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A COMPACTNESS RESULT FOR A SET OF SUBSET-SUM-DISTINCT SEQUENCES

JAEGUG BAE

ABSTRACT. In this paper we obtain a "compactness" result that asserts the existence, in certain sets of sequences, of a sequence which has a maximal reciprocal sum. We derive this result from a much more general theorem which will be proved by introducing a metric into the set of sequences and using a topological argument.

1. Introduction

A subset-sum-distinct set of integers is one in which each subset is uniquely determined by its sum. It is intuitively reasonable that such a set must be rather "sparse". In fact, problems related to density of a subset-sum-distinct set have been considered by many mathematicians in various contexts (see [1, pp. 47-48], [3], [4], [5], [6], [7], [9], [10], [11, pp. 59-60], [12, p. 114, problem C8], [13], [14], [15]). Some of them ([2], [3], [6], [14]) involved, for a set C of subset-sum-distinct sequences $\{a_n\}_{n=1}^{\infty}$, the supremum of

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{{a_n}^s}$$

and the determination of an extremal sequence which obtains the supremum. The purpose of this paper is to establish the existence of such an extremal sequence for a quite general set C of subset-sum-distinct sequences by means of topological arguments. One of the most interesting examples of such C is the set of all subset-sum-distinct sequences such that no subset sum is congruent to a modulo q which is dealt in detail in [2]. To be more precise, we begin with our notation and some formal definitions.

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DEFINITION 1.1.

(i) We denote the set of all positive integers by $\mathbf{N}$ and $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ with the convention that, for any real number r,

$$r < \infty, \qquad \frac{r}{\infty} = 0.$$

(ii) We define

$$\mathcal{W} = \{(a_1, a_2, a_3, \cdots) : a_i \in \overline{\mathbf{N}}\},$$

$$\mathcal{I} = \{(a_1, a_2, a_3, \cdots) \in \mathcal{W} : a_i \leq a_{i+1} \text{ and } a_i < a_{i+1} \text{ if } a_{i+1} < \infty\}$$

and denote elements of $\mathcal{W}$ by $\{a_1, a_2, a_3, \cdots\}$ instead of $(a_1, a_2, a_3, \cdots)$.

DEFINITION 1.2.

(i) For a sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty} \in \mathcal{I}$,

$$rs(\mathbf{a}) = \sum_{n=1}^{\infty} \frac{1}{a_n}. \tag{1.1}$$

We call this the reciprocal sum of $\mathbf{a}$.

(ii) Let $\mathcal{C} \subseteq \mathcal{I}$. A sequence $\mathbf{m}$ in $\mathcal{C}$ is called a maximal sequence of $\mathcal{C}$ if $rs(\mathbf{m}) = \sup\{rs(\mathbf{a}) : \mathbf{a} \in \mathcal{C}\}$. We denote the set of all maximal sequences of $\mathcal{C}$ by $\mathcal{M}(\mathcal{C})$.

(iii) Let A be a set of real numbers. We say that A is a subset-sum-distinct set (briefly, A is an SSD-set or A is SSD) if for any two finite subsets X, Y of A,

$$\sum_{x \in X} x = \sum_{y \in Y} y \qquad \text{implies} \qquad X = Y.$$

Also, we say that a sequence $\{a_n\}_{n=1}^{\infty} \in \mathcal{I}$ is an SSD-sequence if $\mathbf{N} \cap \{a_n : n \in \mathbf{N}\}$ is SSD. We denote the set of all SSD-sequences by $\mathcal{S}$. Note that ϕ is SSD and the sequence $\{\infty, \infty, \infty, \cdots\} \in \mathcal{S}$.

(iv) For $1 \leq a < q$, $\mathcal{S}(a, q)$ denotes the set of all SSD-sequences such that no subset-sum is congruent to a modulo q. In other words, $\{a_n\}_{n=1}^{\infty} \in \mathcal{S}(a, q)$ if and only if

$$\{a_n\}_{n=1}^{\infty} \in \mathcal{S} \qquad \text{and} \qquad \sum_{i \in I} a_i \not\equiv a \pmod{q}$$

for any subset I of the positive intergers.

We now introduce a metric into $\mathcal{W}$ by means of a metric on $\overline{\mathbf{N}}$.

LEMMA 1.3. *Let* $n \in \mathbf{N}$ *be fixed. For any* $x, y \in \overline{\mathbf{N}}$, *define*

$$d_n(x,y) = \frac{1}{n} \cdot \left| \frac{1}{x} - \frac{1}{y} \right| .$$

Then $\left(\overline{\mathbf{N}}, d_n\right)$ *is a metric space. Furthermore, any subset of* $\mathbf{N}$ *is open in* $\overline{\mathbf{N}}$.

Proof. Obviously d_n defines a metric on $\overline{\mathbf{N}}$. For the second claim of the lemma, we show that, for any $a \in \mathbf{N}$,

$$(1.2) \qquad B_{d_n}(a,\epsilon) = \{a\} \quad \text{if} \quad \epsilon < \frac{1}{na} \cdot \min\left\{1 - \frac{a}{a+1},\ \frac{a}{a-1} - 1\right\} .$$

If $b \in \overline{\mathbf{N}}$ and $a \neq b$, then

$$n\, d_n(a,b) \;=\; \left|\frac{1}{a} - \frac{1}{b}\right| \;=\; \begin{cases} \frac{1}{a}\left(1 - \frac{a}{b}\right) \geq \frac{1}{a}\left(1 - \frac{a}{a+1}\right), & \text{if } a < b \\ \frac{1}{a}\left(\frac{a}{b} - 1\right) \geq \frac{1}{a}\left(\frac{a}{a-1} - 1\right), & \text{if } a > b . \end{cases}$$

Therefore we have (1.2). □

THEOREM 1.4. *For any two sequences* $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$ *in* $\mathcal{W}$, *let*

$$\rho(\mathbf{a},\mathbf{b}) = \sup\{d_n(a_n,b_n) : n = 1,2,3,\cdots\} .$$

Then ρ *metrizes the cartesian product topology of* $\mathcal{W}$.

Proof. See [8, p. 190, Theorem 7.2 (2)]. □

After preliminary results in the next section, we will establish the compactness of $\mathcal{W}$, $\mathcal{I}$, $\mathcal{S}$ and show that $\mathcal{M}(\mathcal{C})$, especially $\mathcal{M}(\mathcal{S}(a,q))$, is nonempty for any closed subset $\mathcal{C}$ of $\mathcal{S}$ in the third section.

2. An upper bound for $\sum_{n=1}^{\infty} 1/a_n$

In this section, we prove theorems about upper bounds, in terms of a_1, for $\sum_{n=1}^{\infty} 1/a_n$ where $\{a_n\}_{n=1}^{\infty} \in \mathcal{S}$. These results will also be of use in the next section. We begin with two lemmas.

LEMMA 2.1. *Let* $\{a_1, a_2, \cdots, a_n\}$ *be an SSD-set of positive integers. Then*

$$a_1 + a_2 + \cdots + a_n \ \geq \ 2^n - 1 .$$

Proof. Let $A = \{a_1, a_2, \cdots, a_n\}$ and $J = \left\{\sum_{b \in B} b \ : \ \phi \neq B \subset A\right\}$. We claim that $|J| = 2^n - 1$. Since A is an SSD-set, we obtain

$$B, B' \subset A \quad \text{and} \quad B \neq B' \qquad \text{imply} \qquad \sum_{b \in B} b \neq \sum_{b' \in B'} b'$$

from which the claim follows. Because $a_1 + a_2 + \cdots + a_n$ is the greatest element in J and $J \subseteq \mathbf{N}$, we have the lemma. □

LEMMA 2.2. *Let* $\{b_1, b_2, b_3, \cdots, b_m\}$ *be SSD. Then also the set*

$$A := \{K + b_1, K + b_2, K + b_3, \cdots, K + b_m\}$$

is SSD if $K > b_1 + b_2 + \cdots + b_m$.

Proof. Suppose that A is not SSD. Then there are two distinct subsets I, J of $\{1, 2, 3, \cdots, m\}$ such that $\sum_{i \in I}(K + b_i) = \sum_{j \in J}(K + b_j)$. Since $\{b_1, b_2, \cdots, b_m\}$ is SSD, we have $|I| \neq |J|$. So, we may assume that $|J| > |I|$. But then we have

$$K \ \leq \ (|J| - |I|) \, K \ = \ \sum_{i \in I} b_i - \sum_{j \in J} b_j \ \leq \ b_1 + b_2 + \cdots + b_m \ < \ K ,$$

a contradiction. □

THEOREM 2.3. *Let* $\mathbf{a} = \{a_n\}_{n=1}^{\infty} \in \mathcal{S}$ *and* $a_1 > 1$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \ \leq \ C \cdot \frac{\log a_1}{a_1}$$

where C *is an absolute constant.*

Proof. Let $b_k = a_{2k} - a_{2k-1}$ for $k = 1, 2, 3, 4, \cdots$. Since the sequence $\mathbf{a}$ is SSD, the set $\{b_1, b_2, b_3, \cdots\}$ is SSD also. Now, we claim that

$$a_{2k+1} \geq a_1 + b_1 + b_2 + \cdots + b_k, \qquad k = 1, 2, 3, \cdots. \tag{2.1}$$

We use induction on k. Since, by definition, $b_1 = a_2 - a_1$, we have $a_2 = a_1 + b_1$ which satisfies the claim (2.1) for $k = 1$. Now assume that

$$a_{2k+1} \geq a_1 + b_1 + b_2 + \cdots + b_k .$$

By definition, $b_{k+1} = a_{2k+2} - a_{2k+1}$, and so $a_{2k+2} = a_{2k+1} + b_{k+1}$. Thus

$$a_{2k+3} \geq a_{2k+2} = a_{2k+1} + b_{k+1} \geq a_1 + b_1 + b_2 + \cdots + b_k + b_{k+1}$$

and this completes the proof of the claim (2.1). Applying Lemma 2.1 to the set $\{b_1, b_2, b_3, \cdots b_k\}$, we obtain $a_{2k+1} \geq a_1 + b_1 + b_2 + \cdots + b_k \geq a_1 + 2^k - 1$ for $k = 0, 1, 2, 3, \cdots$. Therefore we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{a_n} &= \sum_{k=0}^{\infty} \left(\frac{1}{a_{2k+1}} + \frac{1}{a_{2k+2}} \right) \leq 2 \sum_{k=0}^{\infty} \frac{1}{a_{2k+1}} \\ &\leq 2 \sum_{k=0}^{\infty} \frac{1}{a_1 + 2^k - 1} \leq \frac{2}{a_1} + 2 \cdot \int_0^{\infty} \frac{1}{a_1 + 2^x - 1}\, dx \\ &= \frac{2}{a_1} + \frac{2}{\log 2} \cdot \frac{\log a_1}{a_1 - 1} \leq C \cdot \frac{\log a_1}{a_1}\end{aligned}$$

for some absolute constant C. □

Remark. We interpret $(\log a_1)/a_1 = 0$ when $a_1 = \infty$. Also, note that

$$\frac{a_1}{\log a_1} \left\{ \frac{2}{a_1} + \frac{2}{\log 2} \cdot \frac{\log a_1}{a_1 - 1} \right\}$$

is a decreasing function of a_1, so we may take $C = 6/\log 2$.

Next, we show that the inequality in Theorem 2.3 is essentially best possible in the following sense:

THEOREM 2.4. *Let $f(x)$ be a positive real valued function that is defined on $(1,\infty)$ such that*

$$f(x) \Big/ \frac{\log x}{x} \longrightarrow 0 \tag{2.2}$$

as $x \to \infty$. Then for any $K > 0$, there exists $\{a_n\}_{n=1}^{\infty} \in \mathcal{S}$ such that

$$a_1 > 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{a_n} > K \cdot f(a_1).$$

Proof. For sequences $\mathbf{a}(1)$, $\mathbf{a}(2)$, $\mathbf{a}(3)$, $\cdots$ in $\mathcal{S}$, we use the notations

$$\mathbf{a}(m) = \{a_{mn}\}_{n=1}^{\infty} \quad \text{for} \quad m = 1,2,3,\cdots.$$

We are to construct $\mathbf{a}(m)$ so that $a_{m1} > 1$, $\mathbf{a}(m) \in \mathcal{S}$ for $m = 1,2,3,\cdots$ and

$$\frac{1}{f(a_{m1})} \sum_{n=1}^{\infty} \frac{1}{a_{mn}} \longrightarrow \infty$$

as $m \to \infty$. Clearly $\{1,2,2^2,\cdots,2^{m-1}\}$ is SSD. Applying Lemma 2.2 with $K = 2^m$, we obtain the SSD property of the set

$$\{2^m+1, 2^m+2, 2^m+2^2, \cdots, 2^m+2^{m-1}\}.$$

Now, for a given positive integer m, we define

$$a_{mn} = \begin{cases} 2^m + 2^{n-1}, & \text{if } 1 \le n \le m \\ 2\sum_{i=1}^{n-1} a_{mi}, & \text{if } n > m. \end{cases}$$

From the construction, obviously $a_{m1} > 1$, $\mathbf{a}(m) \in \mathcal{S}$ for all m.

Also, we have

$$\begin{aligned}
\frac{1}{f(a_{m1})}\sum_{n=1}^{\infty}\frac{1}{a_{mn}} &\geq \frac{1}{f(a_{m1})}\sum_{n=1}^{m}\frac{1}{a_{mn}} \\
&= \frac{1}{f(a_{m1})}\sum_{n=1}^{m}\frac{1}{a_{m1}+2^{n-1}-1} \\
&\geq \frac{1}{f(a_{m1})}\int_0^{m-1}\frac{1}{a_{m1}+2^{x}-1}\,dx \\
&= \frac{1}{f(a_{m1})}\cdot\frac{1}{\log 2}\cdot\frac{\log a_{m1}-\log 3}{a_{m1}-1} \\
&\geq C\cdot\frac{1}{f(a_{m1})}\cdot\frac{\log a_{m1}}{a_{m1}}
\end{aligned}$$

for some positive constant C and for $m > 1$. Thus the theorem follows from (2.2) since $a_{m1} = 2^m + 1 \to \infty$ as $m \to \infty$. □

3. A compactness result

THEOREM 3.1. *$(\mathcal{W}, \rho)$ is a compact space.*

Proof. Since every infinite subset of $(\overline{\mathbf{N}}, d_n)$ has ∞ as a limit point, $(\overline{\mathbf{N}}, d_n)$ is limit point compact (see [16, p. 178]) for any $n \in \mathbf{N}$. Note that the compactness is equivalent to the limit point compactness in a metric space (see [16, p. 181, Theorem 7.4]). Hence $(\overline{\mathbf{N}}, d_n)$ is compact for all $n \in \mathbf{N}$. Applying the Tychonoff Theorem with Theorem 1.4, we conclude that $(\mathcal{W}, \rho)$ is compact. □

Now, we show the compactness of $\mathcal{I}$ and $\mathcal{S}$. We need the following two lemmas.

LEMMA 3.2. *Let $\mathbf{x} = \{x_n\}_{n=1}^{\infty} \in \mathcal{W}$. Then for any positive integer m, there exist $\epsilon > 0$ such that $\mathbf{a} = \{a_n\}_{n=1}^{\infty} \in B_\rho(\mathbf{x}, \epsilon)$ implies that, for all $n \in \{1, 2, \cdots, m\}$,*

$$a_n = x_n \ \text{ if } x_n < \infty \quad \text{and} \quad a_n > m \ \text{ if } x_n = \infty.$$

Proof. By the definition of ρ, $\mathbf{a} = \{a_n\}_{n=1}^{\infty} \in B_\rho(\mathbf{x}, \epsilon)$ implies that $d_n(a_n, x_n) < \epsilon$ for all $n \in \mathbf{N}$. Thus we have the conclusion by (1.2) of Lemma 1.3. □

LEMMA 3.3. *Let $\mathcal{C} \subseteq \mathcal{I}$. Then $\mathcal{C}$ is closed if and only if it has the following property: for any sequence $\mathbf{x} = \{x_n\}_{n=1}^{\infty} \in \mathcal{I}$, $\mathcal{C}$ contains $\mathbf{x}$ if, for any positive integer m, we can find $\mathbf{c}(m) = \{c_{mn}\}_{n=1}^{\infty}$ (depending on m) in $\mathcal{C}$ such that*

$$\begin{gathered} n \in \{1, 2, \cdots, m\} \quad \textit{implies} \\ c_{mn} = x_n \ \textit{ if } \ x_n < \infty \quad \textit{and} \quad c_{mn} > m \ \textit{ if } \ x_n = \infty . \end{gathered} \tag{3.1}$$

Proof. (Sufficiency): Assume that $\mathcal{C}$ is closed and $\mathbf{x} = \{x_n\}_{n=1}^{\infty} \in \mathcal{I}$ and for each m there is $\mathbf{c}(m) = \{c_{mn}\}_{n=1}^{\infty} \in \mathcal{C}$ such that (3.1) is true. Then since

$$d_n(c_{mn}, x_n) \ \le \ \frac{1}{n\, c_{mn}} \ \le \ \frac{1}{n(m+1)}$$

for $1 \le n \le m$ and $\sup\{d_n(c_{mn}, x_n) : n \ge m+1\} \le 1/(m+1)$, we have

$$\rho(\mathbf{c}(m), \mathbf{x}) \ = \ \sup\{d_n(c_{mn}, x_n) : n \ge 1\} \ \le \ \frac{1}{m+1} .$$

Thus $\mathbf{x}$ is a limit point of $\mathcal{C}$, and so $\mathbf{x} \in \mathcal{C}$.

(Necessity): Let $\mathbf{x} = \{x_n\}_{n=1}^{\infty} \in \mathcal{I}$ be a limit point of $\mathcal{C}$. Then, for any $\epsilon > 0$, $B_\rho(\mathbf{x}, \epsilon) \cap \mathcal{C}$ is nonempty. Hence, by Lemma 3.2, there is $\mathbf{c}(m) = \{c_{mn}\}_{n=1}^{\infty}$ such that (3.1) holds for each positive integer m. Thus $\mathbf{x} \in \mathcal{C}$ which means $\mathcal{C}$ is closed in $\mathcal{I}$. □

THEOREM 3.4. *$\mathcal{I}$ is compact in $\mathcal{W}$.*

Proof. Let $\mathbf{x} = \{x_n\}_{n=1}^{\infty} \in \mathcal{W}$ be a limit point of $\mathcal{I}$. Then, for any $\epsilon > 0$, there exist $\mathbf{a} = \{a_n\}_{n=1}^{\infty} \in B_\rho(\mathbf{x}, \epsilon) \cap \mathcal{I}$. Suppose that $\mathbf{x} \notin \mathcal{I}$. Then we can find a positive integer k such that

$$x_k > x_{k+1} \quad \text{or} \quad x_k = x_{k+1} < \infty . \tag{3.2}$$

If $x_k < \infty$, then apply Lemma 3.2 with $m > k$ to find ϵ so that $a_k = x_k$ and $a_{k+1} = x_{k+1}$. Since $\mathbf{a} \in \mathcal{I}$ we have $x_k = a_k < a_{k+1} = x_{k+1}$ which contradicts (3.2). If $x_k = \infty$, then apply Lemma 3.2 with $m > x_{m+1}$ to find ϵ so that $a_k > m > x_{k+1}$ and $a_{k+1} = x_{k+1}$. Again we have $x_{k+1} < m < a_k \leq a_{k+1} = x_{k+1}$ which is impossible. Hence we may conclude $\mathbf{x} \in \mathcal{I}$ and so $\mathcal{I}$ is closed in $\mathcal{W}$. Now the theorem follows since $\mathcal{W}$ is compact by Theorem 3.1. □

THEOREM 3.5. *$\mathcal{S}$ is compact in $\mathcal{W}$.*

Proof. By Lemma 3.3, $\mathcal{S}$ is closed in $\mathcal{I}$. Hence the proof follows immediately from Theorem 3.1 and 3.4. □

THEOREM 3.6. *Let $\mathbf{R}$ be the set of all real numbers with the usual topology. Then the function*

$$rs : \mathcal{S} \longrightarrow \mathbf{R}$$

defined by (1.1) *in* Definition 1.2 (i) *is continuous.*

Proof. Let ϵ be given. We are supposed to find $\delta > 0$ such that

$$\mathbf{x} \in B_\rho(\mathbf{a}, \delta) \cap \mathcal{S} \quad \text{implies} \quad |rs(\mathbf{a}) - rs(\mathbf{x})| < \epsilon .$$

Take a positive integer m large enough so that

$$3C \cdot \frac{\log m}{m} < \epsilon \tag{3.3}$$

where C is the constant of Theorem 2.3. By Lemma 3.2, there exists $\delta > 0$ such that, if $\mathbf{x} \in B_\rho(\mathbf{a}, \delta) \cap \mathcal{S}$, then for all $n \in \{1, 2, \cdots, m\}$,

$$a_n = x_n \text{ if } a_n < \infty \quad \text{and} \quad x_n > m \text{ if } a_n = \infty . \tag{3.4}$$

Let $I = \{i \in \mathbf{N} : 1 \leq i \leq m, \ a_i = \infty\}$ and i_0 the smallest element of I if I is nonempty. By (3.4) and Theorem 2.3, we have

$$\sum_{n=1}^{m} \left| \frac{1}{a_n} - \frac{1}{x_n} \right| = \sum_{i \in I} \frac{1}{x_i} \leq C \cdot \frac{\log x_{i_0}}{x_{i_0}} \leq C \cdot \frac{\log m}{m} . \tag{3.5}$$

Apply Theorem 2.3 again with the fact that $a_m \geq m$ and $x_m \geq m$ to obtain

$$(3.6) \qquad \begin{aligned} \sum_{n=m}^{\infty} \frac{1}{a_n} &\leq C \cdot \frac{\log a_m}{a_m} \leq C \cdot \frac{\log m}{m}, \\ \sum_{n=m}^{\infty} \frac{1}{x_n} &\leq C \cdot \frac{\log x_m}{x_m} \leq C \cdot \frac{\log m}{m}. \end{aligned}$$

Thus, combining (3.3), (3.5), and (3.6), we have

$$\begin{aligned} |rs(\mathbf{a}) - rs(\mathbf{x})| &\leq \sum_{n=1}^{\infty} \left| \frac{1}{a_n} - \frac{1}{x_n} \right| \\ &\leq \sum_{n=1}^{m} \left| \frac{1}{a_n} - \frac{1}{x_n} \right| + \sum_{n=m}^{\infty} \left| \frac{1}{a_n} - \frac{1}{x_n} \right| \\ &\leq \sum_{n=1}^{m} \left| \frac{1}{a_n} - \frac{1}{x_n} \right| + \sum_{n=m}^{\infty} \frac{1}{a_n} + \sum_{n=m}^{\infty} \frac{1}{x_n} \leq \epsilon. \end{aligned}$$ □

THEOREM 3.7. *If* $\mathcal{C}$ *is closed in* $\mathcal{S}$, *then* $\mathcal{M}(\mathcal{C})$ *is nonempty.*

Proof. By Theorem 3.5 and Theorem 3.6, $rs(\mathcal{C})$ is compact in $\mathbf{R}$, the set of all real numbers. Hence $\mathcal{M}(\mathcal{C})$ is nonempty. □

COROLLARY 3.8. $\mathcal{M}(\mathcal{S}(a,q))$ *is nonempty.*

Proof. By Lemma 3.3, it is obvious that $\mathcal{S}(a,q)$ is closed. Thus the corollary follows from Theorem 3.7. □

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Department of Applied Mathematics, Korea Maritime University, Pusan 606-791, Korea
E-mail: jgbae@hanara.kmaritime.ac.kr