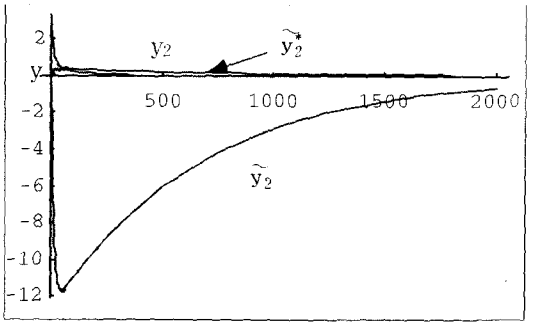

Figure 12. Aggregated Outputs asociated with y_2, $\tilde{y}_2$, & $\tilde{y}_2^*$

$\tilde{x}(t)$ using the first approach and the modified aggregated output $\tilde{y}^*(t)$ in eq.(55). Similarly figure 10 shows a comparison of the exact output $y(t)$ in the eq.(49) with $\tilde{y}(t)$ & $\tilde{y}^*(t)$ based upon the direct aggregated model in eq. (53) and the MAS model as given by the eq. (55).

It is seen from figure 11 and 12 that they result in better approximations to the exact output $y_i(t)$ associated with the direct aggregated output $\tilde{y}_i(t)$ using first approach and the modified aggregated output $\tilde{y}^*_i(t)$ using the second augmented approach. Furthermore, as it was mentioned earlier, $\tilde{y}^*_2(t)$ in Figure 11 and 12 become superior over $\tilde{y}_i(t)$ as time increases and approximates closely the exact output $y(t)$ asymptotically. Finally, the aggregated output $\tilde{y}_1(t)$ and $\tilde{y}^*_1(t)$ of the MAS shows practically less improvements over $\tilde{x}_1(t)$ in Figure 11. The augmentation of the $\tilde{S}_a$ system for this output does give proper excitation model.

V. CONCLUDING REMARK

The aggregation of linear large-scale dynamic system is investigated in this paper and a "two-step" approach is proposed. This approach has been shown, through analysis and numerical example, to be an improvement over existing one-step procedures, Futhermore, the aggregation procedure permits an estimation of the aggregation error directly, thus makes it possible to give a second correction.

One optimization technique is discussed for implementing the aggregation procedure in both single input - single output and multiple input - multiple output cases.

This technique uses minimization principles of certain predetermined performance indices together with Luenberger-type constraint matrix equations.

Numerical examples are given and the result are compared with the exact output. A pharmacokinetics problem is also studied from the aggregation point of view. From these numerical examples, it has been shown that (i) if the aggregated matrix F_{11} is chosen to retain the predominant modes and is given in the phase variable form, the aggregated output $\tilde{y}_i^*(t)$ (or $\tilde{y}_i(t)$) retains either the dominant modes or their approximate values, (ii) the aggregated output $\tilde{y}_i^*(t)$ (or $\tilde{y}_i(t)$) from the MAS model not only retain the dominant modes, but also contain

modes corresponding to the unretained modes by properly selecting the eigenvalues of the fundamental D matrix, and (iii) the aggregated output improves with the oder of the aggrgated system over one-step procedures.

On the other hand, the proposed aggregation procedure introduces some inherent difficulties. First, being nonlinear optimization problems, the optimization techniques employed for the determining the unknown matrices are sensitive to initial estimates. Global minimum may not be obtainable with poorly chosen initial estimates.

Appendix

〈Table 1〉 Outputs associated with the y, $\tilde{y}_i$ and $\tilde{y}_i^*$, y_2 under a unit impulse input

m		System Responses
4	exact solution	$y(t) = 0.0278\ e^{-t} + 0.9286\ e^{-3t} - 3.15\ e^{-5t} + 3.1937\ e^{-10t}$
2	direct aggregation	$\tilde{y}_{2(t)} = -0.520594\ e^{-t} + 1.64189\ e^{-3t}$
	modified aggregation	$y_2^*(t) = -0.458138\,e + 1.45444\ e^{-3t} + 0.124997\ e^{-132.012t}$
	other research's aggregration	$y_2(t) = -0.0949115 - 0.072723\,e^{-t} + 0.0982544\ e^{-3t}$
3	direct aggregation	$\tilde{y}_3(t) = 0.866571\ e^{-t} - 5.03932\ e^{-3t} + 5.84359\ e^{-5t}$
	modified aggregation	$\tilde{y}_3^*(t) = 0.392157\ e^{-t} - 1.87991\ e^{-3t} + 1.79129\ e^{-5t} + 1.3777\ e^{-15.9985t}$

〈Table 2〉 Outputs associated with the y, $\tilde{y}$ and $\tilde{y}_i^*$, y_2 under a unit impulse input

m		System Responses
4	exact solution	$y(t) = 0.0267033 - 0.0278\ e^{-t} - 0.309533\ e^{-3t} + 0.63\ e^{-5t} - 0.31937\ e^{-10t}$
2	direct aggregation	$\tilde{y}_2(t) = 0.0267041 + 0.520593\ e^{-t} - 0.547298\ e^{-3t}$
	modified aggregation	$\tilde{y}_2^*(t) = 0.0266803 + 1.79554\ e^{-t} - 1.85522\ e^{-3t} + 0.0000002277803\ e^{-132.012t}$
	other research's aggregration	$y_2(t) = 0.0368587 + 0.0172723\,e^{-t} + -0.03275\ e^{-3t}$
3	direct aggregation	$\tilde{y}_3(t) = 0.035505 - 0.866567\ e^{-t} + 1.67978\ e^{-3t} - 0.0117619\ e^{-5t}$
	modified aggregation	$\tilde{y}_3^*(t) = 0.0209881 - 0.392153\ e^{-t} + 0.62664\ e^{-3t} - 0.358253\ e^{-5t} - 0.0861142\ e^{-15.9985t}$

〈Table 3〉 Outputs associated with MAS model

	System Responses
exact solution $y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$	$y_1 = 3.25546\ e^{-0.00148157t} - 2.03969\ e^{-0.0861928t} 1 - 0.214439\ e^{-2.09899t} - 0.00133591\ e^{-10.4133t}$
	$y_2 = 0.44448\ e^{-0.00148157t} + 2.25241\ e^{-0.0861928t} + 0.247302\ e^{-2.09899t} + 0.0558051\ e^{-10.4133t}$
direct aggregation $\tilde{y}=\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}$	$\tilde{y}_1 = 25.6446\ e^{-0.00148157t} - 23.6446\ e^{-0.0861928t}$
	$\tilde{y}_2 = -12.7637\ e^{-0.00148157t} + 12.7394\ e^{-0.0861928t}$
modified aggregation $\tilde{y}^*=\begin{bmatrix} \tilde{y}^*_1 \\ \tilde{y}^*_2 \end{bmatrix}$	$\tilde{y}_1^* = -17.6759\ e^{-0.00747714t} + 21.3946\ e^{-0.169153t} - 1.94969\ e^{-0.444167t} + 0.231019\ e^{-3.35637t}$
	$\tilde{y}_2^* = 0.496437\ e^{-0.00747714t} - 0.935463\ e^{-0.169153t} + 0.827922\ e^{-0.444167t} - 0.413137\ e^{-3.35637t}$

Secondly, since approximation is in establishing the subsystem governing the aggregation error matrix, it is not an easy task to assess the relative accuracy of $\tilde{y}_i^*(t)$ and $\tilde{y}_i(t)$ derived from the MAS system. This particularly true for multiple input - multiple output situations such as pharmacokinetics problem. In general, however, $\tilde{y}_i^*(t)$ is expected to give better approximation of the original output asymptotically.

In closing, it is worth mentioning that the proposed aggregation procedure is useful in control of large scale systems under general conditions. This topic, however, is beyond the scope of this paper.

REFERENCES

[1] Anderson, J. H., "Geometrical Approach to the Reduction of Dynamical Systems," Proceedings, *IEEE*, 114, No.7(1967), pp.1014-1018.

[2] Aoki, M., "Control of Large Scale Dynamic Systems by Aggregation," *IEEE Transactions on Automatic Control*, Vol.AC-13 (1968), pp.246-253.

[3] Athans, M. and D. L. Kleinman, "The Design of Suboptimal Linear Time-Varying System", *IEEE on AC*, Vol.AC-13, No.2 (1968), pp.150-159.

[4] Castelan, E. B., Gomes, da Silva Jr., J. M. and J. E. R. Cury, "A Reduced-Order Framework Applied to Linear Systems with Constrained Controls," *IEEE Trans. on Automatic Control*, Vol.41, No.2(1996. 2), pp.249-255.

[5] Chidambara, M. R. and R. B. Schainker, "Lower Order Generalized Aggregated Model and Suboptimal Control," *IEEE Transactions Automatic Control*, Vol.AC-16 (1971), pp.175-180.

[6] Chiu, T. Y., "Model Reduction for Unstable Linear Systems," *Control Theory Advanced Technology*, Vol.9, No.2(1993), pp.467-480.

[7] Curry, H., "Methods of Steepest Descent for Nonlinear Minimization Problems," *Applied Mathematics*, Vol.2(1954), pp.258-261.

[8] Davision, E. J, "A Method for Simplifying Linear Dynamic Systems," *IEEE Transactions*, Vol.AC-11(1966), pp.93-101.

[9] Enns, D. F., Model Reduction for Control System Design, Ph. D. dissertation, Stanford University, 1984.

[10] Fellow, D. M., Sinha, N. K. and J. C. Wismath, "Reduction of the Order of Dynamic Models," *1970 National Conference on Automatic Control*, University of Waterloo, Canada, 1970.

[11] Hinrichsen, D. and A. J. Printchard, "An Improved Error Estimate for Reduced-Order Models of Discrete-Time Systems," *IEEE, Trans. Automatic Control*, Vol.35 (1990), pp.317-320.

[12] Kalman, R. E., "Mathmatical Description of Linear Dynamical Systems," *SIAM, Journal on Control*, Vol.1, No.2(1963), pp.152-192.

[13] Kleinman, D. L. and M. Athans, "The Design of Suboptimal Linear Time-Varying Systems," *IEEE Trans. Automat. Control*, Vol.AC-13(1968. 6), pp.150-159.

[14] Kleinman, D. L., "On the Linear Regulator Problem and the Matrix Riccati Equation," M.I. T. Electronic System Lab., Rep. ESL-R-271, June 1966.

[15] Kolla, S. R. and J. B. Farison, "Reduced

-Order Dynamic Compensator Design for Stability Robustness of Linear Discrete Time System," *IEEE Transactions on Automatic Control Systems Quality*, Vol.36, No.9(1991. 9), pp.1077-1081.

[16] Lucas, T. N., "Scaled Impulse Energy Approximation for Model Reduction," *IEEE Transactions on Automatic Control Systems Quality*. Vol.33, No.8(1988. 8), pp. 791-793.

[17] Meier, L. and D. G. Luenberger, "Approximation of Linear Constant Systems," *Proceeding, 7th Joint Automatic Control Conference*(1966), University of Washington, pp.728-735.

[18] Mitra, D., "On the Reduction of Complexity of Linear Dynamic Model," *Technical Report AEEW*-R520(1967), United Kingdom.

[19] Mitra, D., "W-Matrix and the Geometry of Model Equivalence and Reduction," Proceedings, *IEEE*, Vol.166(1969), pp.1101-1106.

[20] Mitra, D., "Analytic Results on the Use of Reduced Models in the Control of Linear dynamical Systems," *Proceedings, IEEE*, Vol.116(1969), Atomic Anergy Authority, pp.1439-1444.

[21] Mitra, D., "The Reduction of Complexity of Linear, Time Invariant Dynamic Systems," *Proceedings*, 5th IFAC, *Technical Series* 67(Warsaw, 1969), pp.19-33.

[22] Moore, B. C., "Principal Component Analysis in Linear Systems : Controllability, Observbility and Model Reduction," *IEEE Trans. Automatic Control*, Vol.AC-26, No.1 (1981. 2), pp.17-32.

[23] Neudecker, H., "A Note on Kronecker Matrix Products and Matrix Equation System," *SIAM, Journal of Applied Mathematics*, Vol.17, No.3(1969), pp.603-606.

[24] Phang, J. M., Finerman, G. A. A., Singh, B., Rosenberg, L. E. and M. Berman, "Compartmental Analysis of Collagen Synthesis in Fetal Rat Calvaria," *Biochemistry Biophysics*, Acta. 230(1971), pp.146-159.

[25] Powell, M. J. D., "An Efficient Method for Finding the Minimum of a Function of Several Variables without Using Derivatives," *British Computer Journal*, Vol.7, (1964), pp.155-162.

[26] Saidahmed, M. T. F., "A New Approch for Designing a Reduced-Order Controller of Linear Singular Systems," *IEEE Transactions on Automatic Control Systems Quality*, Vol.35, No.4(1990,4), pp.492-495.

[27] Shen, X-M.and Z. Gajic, "Optimal Reduced -Order Solution of the Weakly Coupled Discrete Riccati Equation", *IEEE Transactions on Automatic Control Systems Quality*, Vol.35, No.10(1990. 10), pp.1160-1162.

[28] Su, W., Gajic, Z. and X. Shen, "The Recursive Reduced-Order Solution of an Open -Loop Control Problem of Linear Singularly Perturbed Systems," *IEEE Transactions on Automatic Control Systems Quality*, Vol.37, No.2(1992. 2), pp.279-281.

[29] Teorell, T., "Kinetics of Distribution of Substances Administered in the Body," *Arch. Intern. Pharmacodynamics, Therapy*, No.57(1837), pp.205-240.

[30] Wiberg, D. M. and D. G. Dewolf, "A convergent Approximation of the Continuous -Time Optimal Parameter Estimater," *IEEE Transactions on Automatic Control Systems Quality*, Vol.38, No.4(1993,4), pp.529-545.

[31] Wilson, D. A., "Optimum Solution of Model

Reduction Problem," *Proceedings, IEEE*, Vol.117, No.6(1970), pp.1161-1165.

[32] Wismer, D. A., *Optimization Method for Large-Scale Systems*, McGraw-Hill Co., 1971.

[33] Yoo, B. W., *On Aggregation of Large-Scale Linear Dynamic System*, State University of New York at Buffalo, Ph. D. dissertation, September 1975.