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SCHUR GROUPS OF COMMUTATIVE RINGS

EUNMI CHOI, HEISOOK LEE, KYUNGHEE SHIN

ABSTRACT. We study some properties of Schur functor and its subfunctors related to separable algebras and cyclotomic algebras.

1. Introduction

Let R be a commutative ring and $B(R)$ denote the Brauer group of equivalence classes $[A]$ of Azumaya R-algebra A as defined in [1]. If an Azumaya R-algebra A is the homomorphic image of a group ring RG for some finite group G then A is called a Schur algebra. Equivalence classes of Schur algebras form a subgroup $S(R)$ of Brauer group $B(R)$ and $S(R)$ is called the Schur group of R. If R is a field, every element of $S(R)$ is represented by a cyclotomic R-algebra by a consequence of the Brauer-Witt theorem [7]. The classes in $B(R)$ represented by cyclotomic algebras form a subgroup $S'(R)$ of $S(R)$. Another subgroup $S''(R)$ of $S(R)$ consists of every algebra class which is a homomorphic image of a separable group ring RG. It is known that $[A] \in S''(R)$ if and only if A is a homomorphic image of a group ring RG for some finite group G whose order is a unit in R. If R is a field then $S(R)$, $S'(R)$ and $S''(R)$ are same.

For a homomorphism $f : R \to T$ of commutative rings, the map $S(f) : S(R) \to S(T)$ defined by $A \to A \otimes_R T$ (for $[A] \in S(R)$) is a well-defined group homomorphism and hence $S(-)$ is a functor from the category of commutative rings R to the category of abelian groups $S(R)$.

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In this paper we consider the Schur functor under the change of rings. We further consider subgroups of $S'(R)$ and $S''(R)$ of Schur group $S(R)$ of commutative ring R.

2. Preliminaries

Throughout the paper, we assume every ring is commutative with multiplicative identity. We also assume that for any ring homomorphism $f : R \to S$, we have $f(1_R) = 1_S$. We denote ε_n for a primitive n-th root of unity for $n > 0$.

Let G be a finite group of automorphisms of a commutative ring S, and let R be the subring $S^G = \{x \in S | \sigma x = x \text{ for all } \sigma \in G\}$ of S. We say S is a Galois extension of R with respect to G if there exist elements $x_1, \cdots, x_n, y_1, \cdots, y_n$ in S with

$$\sum_{i=1}^{m} x_i y_i = 1, \qquad \sum_{i=1}^{m} x_i \sigma_i(y_i) = 0$$

for all $\sigma(\neq 1)$ in G .

PROPOSITION 1. [2]. *An extension S of R is a Galois extension with Galois group G if and only if $S^G = R$ and for each maximal ideal M of S and for each $\sigma(\neq 1) \in G$, there exists an element x in S with $\sigma(x) - x \notin M$.*

For a connected commutative ring R, if $R(\varepsilon_n)$ is a Galois extension then a crossed product algebra $[R(\varepsilon_n)/R,\ H,\ f]$ where H is the Galois group of $R(\varepsilon_n)$ over R and f is a 2-cocycle over H is called a cyclotomic R-algebra. The cyclotomic algebra $[R(\varepsilon_n)/R,\ H,\ f]$ is a homomorphic image of RG, where G is a central extension of $\langle \varepsilon_n \rangle$ by H determined by the cocycle f. It is easy to see that the classes in $S(R)$ which are equivalent to cyclotomic algebras form a subgroup of $S(R)$ which is denoted by $S'(R)$.

Let $R \to T$ be a homomorphism of commutative rings and let S be a Galois extension of R with repect to group G. Then $T \otimes_R S$ is not necessarily a Galois extension of T. However, if $R(\varepsilon_n)$ is a Galois extension of R then it is easy to see that $T(\varepsilon_n)$ is a Galois extension of

T. Hence $S'(-)$ is a functor from the category of commutative rings to the category of abelian groups. On the other hand, if characteristic of R is positive then $S(R) = 0$ (refer to [3]). Let characteristic of R and T be zero. Given any homomorphism $R \to T$, if the positive integer n is a unit in R then n is a unit in T. Thus $S''(-)$ is also a functor from the category of rings of characteristic zero to the category of abelian groups.

3. Schur functors under change of rings

In this section we consider the behavior of the Schur functors $S(-)$ and its subgroups $S'(-)$ and $S''(-)$ under the change of rings.

It is not clear whether the isomorphism $B(R) \cong B(T)$ induced from a homomorphism $R \to T$ implies $S(R) \cong S(T)$. It is also not known that if $S(R) \cong S(T)$ then $S'(R) \cong S'(T)$ and $S''(R) \cong S''(T)$.

For an ideal I of R, we assume R is I-adically complete. Then $B(R) \cong B(R/I)$ and hence $S(R) \subset S(R/I)$ (refer to [4]). However it is not clear whether $S(R) \cong S(R/I)$. In particular, in case of (R, M) is a complete local ring, the isomorphism $S(R) \cong S(R/M)$ follows immediately, since $R \cong (R/M)[[x]]$ as in [4].

PROPOSITION 2. *Let I be an ideal of R such that $S(R) \cong S(R/I)$. Then $S'(R) \cong S'(R/I)$ and $S''(R) \cong S''(R/I)$.*

Proof. If $[A] \in S'(R)$ then there exists ε_n for $n > 0$ such that $R(\varepsilon_n)$ is a Galois extension of R and $[A] = [(R(\varepsilon_n)/R,\ G,\ f)]$ for a factor set f. Since $S(R) \cong S(R/I)$ and $(R/I)(\varepsilon_n)$ is a Galois extension of R/I, $S'(R)$ is contained in $S'(R/I)$. Conversely, let $(R/I)(\varepsilon_n)$ be a Galois extension of R/I with Galois group G. Then clearly $R(\varepsilon_n)^G = R$. For any maximal ideal M of R, $(M + I)/I$ is a maximal ideal of R/I and hence for each nontrivial $\sigma \in G$ there exists $\bar{x} \in (R/I)(\varepsilon_n)$ with $\sigma(\bar{x}) - \bar{x} \notin (M + I)/I$. Let y be an element in M such that $y + I = \bar{x}$. Then $\sigma(y) - y \notin M$ and this implies that $R(\varepsilon_n)$ is a Galois extension of R. If $[A] \in S'(R/I)$, then $[A]$ forms $[((R/I)(\varepsilon_n)/(R/I),\ G,\ f)]$. Since $R(\varepsilon_n)$ is a Galois extension of R, $[B] = [(R(\varepsilon_n),\ G,\ f)]$ is contained in $S'(R)$ and $[B \otimes R/I] = [A]$. This concludes $S'(R) \cong S'(R/I)$.

If characteristic of R/I is positive then $S(R/I) = 0$ and hence

$S''(R) \cong S''(R/I) = 0$, as $S(R) \cong S(R/I)$. If characteristic of R/I is zero, it suffices to show that the positive integer units in R and R/I are same. Any positive integer unit in R is clearly a unit in R/I. Conversely let n be a unit in R/I then $nm - 1 = 0$ in R/I for some positive integer m. Since characteristic of R/I is zero, $nm - 1 = 0$ in R, thus n is a unit in R. □

COROLLARY 3. *Let x be an indeterminate over a ring R. Then $B(R) \cong B(R[[x]]), S(R) \cong S(R[[x]])$, $S'(R) \cong S'(R[[x]])$ and $S''(R) \cong S''(R[[x]])$.*

Proof. Since $R[[x]]$ is (x)-adically complete, $B(R[[x]]) \cong B(R)$ (refer to [7]). From the canonical maps $R \to R[[x]] \to R$, we also have $S(R[[x]]) \cong S(R)$. Furthermore Proposition 2 gives rise to isomorphisms $S'(R[[x]]) \cong S'(R)$ and $S''(R[[x]]) \cong S''(R)$. □

If (R, M) is a complete local ring then $S(R) \cong S(R/M)$ implies $S'(R) \cong S'(R/M)$ and $S''(R) \cong S''(R/M)$, because of $S(R/M) \cong S'(R/M) \cong S''(R/M)$.

In the next proposition we consider the behavior of Schur group under polynomial extension. It is well known that if R is a regular domain of characteristic zero then $B(R) \cong B(R[x])$.

PROPOSITION 4. *Let x be an indeterminate over an integral domain R. If $B(R) \cong B(R[x])$ then $S(R) \cong S(R[x])$, $S'(R) \cong S'(R[x])$ and $S''(R) \cong S''(R[x])$.*

Proof. Since $B(R[x]) \cong B(R)$, the map $S(R[x]) \to S(R)$ induced from the natural epimorphism $R[x] \to R$ is a monomorphism. Since an inclusion map $R \hookrightarrow R[x]$ is split, mappings $B(R) \to B(R[x])$ and $S(R) \to S(R[x])$ are always monomorphisms. Hence it follows that $S(R) \cong S(R[x])$.

Since R is an integral domain, it can be seen easily that cyclotomic extension $R[x](\varepsilon_n)$ of $R[x]$ is a Galois extension if and only if $R(\varepsilon_n)$ is Galois extension of R. Thus $S'(R) \cong S'(R[x])$. Furthermore, it is also clear that $S''(R) \cong S''(R[x])$, because units of R and $R[x]$ are same.□

In [3], DeMeyer and Mollin studied basic properties of $S(R)$ and $S'(R)$, $S''(R)$ and their relationship – $S'(R)$ and $S''(R)$ may be proper subgroups of $S(R)$ and $S'(R)$ is may differ from $S''(R)$.

PRPOSITION 5. *If a ring R contains a field then $S''(R) = S(R)$.*

Proof. If characteristic of R is positive then $S(R) = 0$ and the result holds clearly. If characteristic of R is 0 then R contains the field of rational numbers and hence every nonzero integer is a unit in R and $S''(R) = S(R)$.

Let R be an integral domain and K be its field of quotients. The kernel of $B(R) \to B(K)$ is studied extensively in [6], which is in fact $\{\text{End}_R\,(M) \mid M$ is a finitely generated reflexive $R-$module such that $\text{End}_R(M)$ is projective R-module$\}$. If R is a regular domain and if $\text{End}_R(M)$ is projective with M finitely generated reflexive R-module then M is projective R-module and hence the above kernel is zero. The kernel of $S(R) \to S(K)$ may be contained in $\text{Ker}(B(R) \to B(K))$ properly as shown in [4]. □

REMARK. DeMeyer and Mollin showed in Proposition 2 [3] that $S''(R)$ is contained in $S'(R)$ if R is an integrally closed noetherian domain. However they used the fact that the natural map $S(R) \to S(K)$ is one to one, which is not true in general. Indeed we have the well known example

$$R = \frac{\Re[x, y]}{x^2 + y^2}$$

where $\Re$ is the field of real numbers, due to Auslander and Goldman [1]. Clearly, R is an integrally closed noetherian domain, and the usual quaternion algebra in $S(R)$ becomes trivial in $S(K)$ (refer to [4, (2)]). However the quaternion algebra is both in $S''(R)$ and $S'(R)$; it does not show $S''(R)$ is not contained in $S'(R)$ if R is an integrally closed noetherian domain. We note that $S''(R) = S(R)$ by Proposition 5, but we do not know $S'(R) = S(R)$ if R contains a field.

If R is a regular domain containing field then $S'(R) = S(R)$ by Proposition 5 and Proposition 2 [3]. So we have the next proposition.

PROPOSITION 6. *If R is a regular domain R containing a field then every Schur R-algebra is equivalent to a cyclotomic algebra, that is, the Brauer-Witt theorem holds for the ring R.*

Due to Brauer-Witt theorem, questions on the Schur subgroup $S(k)$ (k: field) are reduced to a treatment for a cyclotomic k-algebra, and almost all detailed results about Schur subgroups depend on it. Proposition 6 shows a sort of generalization of Brauer-Witt theorem to certain rings.

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EUNMI CHOI, DEPARTMENT OF MATHEMATICS, HANNAM UNIVERSITY, TAEJON 300-791, KOREA
E-mail: emc@math.hannam.ac.kr

HEISOOK LEE AND KYUNGHEE SHIN, DEPARTMENT OF MATHEMATICS, EWHA WOMANS UNIVERSITY, SEOUL 120-750, KOREA
E-mail: hsllee@mm.ewha.ac.kr