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THE C^r CLOSING LEMMA FOR CHAIN RECURRENCE IN COMPACT SURFACES

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ABSTRACT. We prove the C^r Closing Lemma for chain recurrence in compact surfaces.

Solving the Closing Lemma is interesting in its own right, but more because it implies generically that a dynamical system already has its periodic orbits dense in its set of nonwandering orbits. The first statement and proof of C^1 Closing Lemma (for nonwandering points) are due to Pugh [5]. There was a gap in his proof which was repaired in [7]. As proved by Pugh and Robinson, the C^1 Closing Lemma states that if p is a nonwandering point of a C^1 vector field X on a compact manifold M then every neighborhood of X in the C^1 topology contains a vector field Y having a periodic orbit through p. The C^r Closing Lemma says that if X is C^r then Y can be found in any C^r neighborhood of X, $r \geq 0$.

For $r > 1$ the C^r Closing Lemma has not yet to be verified, even generically, and is known only for very special cases. For detailed historical comments, see [2, 4, 5, 6, 7].

Peixoto and Pugh [4] improved the above results at the chain recurrent points, which are the weakest type of recurrence in dynamics theory. To be precise, they proved that any chain recurrent point of a C^r vector field X on the plane $\mathbb{R}^2$ can be periodic under a C^r perturbation of X in the C^r Whitney topology if every fixed point of X is hyperbolic.

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Until 1992, there was an open fundamental question in dynamics, reasonably called the C^1 Connecting Lemma:
for a flow ϕ on a manifold M and $p, q \in M$, we suppose

$$\omega(p) \cap \alpha(q) \neq \emptyset$$

where $\alpha(q) = \{x \in M : \phi_{t_n}(q) \to x$ for some $t_n \to -\infty\}$. does there exist a flow that C^1-approximates ϕ for which p, q lie on the same orbit?

Pugh gave an example to show that the C^1 Connecting Lemma is false. The example was constructed in the plane using the concept of the flow plug. Also he constructed a C^1 flow ϕ on the punctured torus, $T^2 - \{0\}$, with chain recurrent orbits which cannot be periodic using C^1 small perturbations (see figure 8 in [6]). Consequently, he disproves the C^1-Closing Lemma for chain recurrence on noncompact 2-manifolds in general, but on compact 2-manifolds it is still an open question (for more details, see [6]).

The purpose of this note is to try to solve the above open question, using the technics which are used to prove the C^0 Closing Lemma for chain recurrence in noncompact n-manifolds .

Throughout the paper, we will follow the definitions and notations given in [2].

THEOREM. *Let M be a compact orientible C^∞-manifold of dimension 2, X a C^r vector field on M with $r \geq 1$, and p a chain recurrent point of X. For each neighborhood $\mathcal{U}$ of X in the space $\mathcal{X}^r(M)$ of C^r vector fields on M with the C^r topology, there exists $Y \in \mathcal{U}$ such that Y has a closed orbit through p.*

Proof. Let $p \in M$ be a nontrivial chain recurrent point for X, and $\varepsilon > 0$ be arbitrary. Using the same technics as in the proof of Lemma 3.5 in [2], we can choose $0 < a < \delta(p)$ such that for any time $t \in [-a, a]$, two points $O(\phi^\perp(p,t)) \cap O^\perp(p, -\delta(p))$ and p can be connected by a trajectry arc of a vector field $Y \in \mathcal{U}(X, \varepsilon)$, with $Y = X$ outside N_p. Choose a continuous function $\xi : M_0 \to (0, \infty)$ such that

(1) $B(p, \xi(p)) \subset \{O(\phi^\perp(p,s)) \cap O^\perp\phi(p,t)) : |t| \leq \delta(p), |s| \leq \frac{a}{3}\}$
(2) $B(x, \xi(x)) \subset \{O(\phi^\perp(x,s)) \cap O^\perp(\phi(x,t)) : |t| \leq \delta(x), |s| \leq \frac{1}{3}c(x)\delta(x)\}$, if $x \neq p$.

Since p is chain recurrent, there exists $(\xi, 1)$-chain $(p_1, t_1), ..., (p_n, t_n)$ from p to p. Since $d(\phi(p_i, t_i), p_{i+1}) < \xi(p_{i+1})$ for $1 \le i \le n-1$, we can choose a C^{r+1}-curve $\alpha_i : [0, b_i] \to M_0$ such that

(1) $\alpha_i(0) = x_i = [p_i, \phi(p_i, t_i)] \cap O^{\perp}(\phi(p_{i+1}, -\delta(p_{i+1}))$
(2) $\alpha_i(b_i) = y_{i+1} = \phi(p_{i+1}, \delta(p_{i+1}))$
(3) $\dot{\alpha}_i(0) = X_{x_i}, \dot{\alpha}_i(b_i) = X_{y_{i+1}}$
(4) $\|\dot{\alpha}_i - X\|_r < \varepsilon$

Similarly we can choose a C^{r+1}-curve $\alpha_n : [0, b_n] \to M_0$ such that

(1) $\alpha_n(0) = x_n = [p_n, \phi(p_n, t_n)] \cap O^{\perp}(\phi(p, -\delta(p))$
(2) $\alpha_n(b_n) = p, \dot{\alpha}_n(0) = X_{x_n}, \dot{\alpha}_n(b_n) = X_p$
(3) $\|\dot{\alpha}_n - X\|_r < \varepsilon$

In this way, we can construct a simple closed C^{r+1}-curve α in M_0 such that

(1) $\alpha(0) = p = \alpha(T), \alpha(t+T) = \alpha(t)$, for some $T > 0$
(2) $\|\dot{\alpha} - X\|_r < \varepsilon$

Even if the flow boxes are overlap, we can select a periodic curve α which we want. Since $X_x^{\perp} \neq 0$ for any $x \in \alpha$, there exists a neighborhood U of α in M_0 such that $X_p^{\perp} \neq 0$ for any $p \in \bar{U}$. Choose $b > 0$ such that $\phi^{\perp}(\alpha \times [-b, b]) \subset U$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^{∞}-bump function such that $f(t) = 0$ for $|t| \ge b$, $f(0) = 1$, and $0 < f(t) < 1$ for $0 < |t| < b$. Define a vector field Y on M as follow; for any $x \in M$,

$$Y(x) = \begin{cases} X_x + f(t)V_y(t), \text{if } x = \phi^{\perp}(y, t), y \in \alpha, |t| \le b \\ X_x, \text{if } x \notin \phi^{\perp}(\alpha \times [-b, b]) \end{cases}$$

where $V_y(t)$ is the parallel transport of $\dot{\alpha}(y) - X_y$ along $O^{\perp}(y)$. Then we have $\|X - Y\|_r < \varepsilon$ and p is a periodic point of Y. This completes the proof. □

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