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DERIVED CUP PRODUCT AND (STRICTLY) DERIVED GROUPS

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ABSTRACT. The purpose of this paper is to construct a ring with unity under the derived cup product on the cochain groups of the inverse system and an isomorphism which is useful as the computation of a derived group by deleting the suitable terms in the directed set D. Moreover we apply these results to the K-theory.

1. Introduction

Növeling [11] introduced the n-th derived functor $\varprojlim_{\alpha}^{(m)}(-)$, $m \geq 0$ on the category of inverse systems and morphisms of systems. Mathematicians have studied the properties of this functor and found the desirable exact sequences with respect to several functors ([4], [5]). Araki and Yoshimura [1] showed that if H is an additive (reduced) cohomology theory on arbitrary CW-complexes, then $E_2^{m,n} = \varprojlim_{\alpha}^{(m)}(H^n(X_\alpha))$. Huber and Meier [4] proved that $\ker(\theta : H^n(X) \to \varprojlim_{\alpha}^{(0)}(H^n(X_\alpha)))$ is isomorphic to the group $\mathrm{Pext}(F_{n-1}(X), A)$, where F_* is a homology theory of finite type and $\varprojlim_{\alpha}^{(m)}(H^n(X_\alpha)) = 0$ for all $m \geq 2$. In 1993 Mdzinarishvili and Spanier [10] established the long exact sequence for the derived functor $\varprojlim_{\alpha}^{(m)}(-)$, $m \geq 0$ with respect to a cohomology

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module $\bar{H}^*(X;A)$. The author [7] also found the 4-term exact sequence using the derived functor, stable homotopy group $\{X_\alpha, M^c(A,n)\}$ from the pointed 0-connected CW-space X_α to the co-Moore space $M^c(A,n)$ of type (A,n) and stable cohomotopy group $\pi_s^n(X_\alpha)$.

The main goal of this paper is to construct a ring $\sum_{m\geq 0} C^m(\mathcal{H}_m;R)$ with unity $e_\mathcal{H}$ under the derived cup product $\cup_d$ and a ring homomorphism between them. And we also construct an isomorphism between derived group $\varprojlim_\alpha^{(m)}(\mathcal{H};A)$ and strictly derived group $\hat{\varprojlim_\alpha}^{(m)}(s\mathcal{H};A)$ which is useful as the computation of a derived group by deleting the suitable terms in the directed set D.

2. The construction of a ring $\sum_{m\geq 0} C^m(\mathcal{H}_m;R)$

Let $\mathcal{X} = (X_\alpha, q_{\alpha\alpha'}, D)$ be a direct system of topological spaces X_α and continuous maps $q_{\alpha\alpha'} : X_\alpha \to X_{\alpha'}$, $\alpha \leq \alpha'$, over a directed set D. Let $D^m, m \geq 0$, be the set of all increasing m-sequences $(\alpha_0, \alpha_1, \cdots, \alpha_m)$, $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m, \alpha_i \in D$ and A be an abelian group. If $(\alpha_0, \alpha_1, \cdots, \alpha_m) \in D^m$, let $d(\alpha_0, \alpha_1, \cdots, \alpha_m)^j$ be an element of $D^{m-1}, m \geq 1$, obtained from $(\alpha_0, \alpha_1, \cdots, \alpha_m)$ by deleting the j-th term $\alpha_j, 0 \leq j \leq m$. We are easily able to obtain an inverse system $\mathcal{H} = (Hom(C_k(X_\alpha), A), \tilde{q}_{\alpha\alpha'}, D)$, via chain maps $q_{\alpha\alpha'\sharp} : C_\sharp(X_\alpha) \to C_\sharp(X_{\alpha'}), \alpha \leq \alpha'$, of abelian groups and group homomorphisms $\tilde{q}_{\alpha\alpha'} : Hom(C_k(X_{\alpha'}), A) \to Hom(C_k(X_\alpha), A), \alpha \leq \alpha'$ over the directed set D, where k is fixed in the set of all non-negative integers.

For each m-sequence $(\alpha_0, \alpha_1, \cdots, \alpha_m) \in D^m$, we associate a topological space $X_{(\alpha_0, \cdots, \alpha_m)}$ by the topological space X_{α_0} of the first index α_0, i.e., $X_{(\alpha_0, \cdots, \alpha_m)} = X_{\alpha_0}$.

For $0 \leq i \leq s$, we define continuous maps $F_i^s, B_i^s : \Delta^i \to \Delta^s$ by

$$F_i^s(t_0, \cdots, t_i) = (t_0, \cdots, t_i, 0, \cdots, 0)$$

and

$$B_i^s(t_0, \cdots, t_i) = (0, \cdots, 0, t_0, \cdots, t_i).$$

We call F_i^s a *front face* and B_i^s a *back face* ([3], [12], [13]) and easily check the following;

(1) $B_{l+m}^{l+m+n} \circ B_l^{l+m} = B_l^{l+m+n}$

(2) $F_{l+m}^{l+m+n} \circ F_l^{l+m} = F_l^{l+m+n}$

(3) $B_{l+m}^{l+m+n} \circ F_l^{l+m} = F_{l+n}^{l+m+n} \circ B_l^{l+n}$.

Let R be a commutative ring with unity 1 and let

$$\mathcal{H}_m = (Hom(C_m(X_{(\alpha_0,\cdots,\alpha_m)}), R), \tilde{q}_{\alpha\alpha'}, D^m)$$

be an inverse system which is different from the inverse system $\mathcal{H}$ in that the dimension m of the chain group $C_m(X_{(\alpha_0,\cdots,\alpha_m)})$ depends only on m-sequences $(\alpha_0,\cdots,\alpha_m) \in D^m, m \geq 0$, whereas the dimension $k(\geq 0)$ of the chain group $C_k(X_{(\alpha_0,\cdots,\alpha_m)})$ in $\mathcal{H}$ is fixed.

Define a group $C^m(\mathcal{H}_m; R)$ of the inverse system $\mathcal{H}_m$ with coefficients in R by

$$C^m(\mathcal{H}_m; R) = \prod_{(\alpha_0,\cdots,\alpha_m)\in D^m} Hom(C_m(X_{(\alpha_0,\cdots,\alpha_m)}), R), m \geq 0.$$

Let $p_{(\alpha_0,\cdots,\alpha_m)}$ be the projection of $C^m(\mathcal{H}_m; R)$ onto the group Hom $(C_m(X_{(\alpha_0,\cdots,\alpha_m)}), R)$ for each $(\alpha_0,\cdots,\alpha_m) \in D^m$. If c is an element of $C^m(\mathcal{H}_m; R)$, then we denote the element $(c)_{(\alpha_0,\cdots,\alpha_m)}$ of $Hom(C_m$ $(X_{(\alpha_0,\cdots,\alpha_m)}), R)$ by

$$(c)_{(\alpha_0,\cdots,\alpha_m)} = p_{(\alpha_0,\cdots,\alpha_m)}(c).$$

DEFINITION 2.1. Define a map

$$\cup_d : C^m(\mathcal{H}_m; R) \times C^n(\mathcal{H}_n; R) \to C^{m+n}(\mathcal{H}_{m+n}; R)$$

by

$$\begin{aligned} &\langle (c^m \cup_d c^n)_{(\gamma_0,\cdots,\gamma_{m+n})}, T_{m+n} \rangle \\ &\quad = \langle (c^m)_{(\gamma_0,\cdots,\gamma_m)}, T_{m+n} \circ F_m^{m+n} \rangle \\ &\qquad \cdot \langle (c^n)_{(\gamma_0,\gamma_{m+1},\cdots,\gamma_{m+n})}, T_{m+n} \circ B_n^{m+n} \rangle \end{aligned}$$

for each $(\gamma_0,\cdots,\gamma_{m+n}) \in D^{m+n}$, and $m, n = 0, 1, 2, \cdots$, where T_{m+n} : $\Delta^{m+n} \to X_{(\gamma_0,\cdots,\gamma_{m+n})} = X_{\gamma_0}$ is a singular $(m+n)$-simplex in X_{γ_0} and

$\langle\ ,\ \rangle$ denotes the evaluation map. The element $c^m \cup_d c^n$ of $C^{m+n}(\mathcal{H}_{m+n}; R)$ is called the *derived cup product* of $c^m \in C^m(\mathcal{H}_m; R)$ and $c^n \in C^n(\mathcal{H}_n; R)$.

LEMMA 2.2. *The map $\cup_d$ is bilinear.*

Proof. Let $c^m \in C^m(\mathcal{H}_m; R)$ and $c_1^n, c_2^n \in C^n(\mathcal{H}_n; R)$. Then we have

$$\begin{aligned}
&\langle (c^m \cup_d (c_1^n + c_2^n))_{(\gamma_0,\cdots,\gamma_{m+n})}, T_{m+n}\rangle \\
&\quad = \langle (c^m)_{(\gamma_0,\cdots,\gamma_m)}, T_{m+n} \circ F_m^{m+n}\rangle \\
&\qquad \cdot \langle (c_1^n + c_2^n)_{(\gamma_0,\gamma_{m+1},\cdots,\gamma_{m+n})}, T_{m+n} \circ B_n^{m+n}\rangle \\
&\quad = \langle (c^m)_{(\gamma_0,\cdots,\gamma_m)}, T_{m+n} \circ F_m^{m+n}\rangle \\
&\qquad \cdot [\langle (c_1^n)_{(\gamma_0,\gamma_{m+1},\cdots,\gamma_{m+n})}, T_{m+n} \circ B_n^{m+n}\rangle \\
&\qquad\quad + \langle (c_2^n)_{(\gamma_0,\gamma_{m+1},\cdots,\gamma_{m+n})}, T_{m+n} \circ B_n^{m+n}\rangle] \\
&\quad = [\langle (c^m)_{(\gamma_0,\cdots,\gamma_m)}, T_{m+n} \circ F_m^{m+n}\rangle \\
&\qquad \cdot \langle (c_1^n)_{(\gamma_0,\gamma_{m+1},\cdots,\gamma_{m+n})}, T_{m+n} \circ B_n^{m+n}\rangle] \\
&\qquad\quad + [\langle (c^m)_{(\gamma_0,\cdots,\gamma_m)}, T_{m+n} \circ F_m^{m+n}\rangle \\
&\qquad\qquad \cdot \langle (c_2^n)_{(\gamma_0,\gamma_{m+1},\cdots,\gamma_{m+n})}, T_{m+n} \circ B_n^{m+n}\rangle] \\
&\quad = \langle (c^m \cup_d c_1^n)_{(\gamma_0,\cdots,\gamma_{m+n})}, T_{m+n}\rangle \\
&\qquad + \langle (c^m \cup_d c_2^n)_{(\gamma_0,\cdots,\gamma_{m+n})}, T_{m+n}\rangle \\
&\quad = \langle ((c^m \cup_d c_1^n) + (c^m \cup_d c_2^n))_{(\gamma_0,\cdots,\gamma_{m+n})}, T_{m+n}\rangle.
\end{aligned}$$

for each $(\gamma_0, \cdots, \gamma_{m+n}) \in D^{m+n}$. The right distributive law is obtained by the similar calculation. □

THEOREM 2.3. *$\sum_{m \geq 0} C^m(\mathcal{H}_m; R)$ is a ring with unity $e_{\mathcal{H}}$ under the derived cup product.*

Proof. If $c^l \in C^l(\mathcal{H}_l; R)$, $c^m \in C^m(\mathcal{H}_m; R)$ and $c^n \in C^n(\mathcal{H}_n; R)$, then we obtain

$$
\begin{aligned}
&\langle (c^l \cup_d (c^m \cup_d c^n))_{(\gamma_0,\cdots,\gamma_{l+m+n})}, T_{l+m+n}\rangle \\
&= \langle (c^l)_{(\gamma_0,\cdots,\gamma_l)}, T_{l+m+n}\circ F_l^{l+m+n}\rangle \\
&\qquad \cdot \langle (c^m \cup_d c^n)_{(\gamma_0,\gamma_{l+1},\cdots,\gamma_{l+m+n})}, T_{l+m+n}\circ B_{m+n}^{l+m+n}\rangle \\
&= \langle (c^l)_{(\gamma_0,\cdots,\gamma_l)}, T_{l+m+n}\circ F_l^{l+m+n}\rangle \\
&\qquad \cdot [\langle (c^m)_{(\gamma_0,\gamma_{l+1},\cdots,\gamma_{l+m})}, T_{l+m+n}\circ B_{m+n}^{l+m+n}\circ F_m^{m+n}\rangle \\
&\qquad\qquad \cdot \langle (c^n)_{(\gamma_0,\gamma_{l+m+1},\cdots,\gamma_{l+m+n})}, T_{l+m+n}\circ B_{m+n}^{l+m+n}\circ B_n^{m+n}\rangle] \\
&= [\langle (c^l)_{(\gamma_0,\cdots,\gamma_l)}, T_{l+m+n}\circ F_l^{l+m+n}\rangle \\
&\qquad \cdot \langle (c^m)_{(\gamma_0,\gamma_{l+1},\cdots,\gamma_{l+m})}, T_{l+m+n}\circ B_{m+n}^{l+m+n}\circ F_m^{m+n}\rangle] \\
&\qquad\qquad \cdot \langle (c^n)_{(\gamma_0,\gamma_{l+m+1},\cdots,\gamma_{l+m+n})}, T_{l+m+n}\circ B_{m+n}^{l+m+n}\circ B_n^{m+n}\rangle \\
&= [\langle (c^l)_{(\gamma_0,\cdots,\gamma_l)}, T_{l+m+n}\circ F_l^{l+m+n}\rangle \\
&\qquad \cdot \langle (c^m)_{(\gamma_0,\gamma_{l+1},\cdots,\gamma_{l+m})}, T_{l+m+n}\circ F_{l+m}^{l+m+n}\circ B_m^{l+m}\rangle] \\
&\qquad\qquad \cdot \langle (c^n)_{(\gamma_0,\gamma_{l+m+1},\cdots,\gamma_{l+m+n})}, T_{l+m+n}\circ B_{m+n}^{l+m+n}\circ B_n^{m+n}\rangle \\
&= \langle (c^l \cup_d c^m)_{(\gamma_0,\cdots,\gamma_{l+m})}, T_{l+m+n}\circ F_{l+m}^{l+m+n}\rangle \\
&\qquad \cdot \langle (c^n)_{(\gamma_0,\gamma_{l+m+1},\cdots,\gamma_{l+m+n})}, T_{l+m+n}\circ B_n^{l+m+n}\rangle \\
&= \langle ((c^l \cup_d c^m) \cup_d c^n)_{(\gamma_0,\cdots,\gamma_{l+m+n})}, T_{l+m+n}\rangle
\end{aligned}
$$

for each $(\gamma_0,\cdots,\gamma_{l+m+n}) \in D^{l+m+n}$ and $l,m,n = 0,1,2,\cdots$. Define $e_{\mathcal{H}} \in C^0(\mathcal{H}_0;R)$ by

$$\langle (e_{\mathcal{H}})_{(\gamma_0)}, T_0\rangle = 1$$

for all singular 0-simplex $T_0 : \Delta^0 \to X_{\gamma_0}$ for each $\gamma_0 \in D^0$. Then we check that $e_{\mathcal{H}}$ is a unity on $\sum_{m\geq 0} C^m(\mathcal{H}_m;R)$. It follows from the derived cup product and Lemma 2.2 that $\sum_{m\geq 0} C^m(\mathcal{H}_m;R)$ is a ring with unity $e_{\mathcal{H}}$. □

Let $\mathcal{Y} = (Y_\beta, r_{\beta\beta'}, E)$ be a direct system of topological spaces Y_β and continuous maps $r_{\beta\beta'} : Y_\beta \to Y_{\beta'}, \beta \leq \beta'$ in another directed set E. We also have the inverse system $\mathcal{H}_{Y_m} = (Hom(C_m(Y_{(\beta_0,\cdots,\beta_m)}), A), \tilde{r}_{\beta\beta'}, E^m)$ of abelian groups and dual homomorphisms $\tilde{r}_{\beta\beta'}$ over the directed set E.

DEFINITION 2.4. (Compare [9]) A *map of direct systems* $h : \mathcal{X} \to \mathcal{Y}$ consists of an increasing function $\varphi : E \to D$ and of continuous maps $h_\beta : X_{\varphi(\beta)} \to Y_\beta$ such that

$$r_{\beta\beta'} \circ h_\beta = h_{\beta'} \circ q_{\varphi(\beta)\varphi(\beta')}$$

for $\beta \leq \beta'$ in E.

DEFINITION 2.5. For each $(\beta_0, \cdots, \beta_m) \in E^m$, we define the map $h^* : \sum_{m\geq 0} C^m(\mathcal{H}_{Y_m}; R) \to \sum_{m\geq 0} C^m(\mathcal{H}_m; R)$ by

$$\langle (h^*y)_{(\varphi(\beta_0),\cdots,\varphi(\beta_m))}, \sigma\rangle = \langle y_{(\beta_0,\cdots,\beta_m)}, h_{\beta_0\sharp}(\sigma)\rangle,$$

where σ is an m-chain of $X_{\varphi(\beta_0)}$ and $h_{\beta_0\sharp} : C_\sharp(X_{\varphi(\beta_0)}) \to C_\sharp(Y_{\beta_0})$ is a chain map induced by $h_{\beta_0} : X_{\varphi(\beta_0)} \to Y_{\beta_0}$.

In particular, if σ is an m-simplex T_m of $X_{\varphi(\beta_0)}$, then

$$\langle (h^*y)_{(\varphi(\beta_0),\cdots,\varphi(\beta_m))}, T_m\rangle = \langle y_{(\beta_0,\cdots,\beta_m)}, h_{\beta_0} \circ T_m\rangle.$$

THEOREM 2.6. *The map of direct systems $h : \mathcal{X} \to \mathcal{Y}$ induces a ring homomorphism $h^* : \sum_{m\geq 0} C^m(\mathcal{H}_{Y_m}; R) \to \sum_{m\geq 0} C^m(\mathcal{H}_m; R)$.*

Proof. Let $y^m \in C^m(\mathcal{H}_{Y_m}; R)$ and $y^n \in C^n(\mathcal{H}_{Y_n}; R)$. Then we obtain

$$\begin{aligned}
&\langle (h^*(y^m \cup_d y^n))_{(\varphi(\alpha_0),\cdots,\varphi(\alpha_{m+n}))}, T_{m+n}\rangle \\
&= \langle (y^m \cup_d y^n)_{(\alpha_0,\cdots,\alpha_{m+n})}, h_{\alpha_0} \circ T_{m+n}\rangle \\
&= \langle (y^m)_{(\alpha_0,\cdots,\alpha_m)}, h_{\alpha_0} \circ T_{m+n} \circ F_m^{m+n}\rangle \\
&\qquad \cdot \langle (y^n)_{(\alpha_0,\alpha_{m+1},\cdots,\alpha_{m+n})}, h_{\alpha_0} \circ T_{m+n} \circ B_n^{m+n}\rangle \\
&= \langle (h^*y^m)_{(\varphi(\alpha_0),\cdots,\varphi(\alpha_m))}, T_{m+n} \circ F_m^{m+n}\rangle \\
&\qquad \cdot \langle (h^*y^n)_{(\varphi(\alpha_0),\varphi(\alpha_{m+1}),\cdots,\varphi(\alpha_{m+n}))}, T_{m+n} \circ B_n^{m+n}\rangle \\
&= \langle (h^*y^m \cup_d h^*y^n)_{(\varphi(\alpha_0),\cdots,\varphi(\alpha_{m+n}))}, T_{m+n}\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle (h^*(y_1^m + y_2^m))_{(\varphi(\alpha_0),\cdots,\varphi(\alpha_m))}, T_m\rangle \\
&= \langle (y_1^m + y_2^m)_{(\alpha_0,\cdots,\alpha_m)}, h_{\alpha_0} \circ T_m\rangle \\
&= \langle (y_1^m)_{(\alpha_0,\cdots,\alpha_m)}, h_{\alpha_0} \circ T_m\rangle + \langle (y_2^m)_{(\alpha_0,\cdots,\alpha_m)}, h_{\alpha_0} \circ T_m\rangle \\
&= \langle (h^* y_1^m)_{(\varphi(\alpha_0),\cdots,\varphi(\alpha_m))}, T_m\rangle + \langle (h^* y_2^m)_{(\varphi(\alpha_0),\cdots,\varphi(\alpha_m))}, T_m\rangle \\
&= \langle (h^* y_1^m + h^* y_2^m)_{(\varphi(\alpha_0),\cdots,\varphi(\alpha_m))}, T_m\rangle.
\end{aligned}$$

If $e_{\mathcal{H}}$ and $e_{\mathcal{H}_Y}$ are unities of $\sum_{m\geq 0} C^m(\mathcal{H}_m; R)$ and $\sum_{m\geq 0} C^m(\mathcal{H}_{Y_m}; R)$ respectively, then we see that

$$\begin{aligned}
\langle (h^*(e_{\mathcal{H}_Y}))_{\varphi(\alpha_0)}, T_0\rangle &= \langle (e_{\mathcal{H}_Y})_{(\alpha_0)}, h_{\alpha_0} \circ T_0\rangle \\
&= 1
\end{aligned}$$

for all singular 0-simplex T_0. Thus $h^*(e_{\mathcal{H}_Y}) = e_{\mathcal{H}}$. □

3. The strictly derived group $\hat{\lim\limits_{\leftarrow\alpha}}^{(m)}(s\mathcal{H}; A)$

Following Araki, Yoshimura [1] and Nöbeling [11], we define an m-cochain group $C^m(\mathcal{H}; A)$ of the inverse system $\mathcal{H}$ with coefficients in A by

$$C^m(\mathcal{H}; A) = \prod_{(\alpha_0,\cdots,\alpha_m)\in D^m} Hom(C_k(X_{(\alpha_0,\cdots,\alpha_m)}), A), \; m \geq 0,$$

where k is fixed in the set of all non-negative integers. The coboundary operator $\delta^m : C^{m-1}(\mathcal{H}; A) \to C^m(\mathcal{H}; A)$, $m \geq 1$, which is slightly different from the previous papers, is defined by

$$(\delta^m c)_{(\alpha_0,\cdots,\alpha_m)} = (-1)^m[\tilde{q}_{\alpha_0\alpha_1}(c)_{d(\alpha_0,\cdots,\alpha_m)^0} + \sum_{j=1}^{m}(-1)^j(c)_{d(\alpha_0,\cdots,\alpha_m)^j}],$$

where c is an element of $C^{m-1}(\mathcal{H}; A)$. For $m = 0$, we put $\delta^0 = 0 : 0 \to C^0(\mathcal{H}; A)$.

Then we have a cochain complex $(C^*(\mathcal{H};A),\delta)$

$$0 \xrightarrow{\delta^0} C^0(\mathcal{H};A) \xrightarrow{\delta^1} C^1(\mathcal{H};A) \xrightarrow{\delta^2} C^2(\mathcal{H};A) \to \cdots \to C^{m-1}(\mathcal{H};A) \xrightarrow{\delta^m} C^m(\mathcal{H};A) \to \cdots .$$

The *m-th derived group* $\lim\limits_{\overleftarrow{\alpha}}{}^{(m)}(\mathcal{H};A)$ of the inverse system $\mathcal{H}$ with coefficients in A is defined by the cohomology group of this cochain complex $(C^*(\mathcal{H};A),\delta)$.

Let $\hat{D}^m$, $m \geq 0$ denote the set of all strictly increasing m-sequences $(\alpha_0,\cdots,\alpha_m), \alpha_0 < \alpha_1 < \cdots < \alpha_m$ in D and let

$$s\mathcal{H} = (Hom(C_k(X_{(\alpha_0,\cdots,\alpha_m)}),A), \tilde{q}_{\alpha_0\alpha_1}, D),\ (\alpha_0,\cdots,\alpha_m) \in \hat{D}^m$$

be a subinverse system of $\mathcal{H}$. We now construct a new m-cochain group $\hat{C}^m(s\mathcal{H};A)$ which is a subgroup of $C^m(\mathcal{H};A)$ of the inverse system $s\mathcal{H}$, i.e.,

$$\hat{C}^m(s\mathcal{H};A) = \prod_{(\alpha_0,\cdots,\alpha_m)\in\hat{D}^m} Hom(C_k(X_{(\alpha_0,\cdots,\alpha_m)}),A), m \geq 0,$$

and the coboundary operator $\hat{\delta}^m : \hat{C}^{m-1}(s\mathcal{H};A) \to \hat{C}^m(s\mathcal{H};A)$ is given by the restriction map $\hat{\delta}^m = \delta^m|_{\hat{C}^{m-1}(s\mathcal{H};A)}$ for $m \geq 1$ and $\hat{\delta}^0 = 0$.

DEFINITION 3.1. The *m-th strictly derived group* $\hat{\lim\limits_{\overleftarrow{\alpha}}}{}^{(m)}(s\mathcal{H};A)$ of the inverse system $s\mathcal{H}$ with coefficients in A is defined by

$$\hat{\lim_{\overleftarrow{\alpha}}}{}^{(m)}(s\mathcal{H};A) = \ker\hat{\delta}^{m+1}/\mathrm{im}\hat{\delta}^m, m \geq 0.$$

If we define two maps $f^m : C^m(\mathcal{H};A) \to \hat{C}^m(s\mathcal{H};A)$ and $g^m : \hat{C}^m(s\mathcal{H};A) \to C^m(\mathcal{H};A)$ by

$$(f^m d^m)_{(\alpha_0,\cdots,\alpha_m)} = (d^m)_{(\alpha_0,\cdots,\alpha_m)} \text{ for } (\alpha_0,\cdots,\alpha_m) \in \hat{D}^m$$

and

$$(g^m e^m)_{(\beta_0,\cdots,\beta_m)} = \begin{cases} (e^m)_{(\beta_0,\cdots,\beta_m)} & \text{for } (\beta_0,\cdots,\beta_m) \in \hat{D}^m \\ 0 & \text{for } (\beta_0,\cdots,\beta_m) \in (\hat{D}^m)^c \end{cases}$$

for all $d^m \in C^m(\mathcal{H};A)$ and $e^m \in \hat{C}^m(s\mathcal{H};A)$, then we have the following Lemma;

LEMMA 3.2. *The maps* $f = \{f^m\} : C^*(\mathcal{H}; A) \to \hat{C}^*(s\mathcal{H}; A)$ *and* $g = \{g^m\} : \hat{C}^*(s\mathcal{H}; A) \to C^*(\mathcal{H}; A)$ *are cochain maps.*

Proof. For any element d^{m-1} of $C^{m-1}(\mathcal{H}; A)$ and $(\alpha_0, \cdots, \alpha_m) \in \hat{D}^m$, we obtain that $d(\alpha_0, \cdots, \alpha_m)^j$, $0 \le j \le m$, is an element of $\hat{D}^{m-1}$ and

$$\begin{aligned}
&(\hat{\delta} f d^{m-1})_{(\alpha_0,\cdots,\alpha_m)} \\
&= (-1)^m[\tilde{q}_{\alpha_0\alpha_1}(fd^{m-1})_{d(\alpha_0,\cdots,\alpha_m)^0} + \sum_{j=1}^{m}(-1)^j(fd^{m-1})_{d(\alpha_0,\cdots,\alpha_m)^j}] \\
&= (-1)^m[\tilde{q}_{\alpha_0\alpha_1}(d^{m-1})_{d(\alpha_0,\cdots,\alpha_m)^0} + \sum_{j=1}^{m}(-1)^j(d^{m-1})_{d(\alpha_0,\cdots,\alpha_m)^j}] \\
&= (\delta d^{m-1})_{(\alpha_0,\cdots,\alpha_m)} \\
&= (f\delta d^{m-1})_{(\alpha_0,\cdots,\alpha_m)}.
\end{aligned}$$

To show that g is a cochain map, we have to show

$$(\delta g e^{m-1})_{(\beta_0,\cdots,\beta_m)} = (g\hat{\delta} e^{m-1})_{(\beta_0,\cdots,\beta_m)}$$

for any element e^{m-1} of $\hat{C}^{m-1}(s\mathcal{H}; A)$ and for each $(\beta_0, \cdots, \beta_m) \in D^m$.

(CASE 1) Let $(\beta_0, \cdots, \beta_m)$ be an element of $\hat{D}^m$. Then we compute

$$\begin{aligned}
&(\delta g e^{m-1})_{(\beta_0,\cdots,\beta_m)} \\
&= (-1)^m[\tilde{q}_{\beta_0\beta_1}(ge^{m-1})_{d(\beta_0,\cdots,\beta_m)^0} + \sum_{j=1}^{n}(ge^{m-1})_{d(\beta_0,\cdots,\beta_m)^j}] \\
&= (-1)^m[\tilde{q}_{\beta_0\beta_1}(e^{m-1})_{d(\beta_0,\cdots,\beta_m)^0} + \sum_{j=1}^{n}(e^{m-1})_{d(\beta_0,\cdots,\beta_m)^j}] \\
&= (-1)^m[(-1)^m(\delta e^{m-1})_{(\beta_0,\cdots,\beta_m)}] \\
&= (\hat{\delta} e^{m-1})_{(\beta_0,\cdots,\beta_m)} \quad (\hat{\delta} = \delta|_{\hat{C}^*(s\mathcal{H};A)}) \\
&= (g\hat{\delta} e^{m-1})_{(\beta_0,\cdots,\beta_m)}.
\end{aligned}$$

(Case 2) Let $(\beta_0, \cdots, \beta_m)$ be an element of $(\hat{D}^m)^c$. Since the m-sequence $(\beta_0, \cdots, \beta_m)$ is in $(\hat{D}^m)^c$, there exists an index $i, 0 \le i \le m-1$ such that $\beta_i = \beta_{i+1}$, i.e., $(\beta_0, \cdots, \beta_i, \beta_i = \beta_{i+1}, \cdots, \beta_m) \in (\hat{D}^m)^c$.

(a) If $\beta_0 = \beta_1$ and $\beta_1 < \beta_2 < \cdots < \beta_n$, then the $(m-1)$-sequence $d(\beta_0, \cdots, \beta_m)^s = (\beta_0, \beta_2, \cdots, \beta_m)$, for $s = 0, 1$ is an element of $\hat{D}^{m-1}$ and $d(\beta_0, \cdots, \beta_m)^j$, $2 \le j \le m$, is an element of $(\hat{D}^{m-1})^c$. We compute

$$\begin{aligned}
&(\delta g e^{m-1})_{(\beta_0,\beta_0,\beta_2,\cdots,\beta_m)} \\
&= (-1)^m [\tilde{q}_{\beta_0\beta_0} (g e^{m-1})_{d(\beta_0,\beta_0,\beta_2,\cdots,\beta_m)^0} \\
&\qquad + \sum_{j=1}^{m} (-1)^j (g e^{m-1})_{d(\beta_0,\beta_0,\beta_2,\cdots,\beta_m)^j}] \\
&= (-1)^m [id (g e^{m-1})_{(\beta_0,\beta_2,\cdots,\beta_m)} - (g e^{m-1})_{(\beta_0,\beta_2,\cdots,\beta_m)} \\
&\qquad + \sum_{j=2}^{m} (-1)^j (g e^{m-1})_{d(\beta_0,\beta_0,\beta_2,\cdots,\beta_m)^j}] \\
&= (-1)^m [(e^{m-1})_{(\beta_0,\beta_2,\cdots,\beta_m)} - (e^{m-1})_{(\beta_0,\beta_2,\cdots,\beta_m)} + 0] \\
&= 0,
\end{aligned}$$

where id is the identity on $C^*(\mathcal{H}; A)$. Since $(\beta_0, \beta_0, \beta_2, \cdots, \beta_m)$ is the element of $(\hat{D}^m)^c$, we have $(g\hat{\delta} e^{m-1})_{(\beta_0,\beta_0,\beta_2,\cdots,\beta_m)} = 0$ which is equal to $(\delta g e^{m-1})_{(\beta_0,\beta_0,\beta_2,\cdots,\beta_m)}$.

(b) If $\beta_i = \beta_{i+1}, 1 \le i \le m-1$, i.e., $\beta_0 < \beta_1 < \cdots < \beta_i = \beta_{i+1} < \beta_{i+2} < \cdots < \beta_m$, then $d(\beta_0, \cdots \beta_i, \beta_i, \beta_{i+2}, \cdots, \beta_m)^j \in (\hat{D}^{m-1})^c$, for $0 \le j \le i-1$, $i+2 \le j \le m$ and

$$\begin{aligned}
&(\delta g e^{m-1})_{(\beta_0,\beta_1,\cdots,\beta_i,\beta_i,\beta_{i+2},\cdots,\beta_m)} \\
&= (-1)^m [\tilde{q}_{\beta_0\beta_1} (g e^{m-1})_{d(\beta_0,\cdots,\beta_i,\beta_i,\beta_{i+2},\cdots,\beta_m)^0} \\
&\qquad + \sum_{j=1}^{m} (-1)^j (g e^{m-1})_{d(\beta_0,\cdots,\beta_i,\beta_i,\beta_{i+2},\cdots,\beta_m)^j}] \\
&= (-1)^m [\tilde{q}_{\beta_0\beta_1}(0) + 0 + (-1)^i (e^{m-1})_{d(\beta_0,\cdots,\beta_i,\beta_i,\beta_{i+2},\cdots,\beta_m)^i} \\
&\qquad + (-1)^{i+1} (e^{m-1})_{d(\beta_0,\cdots,\beta_i,\beta_i,\beta_{i+2},\cdots,\beta_m)^{i+1}} + 0]
\end{aligned}$$

$$= (-1)^m[0 + (-1)^i(e^{m-1})_{(\beta_0,\cdots,\beta_i,\beta_{i+2},\cdots,\beta_m)} + (-1)^{i+1}(e^{m-1})_{(\beta_0,\cdots,\beta_i,\beta_{i+2},\cdots,\beta_m)}]$$
$$= 0$$
$$= (g\hat{\delta}e^{m-1})_{(\beta_0,\cdots,\beta_i,\beta_i,\beta_{i+2},\cdots,\beta_m)} \text{ (by the definition of } g).$$

(CASE 3) If the number of equal terms of the m-sequence $(\beta_0,\cdots,\beta_m) \in (\hat{D}^m)^c$ is greater than 2, then we have the result from the definition of the map g. □

Let $f^* : \varprojlim_\alpha{}^{(m)}(\mathcal{H};A) \to \hat{\varprojlim_\alpha}{}^{(m)}(s\mathcal{H};A)$ be a homomorphism induced by the cochain map $f = \{f^m\} : C^*(\mathcal{H};A) \to \hat{C}^*(s\mathcal{H};A)$ and let $a(\alpha_0,\cdots,\alpha_m)^j$ be an element of D^{m+1} obtained from $(\alpha_0,\cdots,\alpha_m) \in D^m$ by adding the j-th term $\alpha_j, 0 \le j \le m$, i.e., $a(\alpha_0,\cdots,\alpha_m)^j = (\alpha_0,\cdots,\alpha_j,\alpha_j,\alpha_{j+1},\cdots,\alpha_m) \in D^{m+1}$. Define a map $u : (\hat{D}^m)^c \to \{0,1,\cdots,m-1\}$ by

$$u(\alpha_0,\cdots,\alpha_i,\alpha_i = \alpha_{i+1},\alpha_{i+2},\cdots,\alpha_m) = u(i),\ 1 \le i \le m-1,$$

where $u(i)$ is the unique index i such that $\alpha_0 < \alpha_1 < \cdots < \alpha_i = \alpha_{i+1}$. Then we have the following Theorem;

THEOREM 3.3. *The map* $f^* : \varprojlim_\alpha{}^{(m)}(\mathcal{H};A) \to \hat{\varprojlim_\alpha}{}^{(m)}(s\mathcal{H};A)$ *is an isomorphism.*

Proof. Define a map $H : C^{m+1}(\mathcal{H};A) \to C^m(\mathcal{H};A)$ by

$$(Hd^{m+1})_{(\alpha_0,\cdots,\alpha_m)} = \begin{cases} 0 & \text{for } (\alpha_0,\cdots,\alpha_m) \in \hat{D}^m \\ (-1)^{m+u(i)}(d^{m+1})_{a(\alpha_0,\cdots,\alpha_m)^i} & \text{for } (\alpha_0,\cdots,\alpha_m) \in (\hat{D}^m)^c, \end{cases}$$

where $d^{m+1} \in C^{m+1}(\mathcal{H};A)$. We now show that H is a cochain homotopy between $g \circ f$ and identity $id_{C^*(\mathcal{H};A)}$.

(CASE 1) Let $(\alpha_0, \cdots, \alpha_m)$ be an element of $\hat{D}^m$. We compute

$$(H\delta d^m)_{(\alpha_0,\cdots,\alpha_m)} + (\delta H d^m)_{(\alpha_0,\cdots,\alpha_m)}$$

$$= 0 + (-1)^m[\tilde{q}_{\alpha_0\alpha_1}(Hd^m)_{d(\alpha_0,\cdots,\alpha_m)^0} + \sum_{j=1}^{m}(-1)^j(Hd^m)_{d(\alpha_0,\cdots,\alpha_m)^j}]$$

$$= 0.$$

On the other hand,

$$\begin{aligned}(gfd^m)_{(\alpha_0,\cdots,\alpha_m)} - (id\ d^m)_{(\alpha_0,\cdots,\alpha_m)} &= (fd^m)_{(\alpha_0,\cdots,\alpha_m)} - (d^m)_{(\alpha_0,\cdots,\alpha_m)} \\ &= (d^m)_{(\alpha_0,\cdots,\alpha_m)} - (d^m)_{(\alpha_0,\cdots,\alpha_m)} \\ &= 0.\end{aligned}$$

(CASE 2) Let $(\alpha_0, \cdots, \alpha_m)$ be an element of $(\hat{D}^m)^c$.

(a) If $\alpha_0 = \alpha_1, \alpha_1 < \alpha_2 < \cdots < \alpha_m$, then we have

(1) $u(\alpha_0, \cdots, \alpha_m) = u(0) = 0$

(2) $d(\alpha_0, \cdots, \alpha_m)^s = (\alpha_0, \alpha_2, \cdots, \alpha_m) \in \hat{D}^{m-1}$, for $s = 0, 1$

(3) $d(\alpha_0, \cdots, \alpha_m)^j \in (\hat{D}^{m-1})^c$, for $2 \leq j \leq m$.

and

$$\begin{aligned}&(\delta H d^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)} \\ &= (-1)^m[\tilde{q}_{\alpha_0\alpha_0}(Hd^m)_{d(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)^0} \\ &\qquad + \sum_{j=1}^{m}(-1)^j(Hd^m)_{d(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)^j}] \\ &= (-1)^m[id(0) - (Hd^m)_{(\alpha_0,\alpha_2,\cdots,\alpha_m)} \\ &\qquad + \sum_{j=2}^{m}(-1)^j(Hd^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_m)}] \\ &= (-1)^m[0 - 0 \\ &\qquad + \sum_{j=2}^{m}(-1)^j(-1)^{m-1}(d^m)_{a(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_m)^0}] \\ &= \sum_{j=2}^{m}(-1)^{2m+j-1}(d^m)_{(\alpha_0,\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots\alpha_m)}.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&(H\delta d^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)}\\
&\quad= (-1)^m(\delta d^m)_{a(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)^0}\\
&\quad= (-1)^m(\delta d^m)_{(\alpha_0,\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)}\\
&\quad= (-1)^m(-1)^{m+1}[\tilde{g}_{\alpha_0\alpha_0}(d^m)_{d(\alpha_0,\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)^0}\\
&\qquad+\sum_{j=1}^{m+1}(-1)^j(d^m)_{d(\alpha_0,\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)^j}]\\
&\quad= (-1)^{2m+1}[(d^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)} - (d^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)}\\
&\qquad+(d^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)}\\
&\qquad+\sum_{j=3}^{m+1}(-1)^j(d^m)_{(\alpha_0,\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_m)}]\\
&\quad= (-1)^{2m+1}(d^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)}\\
&\qquad+\sum_{j=3}^{m+1}(-1)^{2m+j+1}(d^m)_{(\alpha_0,\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_m)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(\delta H d^m)_{(\alpha_0,\cdots,\alpha_m)} + (H\delta d^m)_{(\alpha_0,\cdots,\alpha_m)}\\
&\quad= (-1)^{2m+1}(d^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)}\\
&\quad= (gfd^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)} - (id\, d^m)_{(\alpha_0,\alpha_0,\alpha_2,\cdots,\alpha_m)}
\end{aligned}
$$

so that H is a cochain homotopy between $g \circ f$ and identity map $id_{C^*(\mathcal{H};A)}$.

(b) If $\alpha_i = \alpha_{i+1}$ for $1 \leq i \leq m-1$, $\alpha_0 < \alpha_1 < \cdots < \alpha_i = \alpha_{i+1} < \alpha_{i+2} < \cdots < \alpha_m$, then we can compute the following;

$$
\begin{aligned}
&(\delta H d^m)_{(\alpha_0,\alpha_1,\cdots,\alpha_i,\alpha_i,\cdots,\alpha_m)} \\
&\quad = (-1)^m[\tilde{q}_{\alpha_0\alpha_1}(Hd^m)_{d(\alpha_0,\cdots,\alpha_i,\alpha_i,\cdots,\alpha_m)^0} \\
&\qquad + \sum_{j=1}^{m}(-1)^j(Hd^m)_{d(\alpha_0,\cdots,\alpha_i,\alpha_i,\cdots,\alpha_m)^j}] \\
&\quad = (-1)^m[\tilde{q}_{\alpha_0\alpha_1}(-1)^{m+i-2}(d^m)_{(\alpha_1,\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_m)} \\
&\qquad + \sum_{j=1}^{i-1}(-1)^j(-1)^{m+i-2}(d^m)_{(\alpha_0,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_m)} \\
&\qquad + \sum_{j=i+2}^{m}(-1)^j(-1)^{m+i-1}(d^m)_{(\alpha_0,\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_m)}]
\end{aligned}
$$

and

$$
\begin{aligned}
&(H\delta d^m)_{(\alpha_0,\alpha_1,\cdots,\alpha_i,\alpha_i,\cdots,\alpha_m)} \\
&\quad = (-1)^{m+i}(\delta d^m)_{(\alpha_0,\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_m)} \\
&\quad = (-1)^{m+i}(-1)^{m+1}[\tilde{q}_{\alpha_0\alpha_1}(d^m)_{d(\alpha_0,\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_m)^0} \\
&\qquad + \sum_{j=1}^{m+1}(-1)^j(d^m)_{d(\alpha_0,\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_m)^j}] \\
&\quad = (-1)^{2m+i+1}[\tilde{q}_{\alpha_0\alpha_1}(d^m)_{(\alpha_1,\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_m)} \\
&\qquad + \sum_{j=1}^{i-1}(-1)^j(d^m)_{(\alpha_0,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_m)} \\
&\qquad + (-1)^{i+2}(d^m)_{(\alpha_0,\cdots,\alpha_i,\alpha_i,\alpha_{i+2},\cdots,\alpha_m)} \\
&\qquad + \sum_{j=i+3}^{m+1}(-1)^j(d^m)_{(\alpha_0,\cdots,\alpha_i,\alpha_i,\alpha_i,\cdots,\alpha_{j-1},\alpha_{j+1},\cdots,\alpha_m)}].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&(\delta H d^m)_{(\alpha_0,\alpha_1,\cdots,\alpha_i,\alpha_i,\cdots,\alpha_m)} + (H\delta d^m)_{(\alpha_0,\alpha_1,\cdots,\alpha_i,\alpha_i,\cdots,\alpha_m)} \\
&\quad = -(d^m)_{(\alpha_0,\alpha_1,\cdots,\alpha_i,\alpha_i,\alpha_{i+2},\cdots,\alpha_m)} \\
&\quad = (gfd^m)_{(\alpha_0,\alpha_1,\cdots,\alpha_i,\alpha_i,\cdots,\alpha_m)} - (id\,d^m)_{(\alpha_0,\alpha_1,\cdots,\alpha_i,\alpha_i,\cdots,\alpha_m)}.
\end{aligned}
$$

(CASE 3) Let the number of equal terms of the m-sequence $(\alpha_0, \alpha_1, \cdots, \alpha_m) \in (\hat{d}^m)^c$ be greater than 2. Since the computation of $\delta H d^m + H\delta d^m$ depends only on the first two equal terms $\alpha_i = \alpha_{i+1}, 0 \leq i \leq m-1$, we obtain the same result so that H is a cochain homotopy between $g \circ f$ and identity map $id_{C^*(\mathcal{H};A)}$.

Clearly, if $e^n \in \hat{C}^n(s\mathcal{H}; A)$, then

$$\begin{aligned}(fge^n)_{(\alpha_0,\cdots,\alpha_m)} &= (ge^n)_{(\alpha_0,\cdots,\alpha_m)} \\ &= (e^n)_{(\alpha_0,\cdots,\alpha_m)} \\ &= (id\, e^n)_{(\alpha_0,\cdots,\alpha_m)}.\end{aligned}$$

Thus, we complete the proof. □

(APPLICATION TO THE K-THEORY) Let $s\alpha\alpha' : X_\alpha \to X_{\alpha'}$, $\alpha \leq \alpha'$ be a continuous map. Then the map $Vect_F(s_{\alpha\alpha'}) : Vect_F(X_{\alpha'}) \to Vect_F(X_\alpha)$ between the set of all isomorphism classes of vector bundles on $X_{\alpha'}$ and X_α respectively is defined by $Vect_F(s_{\alpha\alpha'})([\xi]) = [s^\sharp_{\alpha\alpha'}(\xi)]$, where $[\xi]$ means the isomorphism class of ξ over $X_{\alpha'}$ and F is a field over $\mathbb{R}$ or $\mathbb{C}$ and $s^\sharp_{\alpha\alpha'}(\xi)$ is the induced bundle of ξ under $s_{\alpha\alpha'}$. Moreover we obtain the group homomorphism $s^*_{\alpha\alpha'} : K(X_{\alpha'}) \to K(X_\alpha)$ of K-groups([2],[6],[8]) and take the m-cochain group $C^m(\mathcal{K})$ of the inverse system $\mathcal{K} = (K(X_\alpha), s^*_{\alpha\alpha'}, D)$, instead of the inverse system $\mathcal{H}$, of K-groups $K(X_\alpha)$ and group homomorphisms $s^*_{\alpha\alpha'}$ over the directed set D by

$$C^m(\mathcal{K}) = \prod_{(\alpha_0,\cdots,\alpha_m)\in D^m} K(X_{(\alpha_0,\cdots,\alpha_m)}),\ m \geq 0.$$

Let $\tilde{D}^m, m \geq 0$ be the set of all strictly increasing non-negative integral m-sequence $(\gamma_0, \gamma_1, \cdots, \gamma_m)$ and let $s\tilde{\mathcal{K}} = (K(X_{(\gamma_0,\gamma_1,\cdots,\gamma_m)}), s^*_{\gamma_0\gamma_1}, D)$, $(\gamma_0, \gamma_1, \cdots, \gamma_m) \in \tilde{D}^m$ be a subinverse system of $\mathcal{K}$. Now we also construct a cochain group $\tilde{C}^m(s\tilde{\mathcal{K}})$ by

$$\tilde{C}^m(s\tilde{\mathcal{K}}) = \prod_{(\gamma_0,\cdots,\gamma_m)\in \tilde{D}^m} K(X_{(\gamma_0,\cdots,\gamma_m)}),\ m \geq 0.$$

The coboundary operator $\tilde{\delta}^m : \tilde{C}^{m-1}(s\tilde{\mathcal{K}}) \to \tilde{C}^m(s\tilde{\mathcal{K}})$ is given by the corresponding δ^m in section 2.

DEFINITION 3.4. The *m-th strictly integral derived group* $\underset{\overleftarrow{\alpha}}{\tilde{\lim}}^{(m)}(s\tilde{\mathcal{K}})$ of the inverse system $s\tilde{\mathcal{K}} = (K(X_{(\gamma_0,\gamma_1,\cdots,\gamma_m)}), s^*_{\gamma_0\gamma_1}, D), (\gamma_0,\gamma_1,\cdots,\gamma_m) \in \tilde{D}^m$ is given by

$$\underset{\overleftarrow{\alpha}}{\tilde{\lim}}^{(m)}(s\tilde{\mathcal{K}}) = \ker\tilde{\delta}^{m+1}/\mathrm{im}\tilde{\delta}^m, \ m \geq 0.$$

COROLLARY 3.5. $\underset{\overleftarrow{\alpha}}{\lim}^{(m)}(\mathcal{K}) \cong \underset{\overleftarrow{\alpha}}{\hat{\lim}}^{(m)}(s\mathcal{K}) \cong \underset{\overleftarrow{\alpha}}{\tilde{\lim}}^{(m)}(s\tilde{\mathcal{K}})$, *where* $s\mathcal{K}$ *is an inverse system indexed by* $\hat{D}^m$.

Proof. By Theorem 3.3 , we can also construct two cochain maps $f_1 : \hat{C}^*(s\mathcal{K}) \to \tilde{C}^*(s\tilde{\mathcal{K}})$ and $g_1 : \tilde{C}^*(s\tilde{\mathcal{K}}) \to \hat{C}^*(s\mathcal{K})$ defined by the same method of f and g. Therefore we have the proof. □

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