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SINGLY GENERATED DUAL OPERATOR ALGEBRAS WITH PROPERTIES $(\mathbb{A}_{m,n})$

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ABSTRACT. We discuss dual algebras generated by a contraction and properties $(\mathbb{A}_{m,n})$ which arise in the study of the problem of solving systems of the predual of a dual algebra. In particular, we study membership for the class $\mathbb{A}_{1,\aleph_0}$. As some examples we consider dual algebras generated by a Jordan block.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Suppose that $\mathcal{C}_1 = \mathcal{C}_1(\mathcal{H})$ be the trace class in $\mathcal{L}(\mathcal{H})$. Then it is well known that the dual space $\mathcal{C}_1^*$ is isometrically isomorphic to $\mathcal{L}(\mathcal{H})$ under the pairing $\langle T, L\rangle = tr(TL)$, $T \in \mathcal{L}(\mathcal{H}), L \in \mathcal{C}_1$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the weak*-topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the dual algebra generated by T. The theory of dual algebras is applied to the study of invariant subspaces, reflexivity and dilation theory. This theory is closely related to properties $(\mathbb{A}_{m,n})$ which arise in the study of the problem of solving systems of the predual of a dual algebra (cf. [2]). In this paper we discuss dual algebras generated by a contraction and properties $(\mathbb{A}_{m,n})$.

A brief outline of this work is as follows: in Section 2 we discuss some sufficient conditions for the membership of the classes $\mathbb{A}_{m,n}$ which will

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be defined below. In Section 3 we study dual algebras generated by Jordan operators and properties $(\mathbb{A}_{m,n})$.

Now let us recall some notation and terminology from [2]. Suppose that $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$ and let ${}^{\perp}\mathcal{A}$ denote the preannihilator of $\mathcal{A}$ in $\mathcal{C}_1$. Let $\mathcal{Q}_{\mathcal{A}}$ denote the quotient space $\mathcal{C}_1/{}^{\perp}\mathcal{A}$. Then one knows that $\mathcal{A}$ is the dual space of $\mathcal{Q}_{\mathcal{A}}$ and that the duality is given by $\langle T, [L]_{\mathcal{A}}\rangle = tr(TL)$, $T \in \mathcal{A}$, $[L]_{\mathcal{A}} \in \mathcal{Q}_{\mathcal{A}}$. Without any confusion, we write $[L]_{\mathcal{A}} = [L]$. For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $\mathcal{C}_1$ defined by $(x \otimes y)(u) = (u, y)x$, for $u \in \mathcal{H}$.

Throughout this paper, we write $\mathbb{N}$ for the set of natural numbers, $\mathbb{D}$ for the unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ for the boundary of $\mathbb{D}$. For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K})$, $i = 1, 2$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2. For $T \in \mathcal{L}(\mathcal{K})$ we write the n-th *ampliation* of T by

$$(1.1) \qquad T^{(n)} = \overbrace{T \oplus \cdots \oplus T}^{(n)}, \quad 1 \leq n \leq \infty.$$

Suppose that m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$(1.2) \qquad [x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, \ 0 \leq j < n,$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m}$, $\{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if m and n are positive integers and r is a fixed real number satisfying $r \geq 1$, a dual algebra $\mathcal{A}$ (with property $(\mathbb{A}_{m,n})$) is said to have property $(\mathbb{A}_{m,n}(r))$, if for every $s > r$ and every $m \times n$ array $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ from $\mathcal{Q}_{\mathcal{A}}$, there exist sequences $\{x_i\}_{0 \leq i < m}$ and $\{y_j\}_{0 \leq j < n}$ from $\mathcal{H}$ that satisfy (1.2) and also satisfy the following conditions:

$$(1.3a) \qquad \|x_i\|^2 \leq s \sum_{0 \leq j < n} \|[L_{ij}]\|, \quad 0 \leq i < m$$

and

$$\|y_j\|^2 \leq s \sum_{0\leq i<m} \|[L_{ij}]\|, \quad 0 \leq j < n. \tag{1.3b}$$

Finally, a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has property $(\mathbb{A}_{m,\aleph_0}(r))$ (for some real number $r \geq 1$) if, for every $s > r$ and every array $\{[L_{ij}]\}_{\substack{0\leq i<m \\ 0\leq j<\infty}}$ from $\mathcal{Q}_\mathcal{A}$ with summable rows, there exist sequences $\{x_i\}_{0\leq i<m}$ and $\{y_j\}_{0\leq j<\infty}$ of vectors from $\mathcal{H}$ that satisfy (1.2) and (1.3a,b) with the replacement of n by $\aleph_0$. Properties $(\mathbb{A}_{\aleph_0,n}(r))$ and $(\mathbb{A}_{\aleph_0,\aleph_0}(r))$ are defined similarly.

For the sake of brevity we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$.

We denote by $\mathcal{Q}_T$ the predual of $\mathcal{A}_T$. We denote by $\mathbb{A} = \mathbb{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Foiaş-Sz.-Nagy functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ is an isometry. Let $\phi_T : \mathcal{Q}_T \to L^1/H_0^1$ be the isometric corresponded by Φ_T such that $\phi_T^* = \Phi_T$. Furthermore, if m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$, we denote by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$.

Let P_λ be the Poisson kernel function in L^1, for each $\lambda \in \mathbb{D}$. For a given contraction $T \in \mathbb{A}$, let us write $\phi_T^{-1}([P_\lambda]) = [C_\lambda]$. Then we have $\langle f(T), [C_\lambda]\rangle = \widetilde{f}(\lambda)$, for $f \in H^\infty$.

Recall (e.g., from [12]) that the class $C_{0\cdot}$ consists of those operators T such that $\|T^n x\| \to 0$ for all $x \in \mathcal{H}$, $C_{\cdot 0} = (C_{0\cdot})^*$, and $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

For $\mathcal{M} \in \operatorname{Lat}(T)$, the class of invariant subspaces for an operator $T \in \mathcal{L}(\mathcal{H})$, we denote by $T|\mathcal{M}$ the restriction of T to $\mathcal{M}$. If $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{K}$ is a semi-invariant subspace for T (i.e., there exist $\mathcal{K}_1, \mathcal{K}_2 \in \operatorname{Lat}(T)$ with $\mathcal{K}_1 \supset \mathcal{K}_2$ such that $\mathcal{K} = \mathcal{K}_1 \ominus \mathcal{K}_2$), we shall write $T_\mathcal{K} = P_\mathcal{K} T|\mathcal{K}$ for the compression of T to $\mathcal{K}$, where $P_\mathcal{K}$ is the orthogonal projection whose range is $\mathcal{K}$.

2. Membership for the classes $\mathbb{A}_{m,n}$

For $T \in \mathcal{L}(\mathcal{H})$, we write $\mathcal{F}_+{}'(T)$ for the set of all points λ in $\mathbb{C}$ such that $T - \lambda$ is a Fredholm operator with positive index.

The following theorem may be compared with [5, Theorem 3.1].

THEOREM 2.1. *Suppose that $T \in \mathbb{A}(\mathcal{H})$, $m \in \mathbb{N}$, $\Lambda \subset \mathbb{D}$ is dominating for $\mathbb{T}$ and can be written as $\Lambda = \bigcup_{1 \le i \le m} \Lambda_i$, where $\Lambda_1 \subset \sigma_e(T)$. In the case of $m \geq 2$, assume that for $2 \leq i \leq m$, there exists a semi-invariant subspace $\mathcal{M}_i$ for T such that $T_{\mathcal{M}_i} \in C_{\cdot 0}$ (or $T_{\mathcal{M}_i} \in C_{0\cdot}$, resp.) and that for every $\lambda \in \Lambda_i$ and $i = 2, \cdots, m$, there exists a sequence $\{x_n^\lambda\}_{n=1}^\infty$ of unit vectors in $\mathcal{M}_i$ converging weakly to zero and satisfying*

$$\|[C_\lambda]_T - [x_n^\lambda \otimes x_n^\lambda]_T\| \to 0 \quad (n \to \infty). \tag{2.1}$$

Then $T \in \mathbb{A}_{1,\aleph_0}$ (or $T \in \mathbb{A}_{\aleph_0,1}$, resp.).

Proof. Suppose first that $m = 1$. Then $\sigma_e(T) \cap \mathbb{D}$ is dominating for $\mathbb{T}$. Let $\mathcal{F}_+{}'(T)$ be the set of all points λ in $\mathbb{C}$ such that $T - \lambda$ is a Fredholm operator with positive index. So $\mathbb{D} \setminus \mathcal{F}_+{}'(T)$ is dominating for $\mathbb{T}$ and it follows from [4, Theorem 6.2] that $T \in \mathbb{A}_{1,\aleph_0}$. If $m \geq 2$, we note that Λ_1 may be void and we can consider $\mathcal{K} = \mathcal{H} \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_m$ as a semi-invariant subspace for $T^{(m)}$. If we put

$$\widetilde{T} = T \oplus T_{\mathcal{M}_2} \oplus \cdots \oplus T_{\mathcal{M}_m}, \tag{2.2}$$

then $\widetilde{T}$ is unitarily equivalent to the compression $(T^{(m)})_{\mathcal{K}}$. Since $\widetilde{T}$ has T as a direct summand, we have $\widetilde{T} \in \mathbb{A}$. Thus it is sufficient to show that $\widetilde{T} \in \mathbb{A}_{1,\aleph_0}(\mathcal{K})$. Furthermore, it follows from the proof of [5, Theorem 3.1] that for every $\lambda \in \Lambda$, there exists a sequence of unit vectors $\{x_n^\lambda\}_{n=1}^\infty$ from $\mathcal{K}$ such that

$$\|[C_\lambda]_{\widetilde{T}} - [x_n^\lambda \otimes x_n^\lambda]_{\widetilde{T}}\| \to 0 \quad (n \to \infty), \tag{2.3a}$$

and

$$\|[x_n^\lambda \otimes w]_{\widetilde{T}}\| \to 0 \ (n \to \infty), \tag{2.3b}$$

for all $w \in \mathcal{K}$. Thus, by [4, Theorem 6.2], $T \in \mathbb{A}_{1,\aleph_0}$. □

The following is an improvement of [5, Theorem 3.7].

LEMMA 2.2. *Suppose that $T \in C_{0\cdot} \cap \mathbb{A}(\mathcal{H})$. If $(\sigma_r(T) \cap \mathbb{D}) \cup (\mathbb{D} \setminus \mathcal{F}_+{}'(T))$ is dominating for $\mathbb{T}$, then $T \in \mathbb{A}_{\aleph_0}$. Therefore $(C_{0\cdot} \cap \mathbb{A}_{1,\aleph_0}) \cup (C_{\cdot 0} \cap \mathbb{A}_{\aleph_0,1}) \subset \mathbb{A}_{\aleph_0}$.*

Proof. By [4, Theorem 6.2], we have $T \in \mathbb{A}_{1,\aleph_0}$. Since $T \in C_{0\cdot}$, it follows from [5, Proposition 2.7] that $T \in \mathbb{A}_{\aleph_0}$. □

C. Apostol, H. Bercovici, C. Foias and C. Pearcy [2] characterized subnormal operators in $\mathbb{A} \cap C_{00}$. The following theorem is a generalization of their result.

THEOREM 2.3. *If T is a hyponormal operator in $C_{0\cdot}(\mathcal{H})$, then the following two conditions are equivalent:*
(i) $T \in \mathbb{A}$,
(ii) $T \in \mathbb{A}_{\aleph_0}$.
In particular, if T is a subnormal operator, then each of (i) *and* (ii) *is equivalent to*
(iii) $\sigma(T) \cap \mathbb{D}$ *is dominating for* $\mathbb{T}$.

Proof. If T is a hyponormal operator, then $\mathcal{F}_+{}'(T)$ is an empty set. By [4, Theorem 6.2], we have $T \in \mathbb{A}_{1,\aleph_0}$. Hence it follows from Lemma 2.2 that (i) implies (ii). On the other hand, obviously (ii) implies (i).

Under the hypothesis of the second part, it is also obvious from [2, Proposition 4.6] that (iii) implies (i). Now it remains to show that (i) implies (iii). Let N be the minimal normal extension of T acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$. According to [8, Propositions III 2.4 and III 2.11], we have

$$\mathcal{K} = \bigvee_{k=0}^{\infty} N^{*k}\mathcal{H} \tag{2.4}$$

and $\sigma(N) \subset \sigma(T)$. By [8, Proposition III 4.7], we have

$$\rho(N) = \lim \|N^n\|^{\frac{1}{n}} = \|N\| \le \rho(T) \le 1, \tag{2.5}$$

where $\rho(N)$ is the spectral radius of N. Hence N is a contraction operator. To show that $\mathcal{K} \subset \{x \in \mathcal{K} : \|N^n x\| \longrightarrow 0\}$, let us take $N^{*k}h \in \mathcal{K}$ for $h \in \mathcal{H}$ and a nonnegative integer k. Then since $T \in C_{0\cdot}$, we have

$$\|N^n N^{*k} h\| = \|N^{*k} N^n h\| \le \|N^n h\| = \|T^n h\| \longrightarrow 0 \quad (n \to \infty). \tag{2.6}$$

Moreover, since $\{x \in \mathcal{K} : \|N^n x\| \to 0\}$ is a subspace of $\mathcal{K}$, we have $\mathcal{K} \subset \{x \in \mathcal{K} : \|N^n x\| \to 0\}$. Therefore $N \in C_{0\cdot} \cap \mathbb{A}(\mathcal{K})$. Hence N is a completely nonunitary contraction satisfying $\|f(N)\| = \|f\|_\infty$, for all $f \in H^\infty$. Now, according to the usual proof in the theory of dual algebras (cf. [2]), we can obtain

$$(2.7) \qquad \|f(N)\| = \sup_{\lambda \in \sigma(N) \cap \mathbb{D}} |\widetilde{f}(\lambda)|, \quad \text{for all} \quad f \in H^\infty.$$

Hence $\sigma(N) \cap \mathbb{D}$ is dominating for $\mathbb{T}$ (cf. [2, Definition 4.5]). Since $\sigma(N) \subset \sigma(T)$, the proof is completed. □

Recall from [7] (or [11]) that, for every $[L] \in \mathcal{Q}_T$ and every $s > 1$, there exist square summable sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in $\mathcal{H}$ such that

$$(2.8) \qquad [L] = \sum_{n=1}^{\infty} [x_n \otimes y_n],$$

$$(2.9a) \qquad \sum_{n=1}^{\infty} \|x_n\|^2 < s\|[L]\|$$

and

$$(2.9b) \qquad \sum_{n=1}^{\infty} \|y_n\|^2 < s\|[L]\|.$$

THEOREM 2.4. *Let T be an absolutely continuous contraction in $\mathbb{A}(\mathcal{H})$. Let U_T^+ be an isometric dilation of T acting on a Hilbert space $\mathcal{K}_+$. Assume that*

$$(2.10) \qquad U_T^{+(\infty)}|\bigvee_{k=1}^{\infty} {U_T^{+(\infty)}}^k \widetilde{y} \cong U_T^+$$

for some $\widetilde{y}$ in $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H} \oplus \cdots$. Suppose that $\mathcal{H}$ is a hyperinvariant subspace for ${U_T^+}^*$. Then $T \in \cap_{n=1}^\infty \mathbb{A}_{1,n}(1)$.

Proof. For any $n \in \mathbb{N}$, it is sufficient to show that $T^* \in \mathbb{A}_{n,1}(1)$. Suppose that φ_i is a weak*-continuous linear functional on $\mathcal{A}_{T^*}$ and $s > 1$, $1 \le i \le n$. By (2.8) and (2.9a,b), there exist sequences $\{x_k^{(i)}\}_{k=1}^{\infty}$ and $\{y_k^{(i)}\}_{k=1}^{\infty}$ in $\mathcal{H}$ satisfying

$$\varphi_i(A) = \sum_{k=1}^{\infty} (Ax_k^{(i)}, y_k^{(i)}) \tag{2.11}$$

for all A in $\mathcal{A}_{T^*}$ such that

$$\sum_{k=1}^{\infty} \|x_k^{(i)}\|^2 < s\|\varphi_i\| \tag{2.12}$$

and

$$\sum_{k=1}^{\infty} \|y_k^{(i)}\|^2 < s\|\varphi_i\|. \tag{2.13}$$

Let

$$\widetilde{\mathcal{K}} := \underbrace{\mathcal{K}_1^{(1)} \oplus \cdots \oplus \mathcal{K}_1^{(n)}}_{(n)} \oplus \underbrace{\mathcal{K}_2^{(1)} \oplus \cdots \oplus \mathcal{K}_2^{(n)}}_{(n)} \oplus \cdots, \tag{2.14a}$$

$$\widetilde{x}^{(i)} = (\underbrace{\overbrace{0, \cdots, 0}^{(i-1)}, x_1^{(i)}, \cdots, 0}_{(n)}, \underbrace{\overbrace{0, \cdots, 0}^{(i-1)}, x_2^{(i)}, \cdots, 0}_{(n)}, \cdots) \tag{2.14b}$$

and

$$\widetilde{y} = (\underbrace{y_1^{(1)}, \cdots, y_1^{(n)}}_{(n)}, \underbrace{y_2^{(1)}, \cdots, y_2^{(n)}}_{(n)}, \cdots), \tag{2.14c}$$

where $\mathcal{K}_k^{(i)} = \mathcal{K}$, $1 \le i \le n$, $k \in \mathbb{N}$. For brevity, we let

$$\mathcal{M} = \bigvee_{k=1}^{\infty} \widetilde{U}^k \widetilde{y}, \tag{2.15}$$

where $\widetilde{U} := U_T^{+(\infty)}$. By the hypotheses, there exists an isometry W from $\mathcal{K}$ into $\widetilde{\mathcal{K}}$ such that $W\mathcal{K} = \mathcal{M}$ and

$$WU_T^+ = \widetilde{U}W. \tag{2.16}$$

Let $T_k^{(i)} = P_{k,i}W$, where $P_{k,i}$ is the projection from $\widetilde{\mathcal{K}}$ onto $\mathcal{K}_k^{(i)}$. Then, clearly, $T_k^{(i)} \in \mathcal{L}(\mathcal{K})$ and for every $x \in \mathcal{K}$ we have

$$Wx = \underbrace{T_1^{(1)}x \oplus \cdots \oplus T_1^{(n)}x}_{(n)} \oplus \underbrace{T_2^{(1)}x \oplus \cdots \oplus T_2^{(n)}x}_{(n)} \oplus \cdots . \tag{2.17}$$

It follows from (2.16) that

$$T_k^{(i)}U_T^+ = U_T^+T_k^{(i)} \tag{2.18}$$

for any k, i. Let $y_0 = W^*\widetilde{y}$. Then $T_k^{(i)}\ y_0 = P_{k,i}\widetilde{y} = y_k^{(i)}$ for any $k \in \mathbb{N}$, $1 \le i \le n$. Furthermore, by (2.13) we have

$$\|y_0\|^2 = \|\widetilde{y}\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{\infty}\|y_k^{(i)}\|^2 < s\sum_{i=1}^{n}\|\varphi_i\|. \tag{2.19}$$

By (2.17) we have

$$(W^*\widetilde{x}^{(i)}, z) = (\widetilde{x}^{(i)}, Wz) = \sum_{k=1}^{\infty}(x_k^{(i)}, T_k^{(i)}z) = \sum_{k=1}^{\infty}(T_k^{(i)*}x_k^{(i)}, z) \tag{2.20}$$

for every $z \in \mathcal{K}$ and we can assert that the series $\sum_{k=1}^{\infty} T_k^{(i)*}x_k^{(i)}$ converges weakly to some $x_0^{(i)}(= W^*\widetilde{x}^{(i)}) \in \mathcal{K}$, $1 \le i \le n$. Now by (2.12) we have

$$\|x_0^{(i)}\|^2 = \|\widetilde{x}^{(i)}\|^2 = \sum_{k=1}^{\infty}\|x_k^{(i)}\|^2 < s\|\varphi_i\|. \tag{2.21}$$

Since $\mathcal{H}$ is a hyperinvariant subspace for ${U_T^+}^*$,

$${T_k^{(i)}}^*\mathcal{H} \subset \mathcal{H} \tag{2.22}$$

by (2.18). Hence we have $x_0^{(i)} \in \mathcal{H}$. Now for every $n \in \mathbb{N}$, $1 \le i \le n$, we have

(2.23)

$$\begin{aligned}
\varphi_i(T^{*n}) &= \sum_{k=1}^{\infty}(T^{*n}x_k^{(i)}, y_k^{(i)}) = \sum_{k=1}^{\infty}(T^{*n}x_k^{(i)}, T_k^{(i)}y_0) \\
&= \sum_{k=1}^{\infty}(U_T^{+*n}x_k^{(i)}, T_k^{(i)}y_0) = \sum_{k=1}^{\infty}(T_k^{(i)^*}U_T^{+*n}x_k^{(i)}, y_0) \\
&= \sum_{k=1}^{\infty}(U_T^{+*n}T_k^{(i)^*}x_k^{(i)}, y_0) = \sum_{k=1}^{\infty}(T^{*n}T_k^{(i)^*}x_k^{(i)}, y_0) \quad \text{by (2.22)} \\
&= (T^{*n}\sum_{k=1}^{\infty}T_k^{(i)*}x_k^{(i)}, y_0) = (T^{*n}x_0^{(i)}, y_0) \\
&= (T^{*n}x_0^{(i)}, P_{\mathcal{H}}y_0), \qquad \text{since } x_0^{(i)} \in \mathcal{H},
\end{aligned}$$

$i = 1, 2, \cdots, n$, so that $\varphi_i(A) = (Ax_0^{(i)}, P_{\mathcal{H}}y_0)$ for any $A \in \mathcal{A}_{T^*}$. Hence $T^* \in \mathbb{A}_{n,1}(1)$. □

Note that the unilateral shift operator $S^{(n)}$ of multiplicity n and multiplication operator M_Γ on $L^2(\Gamma)$, $\Gamma \subset \mathbb{T}$, satisfy the hypotheses of Theorem 2.4. In particular, if we consider a Jordan block $S(\theta)$ which will be defined in the next section, and if we follow the proof of Theorem 2.4, then $\mathcal{A}_{S(\theta)}$ has property $(\mathbb{A}_{1,n})$ for any $n \in \mathbb{N}$.

3. Examples

For $1 \le p \le \infty$ we write $H^p = H^p(\mathbb{T})$ for the usual Hardy space. Let us recall that a completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is to be of *class* C_0 if there exists a non-zero function $u \in H^\infty(\mathbb{T})$ such that (under the functional calculus) $u(T) = 0$ (cf. [12]). Let S be the unilateral shift of multiplicity one. Then the function $S(\theta)$ defined by $S(\theta) = (S^*|(H^2 \ominus \theta H^2))^*$, for an inner function θ, is called a *Jordan block* and that any operator of the form $S(\theta_1)\oplus S(\theta_2)\oplus\cdots\oplus S(\theta_k)\oplus S^{(l)}$, where $\theta_1, \theta_2, \cdots, \theta_k$ are nonconstant (scalar valued) inner functions and $0 \le k < \infty$, $0 \le l \le \infty$, is called a *Jordan operator* (cf. [9], [10]).

We start this section from the following theorem which is an improvement of [10, Corollary 4.8].

THEOREM 3.1. *Suppose that $T \in \mathbb{A}(\mathcal{H})$ and $1 \leq m, n \leq \aleph_0$. Then the following statements are equivalent:*
(i) $T \in \mathbb{A}_{m,n}(\mathcal{H})$,
(ii) $T \oplus A \in \mathbb{A}_{m,n}(\mathcal{H} \oplus \mathcal{K})$ *for any* $A \in C_0(\mathcal{K})$,
(iii) $T \oplus A \in \mathbb{A}_{m,n}(\mathcal{H} \oplus \mathcal{K})$ *for some* $A \in C_0(\mathcal{K})$.

Proof. It is sufficient to show that (iii) $\Longrightarrow$ (i). Assume that $T \oplus A \in \mathbb{A}_{m,n}$ for some $A \in C_0$. Let θ be an inner function with $\theta(A) = 0$. To show that $T \in \mathbb{A}_{m,n}$, let us consider an $m \times n$ system $\{[L_{ij}]\}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ in Q_T and let $[l_{ij}]$ be the corresponding system in L^1/H_0^1 such that $\phi_{\widehat{T}}([L_{ij}]) = [l_{ij}]$. Consider $\widetilde{\theta} l_{ij} \in L^1$, where $\widetilde{\theta}(e^{it}) = \overline{\theta(e^{-it})}$. Since $\widehat{T} := T \oplus A \in \mathbb{A}_{m,n}$, there exist two vectors $\widehat{u}_i = u_i \oplus u_i'$ and $\widehat{v}_j = v_j \oplus v_j'$ in $\mathcal{H} \oplus \mathcal{K}$ such that $\phi_{\widehat{T}}^{-1}([\widetilde{\theta} l_{ij}]) = [\widehat{u}_i \otimes \widehat{v}_j]$, $1 \leq i \leq m$, $1 \leq j \leq n$. Then, for any $h \in H^\infty$, we have

$$
\begin{aligned}
\int \widetilde{\theta} l_{ij} h \, dm &= \langle h, [\widetilde{\theta} l_{ij}] \rangle = \langle h(\widehat{T}), \phi_{\widehat{T}}^{-1}([\widetilde{\theta} l_{ij}]) \rangle = (h(\widehat{T}) \widehat{u}_i, \widehat{v}_j) \\
&= (h(T) u_i, v_j) + (h(A) u_i', v_j') = (h(T) u_i, v_j).
\end{aligned} \tag{3.1}
$$

Now we replace h by θh in (3.1), and we have

$$
\begin{aligned}
\langle h(T), [L_{ij}]_T \rangle = \langle h, \phi_T([L_{ij}]) \rangle = \langle h, [l_{ij}] \rangle &= \int l_{ij} h \, dm, \\
&= \int l_{ij} h |\theta|^2 dm \quad \text{since } |\theta(e^{it})| = 1 \text{ a.e.} \\
&= \int (\widetilde{\theta} l_{ij})(\theta h) dm \\
&= ((\theta h)(T) u_i, v_j) + (\theta h(A) u_i', v_j') \quad \text{by (3.1)} \\
&= (\theta(T) h(T) u_i, v_j) = (h(T)(\theta(T) u_i), v_j) \\
&= \langle h(T), [\theta(T) u_i \otimes v_j]_T \rangle,
\end{aligned} \tag{3.2}
$$

so that $[L_{ij}]_T = [\theta(T) u_i \otimes v_j]_T$, $1 \leq i \leq m$, $1 \leq j \leq n$. □

The C_0-operator A in the above theorem didn't play any role for properties $(\mathbb{A}_{m,n}(1))$. However, it is well-known that the singly generated dual algebra $\mathcal{A}_{S(\theta)}$ has some property $(\mathbb{A}_{1,1})$. If $\theta_1 = \cdots = \theta_n$, $n \in \mathbb{N}$, then we obtain an interesting result as the following Proposition 3.3. However, if $\theta \neq \theta'$, in general the dual algebra $\mathcal{A}_{S(\theta)\oplus S(\theta')}$ doesn't always have property $(\mathbb{A}_2(1))$ (see Example 3.5).

The following corollary results from the proof of Theorem 2.4.

COROLLARY 3.2. *For an inner function θ and any $m \in \mathbb{N}$, the dual algebra $\mathcal{A}_{S(\theta)}$ has property $(\mathbb{A}_{1,m}(1))$ and property $(\mathbb{A}_{m,1}(1))$.*

Proof. Since $U^+_{S(\theta)}$ is unitarily equivalent to a unilateral shift S of multiplicity one, every nonzero vector in $\mathcal{K}_+$ is an invariant ampliation for $U^+_{S(\theta)}$ itself. Furthermore, it is well-known that the acting space of $S(\theta)$ is a hyperinvariant subspace for S^*. Hence the proof of Theorem 2.4 applies to prove this corollary. □

PROPOSITION 3.3. *If $\mathcal{A}_T$ has property $(\mathbb{A}_{1,m}(1))$ (or $(\mathbb{A}_{m,1})$ resp.), $n, m \in \mathbb{N}$, then $\mathcal{A}_{T^{(n)}}$ has property $(\mathbb{A}_{n,m}(1))$ (or $(\mathbb{A}_{m,n})$ resp.).*

Proof. Suppose that φ_{ij} is a weak*-continuous functional on $\mathcal{A}_{T^{(n)}}$ and $s > 1$, $1 \le i \le n$, $1 \le j \le m$. Define

$$\phi_{ij}(A) = \varphi_{ij}(A^{(n)}) \tag{3.3}$$

for $A \in \mathcal{A}_T$, $1 \le i \le n$, $1 \le j \le m$. Then ϕ_{ij} is a weak*-continuous functional on $\mathcal{A}_T$. Since $\mathcal{A}_T$ has property $(\mathbb{A}_{1,m}(1))$, for every $s > 1$ there exist $x_i \in \mathcal{H}$ and $\{y_j^{(i)}\}_{1\le j\le m}$ in $\mathcal{H}$ such that $\phi_{ij} = x_i \otimes y_j^{(i)}$,

$$\|x_i\|^2 \le s \sum_{j=1}^{m} \|\phi_{ij}\|, \quad 1 \le i \le n \tag{3.4a}$$

and

$$\|y_j^{(i)}\|^2 \le s\|\phi_{ij}\|, \quad 1 \le i \le n, \ 1 \le j \le m. \tag{3.4b}$$

Now we set

$$\widetilde{x_i} = (\overbrace{\underbrace{0, \cdots, 0}_{(i)}, x_i, 0, \cdots}^{(n)}), \quad 1 \le i \le n, \tag{3.5a}$$

and

$$\widetilde{y_j} = (\overbrace{\underbrace{y_{1j}, \cdots y_{ij}}_{(i)}, \cdots y_{nj}}^{(n)}), \quad 1 \le j \le m. \tag{3.5b}$$

Then it is easy to show that $\varphi_{ij} = \widetilde{x_i} \otimes \widetilde{y_j}$ on $\mathcal{A}_{T^{(n)}}$, for $1 \le i \le n$, $1 \le j \le m$

$$\|\widetilde{x_i}\|^2 = \|x_i\|^2 \le s \sum_{j=1}^{m} \|\varphi_{ij}\| \tag{3.6a}$$

and

$$\|\widetilde{y_j}\|^2 = \sum_{i=1}^{n} \|y_{ij}\|^2 \le s \sum_{i=1}^{n} \|\varphi_{ij}\|. \tag{3.6b}$$

Hence the dual algebra $\mathcal{A}_{T^{(n)}}$ has property $(\mathbb{A}_{n,m}(1))$. □

The following result is an immediate consequence of Corollary 3.2 and Proposition 3.3.

THEOREM 3.4. *If* $T = S(\theta)^{(n)}$, $n \in \mathbb{N}$, *then* $\mathcal{A}_T$ *has property* $(\mathbb{A}_{n,m}(1))$ *and property* $(\mathbb{A}_{m,n}(1))$ *for any* $m \in \mathbb{N}$.

Now we provide the example mentioned earlier as follows:

EXAMPLE 3.5. Let $\varphi_n = e^{int}$. Then it follows easily that

$$S(\varphi_n) \cong \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & O & \\ & & \ddots & \ddots & \\ & O & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \tag{3.7}$$

relative to $\mathbb{C}^n$. If $m \neq n$, the dual algebra $\mathcal{A}_{S(\varphi_n) \oplus S(\varphi_m)}$ doesn't have property $(\mathbb{A}_2)$. For example, $\mathcal{A}_{S(\varphi_2)}$ and $\mathcal{A}_{S(\varphi_3)}$ have properties $(\mathbb{A}_{1,m})$ and $(\mathbb{A}_{m,1})$ for any $m \in \mathbb{N}$, but not property $(\mathbb{A}_{2,2})$ (cf. [1, p. 321]).

COROLLARY 3.6. *For $k \in \mathbb{N}$, if θ_i is an inner function, $i = 1, \cdots, k$, then $\mathcal{A}_{S(\theta_1)\oplus\cdots\oplus S(\theta_k)}$ has property $(\mathbb{A}_{1,n}(1))$ and property $(\mathbb{A}_{n,1}(1))$.*

Proof. Since $\mathcal{A}_{S(\theta_1)\oplus\cdots\oplus S(\theta_k)}$ is contained in $\mathcal{A}_{S(\theta_1)} \oplus \cdots \oplus \mathcal{A}_{S(\theta_k)}$ which has property $(\mathbb{A}_{1,n}(1))$, it has property $(\mathbb{A}_{1,n}(1))$. □

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