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ON THE NONVANISHING OF TOR

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ABSTRACT. Using spectral sequences we calculate the highest non-vanishing index of Tor for modules of finite projective dimension.

1. Introduction

Throughout this paper, every ring is assumed to be commutative and noetherian with identity. For an R-module M, the projective dimension of M is written as pd$M$. The purpose of this paper is to investigate the index, $\sup\{i \mid \mathrm{Tor}_i(M,N) \neq 0\}$ for R-modules M and N. From now on we let

$$s := \sup\{i \mid \mathrm{Tor}_i(M,N) \neq 0\}.$$

Our main result(Theorem 2.4) generalizes a formula due to Serre.

THEOREM 1.1. [7, V. Theorem 4] *Let (R,m) be a regular local ring and M and N be finitely generated nonzero R-modules with $l(M \otimes N) < \infty$. Then*

$$s = pdM + pdN - \dim R.$$

Earlier than Theorem 1.1, M. Auslander has connected s with the depth of Tor module.

THEOREM 1.2. [1, Theorem 1.2] *Let M and N be nonzero finite modules over a local ring R such that $pdM < \infty$. If either depth $(Tor_s^R$ $(M,N)) \leq 1$ or $s = 0$, then*

$$s = pdM - depthN + depth\ (Tor_s^R(M,N)).$$

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Recently, S. Iyengar has defined the depth of a complex and has connected it with the depth of homology (Tor) module. Let I be an ideal of R generated by n elements $x_1, \cdots, x_n$ and $K. = K.(x_1, \cdots, x_n)$ be the Koszul complex on $x_1, \cdots, x_n$. For a complex of R-modules $M.$, define the $I - depth\ of\ M.$ by (cf. [M, 16.8])

$$\mathrm{depth}_I(M.) := n - \sup\{i \mid H_i(K. \otimes_R M.) \neq 0\}.$$

If (R, m) is local with maximal ideal m, then write depth $(M.)$ instead of $\mathrm{depth}_m(M.)$. Note that $\mathrm{depth}_I(M.) = \infty$ if and only if $H(K. \otimes_R M.) = 0$.

THEOREM 1.3. [4, Theorem 2.3] *Let $L.$ be a complex over a local ring R. If $q = \sup\{i \mid H_i(L.) \neq 0\}$ is finite, and $depth H_q(L.) - q \leq depth H_i(L.) - i$ for all $i \leq q$, then*

$$q = depth\ (H_q(L.)) - depth\ (L.).$$

Suppose that $\mathrm{pd}M < \infty$ and $F.$ be a free resolution of a finite R-module M over the local ring M. Let $L.$ be the complex $F. \otimes N$. If depth $(\mathrm{Tor}_s^R(M, N)) \leq 1$ or $s = 0$, then depth $(\mathrm{Tor}_s^R(M, N)) - s \leq$ depth $(\mathrm{Tor}_i^R(L.)) - i$ for all $i \leq s$. Hence by Theorem 1.3, $s =$ depth $(\mathrm{Tor}_s^R(M, N))$ − depth $(F. \otimes N)$. It is also due to Iyengar [4, Corollary 2.2] that depth (N) = depth $(F. \otimes N) + \mathrm{pd}M$. Therefore, Auslander's formula (Theorem 1.2) has reproved.

2. Nonvanishing of Tor

In this section we generalize Theorem 1.1 for any local ring. Spectral sequences of double (triple) complexes are main tools of computation.

DEFINITION 2.1. Let L, M and N be R-modules, $P.$, $F.$ and $G.$ be projective resolutions of L, M and N respectively. Define

$$\mathrm{Tor}_i^R(L, M, N) := H_i(P. \otimes F. \otimes G.).$$

Since a projective module is a direct summand of a free module, we can formulate the following lemma.

LEMMA 2.2. *If $C.$ is a complex and P is a projective, then $H_i(P \otimes C.) \cong P \otimes H_i(C.)$.*

Applying Lemma 2.2 to compute Tor of the double complexes in Definition 2.1, we obtain the following spectral sequence.

THEOREM 2.3. $Tor_p^R(L, Tor_q^R(M,N)) \Longrightarrow Tor_{p+q}^R(L,M,N)$.

THEOREM 2.4. *Let (R,m) be a local ring and M, N be finite nonzero R-modules of finite projective dimension. Then*

$$s \geq pdM + pdN - depthR.$$

If $Tor_s^R(M,N)$ has an associated prime whose grade is equal to $depthR$, then the equality holds in the above formula.

Proof. Let $\mathrm{depth}R = n$ and $\underline{x} = (x_1, \cdots, x_n)$ be a maximal R-sequence. Consider the spectral sequences in Theorem 2.3.

$$\mathrm{Tor}_p^R(M, \mathrm{Tor}_q^R(R/\underline{x}, N)) \Rightarrow \mathrm{Tor}_{p+q}^R(R/\underline{x}, M, N),$$

$$\mathrm{Tor}_p^R(R/\underline{x}, \mathrm{Tor}_q^R(M,N)) \Rightarrow \mathrm{Tor}_{p+q}^R(R/\underline{x}, M, N).$$

Note that $\mathrm{Tor}_p^R(M, \mathrm{Tor}_q^R(R/\underline{x}, N)) = 0$ if $p \geq \mathrm{pd}M + 1$ or $q \geq \mathrm{pd}N + 1$. As $x_1, \cdots, x_n$ are a maximal regular sequence, $(0 :_{R/\underline{x}} m) \neq 0$. Computing $\mathrm{Tor}_{\mathrm{pd}N}^R(R/\underline{x}, N)$ from the minimal free resolution of N, we obtain $\mathrm{Tor}_{\mathrm{pd}N}^R(R/\underline{x}, N) \neq 0$, and it is a submodule of finite free $R/(\underline{x})$-module. Hence

$$\mathrm{Tor}_{\mathrm{pd}M}^R(M, \mathrm{Tor}_{\mathrm{pd}N}^R(R/\underline{x}, N)) \neq 0.$$

On the other hand, $\mathrm{Tor}_p^R(R/\underline{x}, \mathrm{Tor}_q^R(M,N)) = 0$ if $p \geq n+1$ or $q \geq s+1$. It is due to the maximal cycle principle [6] that

$$\mathrm{pd}M + \mathrm{pd}N = \sup\{i \mid \mathrm{Tor}_i^R(R/\underline{x}, M, N) \neq 0\} \leq n + s.$$

Suppose that $\mathrm{Tor}_s^R(M,N)$ has an associated prime P of grade n. Choose a maximal R-sequence $\underline{x} = (x_1, \cdots, x_n)$ in P. Then

$$\mathrm{Tor}_n^R(R/\underline{x}, \mathrm{Tor}_s^R(M,N)) = (0 :_{\mathrm{Tor}_s^R(M,N)} \underline{x}) \neq 0.$$

Therefore

$$n + s = \mathrm{pd}M + \mathrm{pd}N = \sup\{i \mid \mathrm{Tor}_i^R(R/\underline{x}, M, N) \neq 0\}.$$

This concludes the proof of Theorem 2.4. □

Notice that the equality, $n+s = \mathrm{pd}M + \mathrm{pd}N$, does *not* depend on the choice of the maximal R-sequence. Thus if $\mathrm{Tor}_n^R(R/\underline{x}, \mathrm{Tor}_s^R(M,N)) \neq 0$, for a maximal R-sequence $\underline{x} = (x_1, \cdots, x_n)$, then for any R-sequence $\underline{y} = (y_1, \cdots, y_n)$,

$$\mathrm{Tor}_n^R(R/\underline{y}, \mathrm{Tor}_s^R(M,N)) \cong \mathrm{Tor}_{\mathrm{pd}M}^R(M, \mathrm{Tor}_{\mathrm{pd}N}^R(R/\underline{y}, N)) \neq 0.$$

In this case, depth $(\mathrm{Tor}_s^R(M,N)) = 0$. We may ask whether there is a natural map between $\mathrm{Tor}_n^R(R/\underline{x}, \mathrm{Tor}_s^R(M,N))$ and $\mathrm{Tor}_n^R(R/\underline{y}, \mathrm{Tor}_s^R$ $(M,N))$ for two maximal R-sequence $\underline{x}$ and $\underline{y}$.

C. Huneke has pointed out that $s \geq \mathrm{pd}M - \mathrm{depth}N$ without assuming that N is of finite projective dimension (cf. [2]). Suppose that $s < \mathrm{pd}M - \mathrm{depth}N$. Let $\mathrm{pd}M = m$, $\mathrm{pd}M - \mathrm{depth}N = l$ and $F.$ be a minimal free resolution of M. Note that

$$0 \longrightarrow F_m \xrightarrow{\phi} \cdots \longrightarrow F_l \longrightarrow F_{l-1}$$

is exact. Since $s < l$,

$$0 \longrightarrow F_m \otimes N \xrightarrow{\phi \otimes 1} \cdots \longrightarrow F_l \otimes N \longrightarrow F_{l-1} \otimes N$$

is also exact. Due to the Buchsbaum-Eisenbud criterion of exactness [3], $\mathrm{depth}_{I(\phi)} N \geq m - l + 1$. Hence $\mathrm{depth}N \geq m - l + 1$. This is a contradiction and $s \geq \mathrm{pd}M - \mathrm{depth}N$.

If $M \otimes N$ has the maximal grade then so does $\mathrm{Tor}_s^R(M,N)$. So we obtain the following corollary

COROLLARY 2.5. *Let (R, m) be a local ring and M and N be finitely generated nonzero R-modules of finite projective dimension. If grade $(M \otimes N) =$ depth$R$, then*

$$s = pdM + pdN - depthR.$$

If $l(M \otimes N) < \infty$, then $\mathrm{ann}M + \mathrm{ann}N$ is m-primary and its grade is equal to depth$R$. Hence we obtain a corollary similar to Theorem 1.1.

COROLLARY 2.6. *Let (R, m) be a local ring and M and N be finitely generated nonzero R-modules of finite projective dimension. If $l(M \otimes N) < \infty$, then*

$$s = \text{pd}M + \text{pd}N - \text{depth}R.$$

The following question have been asked by M. Auslander [1].

PROBLEM 2.7. *Let (R, m) be a local ring and M, N be finite nonzero R-modules. Suppose that M is of finite projective dimension. Is it true that*

$$\text{pd}M - \text{depth}N = j - \text{depth}\,(\text{Tor}_j^R(M, N))$$

for some j?

If the equation in Problem 2.7 holds, then write it as $\text{TF}_j(M, N)$. The depth of Tor is 'unknown' in general and it plays a role. For some restrictive cases the above formula is known:

(1) (Theorem 1.2) If either depth $(\text{Tor}_s^R(M, N)) \leq 1$ or $s = 0$, then $\text{TF}_s(M, N)$ holds.
(2) (Theorem 2.4) If N is of finite projective dimension and Tor_s^R (M, N) has an associated prime whose grade is equal to depth$R$, then $\text{TF}_s(M, N)$ holds.

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