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SYMPLECTICITY OF 4-DIMENSIONAL NIL-MANIFOLDS AND SCALAR CURVATURE

JONGSU KIM AND GABJIN YUN

ABSTRACT. We make an explicit description of compact 4-dimensional nilmanifolds as principal torus bundles and show that they are symplectic. We discuss some consequences of this and give in particular a Seibebrg-Witten-invariant proof of a Gromov-Lawson theorem that if a compact 4-dimensional nilmanifold admits a metric of zero scalar curvature, then it is diffeomorphic to 4-torus, T^4.

1. Introduction

For a given compact smooth manifold $M^n, n \geq 3$, there is no obstruction for the existence of a Riemannian metric of negative scalar curvature (cf. [4], [9], [1]). In fact, Aubin [1] proved that every compact smooth manifold admits a metric of negative constant scalar curvature. However, for nonnegative case, there is an obstruction. For example, Lichnerowicz [10] showed that if M^{4k} is a compact spin manifold with nonzero $\hat{A}$-genus, $\hat{A}(M) \neq 0$, then M does not admit a metric of positive scalar curvature.

On the other hand, using minimal surfaces Schoen and Yau [14] proved that T^3 does not have positive scalar curvature metrics. Shortly thereafter Gromov and Lawson [7] generalized Lichnerowicz's Dirac operator approach and proved that for all n, T^n has no metrics with positive scalar curvature. Furthermore, they showed that every solvmanifold does not admit a metric of positive scalar curvature. Recently new Seiberg-Witten invariants were successfully used to prove non-existence

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of positive scalar curvature metrics on some (mostly symplectic) manifolds. To understand broad applicability of this new invariant argument, we wanted to know if one can use the new argument to yield the same result for the *enlargeable* manifolds of Gromov and Lawson. In this paper we have only dealt with nil-manifolds. We do not know if most of the enlargeable manifolds including solv-manifolds admit symplectic structures, but in any case the Seiberg-Witten-invariant argument may still help partially in studying the scalar curvarture properties of enlargeable manifolds.

In section 2 and 3 we will describe explicitly compact 4-dimensional nilmanifolds. In section 4 we discuss a Gromov-Lawson theorem and a corollary.

THEOREM 3.4. *Any compact 4-dimensional nilmanifold admits a symplectic structure. Furthermore the first Betti number satisfies* $b_1(N) \geq 2$.

THEOREM 4.2. *If a 4-dimensional compact nilmanifold admits a metric of zero scalar curvature, then it is diffeomorphic to 4-torus,* T^4.

This implies

COROLLARY 4.3. *There is a positive number* $\epsilon > 0$ *such that if a 4-dimensional compact manifold* M *admits a metric* g *satisfying*

$$|K_g| \cdot diam(M, g)^2 \leq \epsilon,$$

where $|K_g|$ *is the norm of the sectional curvature* K_g *of* g *and* $diam(M, g)$ *is the diameter of* M *with respect to* g*, then a finite cover of* M *is diffeomorphic to* T^4 *or* M *does not have zero scalar curvature metrics.*

2. Some properties for nilpotent groups

A solvable group Γ is called *polycyclic* if there is a subnormal series

$$\Gamma = \Gamma_o \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{1\},$$

where factors Γ_i/Γ_{i+1} are all infinite cyclic.

A solvable group is *virtually polycyclic* (*or almost polycyclic*) if it contains a subgroup of finite index which is polycyclic. The number of infinite cyclic factors is independent of the choice of finite index subgroup or subnormal series, and is called the *Hirsch length* of the group, denoted by $h(\Gamma)$.

For each natural number $q \geq 1$, let Γ_q be the group with presentation

$$\langle x, y, z : xz = zx, yz = zy, xy = z^q yx \rangle.$$

Every such group Γ_q is torsion free and nilpotent of Hirsch length 3. In fact, Γ_q can be realized by a nilpotent Lie group of 3×3 matrices in the following form:

$$\left\{ \begin{pmatrix} 1 & a & \frac{c}{q} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}.$$

The generators x, y and z correspond to $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & \frac{1}{q} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, respectively. Note that $\Gamma_q/[\Gamma_q, \Gamma_q] \cong \mathbf{Z} \oplus \mathbf{Z}$.

The following theorem shows that torsion free nilpotent groups of Hirsch length 4 are characterized.

THEOREM 2.1 ([8]). *Let Γ be a finitely generated torsion free nilpotent group of Hirsch length $h(\Gamma) = 4$. Then either*

(1) *Γ is a free abelian group; or*

(2) *$C_\Gamma \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\Gamma \cong \Gamma_q \times \mathbf{Z}$ for some $q \geq 1$; or*

(3) *$C_\Gamma \cong \mathbf{Z}$ and $\Gamma / C_\Gamma \cong \Gamma_q$ for some $q \geq 1$,*

where C_Γ denotes the center of Γ.

3. Nilmanifolds and symplectic structures

Let M be a compact 4-dimensional nilmanifold so that there exist a simply connected nilpotent Lie group N and a lattice Γ of N such that

$M = N/\Gamma$. If Γ is a free abelian group, then it is easy to see that M is diffeomorphic to T^4 since $h(\Gamma) = 4$. Assume now Γ is not a free abelian group. Then by Theorem 2.1, either $\Gamma \cong \Gamma_q \times \mathbf{Z}$ with $C_\Gamma \cong \mathbf{Z} \oplus \mathbf{Z}$ or $\Gamma/C_\Gamma \cong \Gamma_q$ with $C_\Gamma \cong \mathbf{Z}$ for some $q \geq 1$.

In any case, we will show

PROPOSITION 3.1. *A compact 4-dimensional nilmanifold M is a principal torus bundle over a lower dimensional nilmanifold.*

Proof. We only need to consider the nilpotent groups $\Gamma = \pi_1(M)$ corresponding to (2) and (3) of the Theorem 2.1. Recall that $C_\Gamma = C_N \cap \Gamma$ from nilpotent group theory [12]. So C_N/C_Γ is a Lie group and a torus. One easily sees that this torus acts freely on $M = N/\Gamma$ induced by left multiplication. Let Q be the quotient manifold $(N/\Gamma)/(C_N/C_\Gamma)$. Now let $N_1 = N/C_N$ and $\Gamma_1 = \Gamma/C_\Gamma$. One can check that N_1 is a nilpotent group with lattice Γ_1. We claim that there is a natural diffeomorphism between Q and N_1/Γ_1. First we will show below in Lemma 3.2 that there is a natural diffeomorphism Φ_1: $Q \to N/(\Gamma \cdot C_N)$ induced by id_N. We define similarly another diffeomorphism $\Phi_2 : N/(\Gamma \cdot C_N) \to (N/C_N)/(\Gamma \cdot C_N/C_N)$. As $\Gamma \cdot C_N/C_N \cong \Gamma/(\Gamma \cap C_N) \cong \Gamma_1$, we also have a diffeomorphism $\Phi_3 : (N/C_N)/(\Gamma \cdot C_N/C_N) \to N_1/\Gamma_1$. Now $\Phi_3 \circ \Phi_2 \circ \Phi_1$ is the desired diffeomorphism between Q and N_1/Γ_1. This proves the Proposition. □

LEMMA 3.2. *There is a well-defined diffeomorphism*

$$\Phi_1 : Q = (N/\Gamma)/(C_N/C_\Gamma) \to N/(\Gamma \cdot C_N),$$

induced by id_N.

Proof. Let $\bar{n} = n\Gamma$ for $n \in N$ be an element of the coset space N/Γ. For convenience H denotes C_N/C_Γ. Note that a point of Q is an orbit of H action on N/Γ. We define $\Phi_1(\bar{n}H)$ to be $n(\Gamma \cdot C_N)$.

We want to show that this is a well-defined map. Suppose that $\bar{n_1}H = \bar{n_2}H$ as orbits for $n_1, n_2 \in N$. Then $\bar{n_1} = \bar{n_2} \cdot \hat{h}$ for some $\hat{h} \in H$. We may write $\hat{h} = h(C_\Gamma) \in H = C_N/C_\Gamma$ with $h \in C_N$. So $\bar{n_1} = n_1\Gamma = hn_2\Gamma$, i.e. $h^{-1}{n_2}^{-1}n_1 \in \Gamma$. This implies that ${n_2}^{-1}n_1 \in \Gamma \cdot C_N$, so that $n_1(\Gamma \cdot C_N) = n_2(\Gamma \cdot C_N)$. So Φ_1 is well-defined.

Next we prove that Φ_1 is one-to-one. If $\Phi_1(\bar{n_1}H) = \Phi_1(\bar{n_2}H)$, then $n_2{}^{-1}n_1 \in \Gamma \cdot C_N$. Set $n_2{}^{-1}n_1 = gh$ for $g \in \Gamma$, $h \in C_N$. So $n_1 = n_2gh$ and $n_1\Gamma = hn_2g\Gamma = hn_2\Gamma = h(C_\Gamma)\cdot(n_2\Gamma)$. We write this as $\bar{n_1} = \bar{n_2}\cdot\hat{h}$. Therefore $\bar{n_1}H = \bar{n_2}H$ as orbits and Φ_1 is one-to-one. Obviously Φ_1 is surjective. Furthermore, Φ_1 and its inverse are clearly smooth maps.□

We need the following theorem by Fernandez, Gotay and Gray [5, Theorem 1.1]

THEOREM 3.3. *Let a compact 4-dimensional manifold M be a principal circle bundle over M_1 which is in turn a principal circle bundle over a torus T^2, so that the first Betti number of M satisfies $2 \leq b_1(M) \leq 4$. Then*

(i) *if $b_1(M)$ is equal to 2 or 3, then M has symplectic structures*
(ii) *if $b_1(M) = 4$ if and only if M is a 4-torus T^4.*

Proposition 3.1 implies that compact 4-dimensional nilmanifolds satisfy the condition of theorem 3.3. So we have

THEOREM 3.4. *Any compact 4-dimensional nilmanifold admits a symplectic structure and satisfies $b_1(N) \geq 2$.*

Suppose now N is a compact 4-dimensional nilmanifold. Then both the Euler-Poincaré characteristic and the signature of N vanish because it is parallelizable. Thus, since $b_1(N) \geq 2$, we have $b_2^+(N) = b_1 - 1 \geq 1$, where b_2^+ denotes the dimension of maximal subspace of 2-forms on which the intersection form is positive definite.

4. Scalar Curvatures on Nilmanifolds

As we proved nilmanifolds to be symplectic, the following Ohta-Ono theorem [11] becomes relevant to our discussion. We make a brief sketch of its proof just enough to explain the difference from Gromov-Lawson's approach [7] which uses an estimate from a Dirac equation.

THEOREM 4.1. *If a 4-dimensional compact symplectic manifold admits a metric of positive scalar curvature, then it is diffeomorphic to either the complex projective plane or a ruled surface up to blow-up and down.*

Sketch of proof. For $b_1 \geq 2$, recall a Taubes theorem [15] that the Seiberg-Witten invariant for $c_1({K_X}^{-1})$ is nonzero. Then a well known estimate for smooth solutions of the Seiberg-Witten equation [13, p. 184] forces any scalar curvature to be non-positive.
For $b_1 = 1$, they use perturbed Seiberg-Witten equations. The wall crossing contribution and the existence of metrics of positive scalar curvature yield solutions of perturbed Seiberg-Witten equations. The solutions admit some estimates or are associated with some pseudo-holomorphic curves. These estimates often rule out possibilities of some particular topological manifolds and the pseudo-holomorphic curves charaterize the symplectic manifolds as stated in the theorem. □

Now we prove that a Gromov-Lawson's theorem can be handled by Seiberg-Witten-invariant argument at least for the small class of nil-manifolds.

THEOREM 4.2. *If a 4-dimensional compact nilmanifold admits a metric of zero scalar curvature, then it is diffeomorphic to 4-torus, T^4.*

Proof. Suppose a compact 4-dimensional nilmanifold M admits a metric g of zero scalar curvature. Then g can be either perturbed to positive scalar-curved metrics or g is ricci-flat [2, Chap 4.F]. If g is ricci-flat on M, then by the Chern-Gauss-Bonnet formula $8\pi^2\chi(M) = \int_M |W|^2 dvol = 0$. As the Weyl curvature tensor W vanishes, g is flat. In this case $\pi_1(M)$ is abelian and M is diffeomorphic to the 4-torus.
So suppose that g can be perturbed to a positive scalar-curved metric. Now M admits a symplectic structure by Theorem 3.3. By theorem 4.1, M has to be either the complex projective plane or a ruled surface up to blow-up and down. It is easy to show that nilmanifolds can not be diffeomorphic to one of these. For instance we compare their fundamental groups as follows. As blow-up or blow-down does not change fundamental groups of complex manifolds, we only need to discuss for minimal complex surfaces i.e. complex 2-dimensional manifolds which have no self-intersection -1 holomorphic curves to be blown down. As $b_1 \geq 2$, M can not be the complex projective plane. The fundamental group of a ruled surface is isomorphic to the fundamental group of a closed orientable real 2-dimensional manifold. But $\pi_1(M)$ of theorem

2.1 can not be such a group. Therefore g can not be perturbed to a positive scalar-curved metric. □

A compact smooth manifold is called *almost flat* if for any $\epsilon > 0$, ther exists a metric g satisfying

$$|K_g| \cdot diam(M, g)^2 \leq \epsilon.$$

We have the following theorem.

COROLLARY 4.3. *There is a positive number $\epsilon > 0$ such that if a 4-dimensional compact manifold M admits a metric satisfying*

$$|K_g| \cdot diam(M, g)^2 \leq \epsilon,$$

then a finite cover of M is diffeomorphic to T^4 or M does not have zero scalar curvature metrics.

Proof. By a Gromov theorem ([6]), there is an $\epsilon > 0$ such that if a compact manifold M admits a metric satisfying

$$|K_g| \cdot diam(M, g)^2 \leq \epsilon,$$

then M is covered by a nilmanifold. More precisely, there is a simply connected nilpotent Lie group N, a finitely generated torsion-free nilpotent subgroup Γ of $\pi_1(M)$ of finite index and a finite cover $\hat{M}$ of M such that $\pi_1(\hat{M}) = \Gamma$ and $\hat{M} \cong N/\Gamma$ (cf. [3]). Thus, it follows from Theorem 4.2 that if $\hat{M}$ is not diffeomorphic to T^4, then $\hat{M}$ does not have zero scalar curvature metrics and so neither does M. □

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DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, KOREA
E-mail: jskim@ccs.sogang.ac.kr

DEPARTMENT OF MATHEMATICS, MYONG-JI UNIVERSITY, KYUNG-GI DO 449-728, KOREA
E-mail: gabjin@wh.myongji.ac.kr or gabjin@math.snu.ac.kr