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ON THE EXISTENCE OF SOLUTIONS OF THE HEAT EQUATION FOR HARMONIC MAP

DONG PYO CHI, HYUN JUNG KIM*, AND WON KUK KIM

ABSTRACT. In this paper, we prove the existence of solutions of the heat equation for harmonic map on a compact manifold with a boundary when the target manifold is allowed to have positively curved parts.

1. Introduction

Let (M, f) and (N, γ) be Riemannian manifolds of dimension m and n respectively. Let $\{x^\alpha\}_{\alpha=1}^m$ and $\{y^i\}_{i=1}^n$ be the local coordinates of M and N, respectively, and let f be defined by $f = \sum_{\alpha\beta} f_{\alpha\beta} dx^\alpha dx^\beta$ in this local expression.

Let $u : M \times [0, \infty) \to N$ be a map which is represented as $u = (u^1, \dots, u^n)$ in terms of the above local coordinates. We say u satisfies the heat equation for harmonic maps if it is a solution of the following nonlinear parabolic system:

$$(\Delta - \frac{\partial}{\partial t})u^i(x,t) = f^{\alpha\beta}(x)\Gamma^i_{jk}(u(x,t))\frac{\partial u^j}{\partial x^\alpha}(x,t)\frac{\partial u^k}{\partial x^\beta}(x,t),$$

for $i = 1, \dots, n$, where $(f^{\alpha\beta}) = (f_{\alpha\beta})^{-1}$ and $\Gamma^i_{jk}(y)$ is the Christoffel symbol at y in N.

Let $\Lambda = (M \times \{0\}) \cup (\partial M \times [0, \infty))$, and $\Lambda_T = (M \times \{0\}) \cup (\partial M \times [0, T))$, for all $T > 0$. Let $\psi : \Lambda \to N$ be a given map. The boundary

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value problem of the heat equation for harmonic map is to find a map $u : M \times [0, \infty) \to N$ which satisfies

(1.1)
$$(\Delta - \frac{\partial}{\partial t})u^i(x,t) = f^{\alpha\beta}(x)\Gamma^i_{jk}(u(x,t))\frac{\partial u^j}{\partial x^\alpha}(x,t)\frac{\partial u^k}{\partial x^\beta}(x,t)$$
$$u|_\Lambda(x,t) = \psi(x,t),$$

for $i = 1, \ldots, n$.

The heat equation for harmonic map has been investigated by many mathematicians for many years. Eells and Sampson proved the existence of the unique solution of (1.1) when the domain manifold is a compact manifold without boundary [5]. R. Hamilton proved it for the case when the domain manifold is a compact manifold with boundary, but he dealt with only the case when the target manifold is negatively curved [7]. W. Kendall showed the existence problem of solutions of the heat equation for harmonic map when the target manifold has positive curvature parts and the domain manifold is a compact manifold with boundary in a similar fashion as R. Hamilton's [10]. He proved it using not the analytic method, which was used in [5] and [7], but the probabilistic method. The goal of this paper is to give an analytic proof of the same results of W. Kendall. And our domain manifold is the same as Hamilton's but the target manifold is different, so our method of proof is different from it. We get the solution of (1.1) by applying the Leray-Schauder degree theory to the nonlinear parabolic system. The main idea of proof comes from the proof in [8], but we get the gradient estimate of the solution of (1.1), which is the important part of the proof in using the Leray Schauder degree theory, in a different way. Hilderbrandt et al. [8] used the distance function on N from a fixed point as a convex function on N, because the target manifold N is only nonpositively curved. Since our target manifold N is allowed to have positive curvature parts as well, we need to define a new convex function instead of the distance function.

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2. Preliminaries

Suppose that (N, γ) is a Riemannian manifold with the sectional curvature bounded above by a positive constant $K > 0$. Without loss of generality, we set $K = 1$. Let $q \in N$ be given and $B_r(q)$ is the geodesic ball with raius $r \leq \min\{\frac{\pi}{2}, \ \tau\}$ and center q, and where τ is the injectivity radius at q. Then $B_r(q)$ is diffeomorphic to a Euclidean ball in $\mathbb{R}^n$ with center $0 = (0, \ldots, 0)$ and radius r, the diffeomorphism being given by any normal coordinate system at q. Hence using the normal coordinates, any map $u : M \times [0, \infty) \to B_r(q)$ can be represented by vector valued functions $u = (u^1, \ldots, u^n) : M \times [0, \infty) \to \mathbb{R}^n$.

Now the notations which will be used through the present paper are introduced. Choose an orthonormal frame $\{e_\alpha, \frac{\partial}{\partial t}\}$ in a neighborhood of $(x, t) \in M \times [0, \infty)$ and a local orthonormal frame $\{f_i\}$ in a neighborhood of $u(x, t) \in N$. Let $\{\theta_\alpha, dt\}$ and $\{\omega_i\}$ be the dual coframes of $\{e_\alpha, \frac{\partial}{\partial t}\}$ and $\{f_i\}$, respectively.

Denote $d = d_M + \frac{\partial}{\partial t} dt$ is a cannonical differential on $M \times [0, \infty)$ where d_M is a differential on M. Let us define $u_{i\alpha}$ by

$$u^*(\omega_i) = \sum_\alpha u_{i\alpha}\theta_\alpha + u_{it}dt.$$

By taking the covariant derivative of the above equation, we get $u_{i\alpha\beta}$ by

$$\sum_\beta u_{i\alpha\beta}\theta_\beta + u_{i\alpha t}dt = du_{i\alpha} + \sum_j u_{j\alpha}u^*\omega_{ji} + \sum_\beta u_{i\beta}\theta_\beta\alpha.$$

Since $du_{i\alpha} = d_M u_{i\alpha} + u_{i\alpha t}dt$,

$$\sum_\beta u_{i\alpha\beta}\theta_\beta = d_M u_{i\alpha} + \sum_j u_{j\alpha}u^*\omega_{ji} + \sum_\beta u_{i\beta}\theta_{\beta\alpha}.$$

It is well known that the heat equation for harmonic map (1.1) is equivalent to

$$u_{it} = u_{i\alpha\alpha}$$

for $i = 1, \ldots, n$.

We define the energy function $e(u)$ of u by $e(u)(x, t) = \sum_{i\alpha} u_{i\alpha}^2(x, t)$.

For $p \in \overline{B_r(q)}$, let us define a function $\phi_p : N \to \mathbb{R}$ by

$$\begin{aligned}\phi_p(y) &= \frac{1-\cos\rho(y,p)}{\cos\rho(y,q)} \\ &= \frac{1-\cos\rho_p(y)}{\cos\rho_q(y)} =: \frac{g(y)}{h(y)}\end{aligned}$$

where $p \in \overline{B_r(q)}$ and ρ_q, ρ_p are the distance functions from q, p on N, respectively [9].

LEMMA 2.1. *ϕ_p is convex for all $p \in \overline{B_r(q)}$. Furthermore, if $u : M \times [0,\infty) \to N$ is a heat equation for harmonic map with $u(M \times [0,\infty)) \subset B_r(q)$, then*

$$\begin{aligned}(\Delta - \frac{\partial}{\partial t})(\phi_p \circ u)(x,t) &\geq \frac{1}{2}\phi_p(u(x,t))e(u)(x,t) \\ &\geq \frac{1}{2}(1-\cos\rho_p(u))e(u)(x,t) \geq 0,\end{aligned}$$

for all $(x,t) \in M \times [0,\infty)$.

Proof. We have

$$\begin{aligned}(\phi_p)_{ij} &= (\frac{g}{h})_{ij} \\ &= \frac{h^2(g_{ij}h - g_i h_j - g_j h_i - g h_{ij}) + 2hgh_ih_j}{h^4}.\end{aligned}$$

We can get $g_{ij} \geq \cos\rho_p\delta_{ij}$ and $h_{ij} \leq -\cos\rho_q\delta_{ij}$, on $B_r(q)$. Inserting these into $(\phi_p)_{ij}$, one can obtain

$$\begin{aligned}&(\phi_p)_{ij} \\ &\geq \frac{h\{\cos\rho_q\delta_{ij} + \sin\rho_p\sin\rho_q((\rho_p)_i(\rho_q)_j + (\rho_q)_i(\rho_p)_j)\}}{h^3} \\ &\quad + \frac{2(1-\cos\rho_p)\sin^2\rho_q(\rho_q)_i(\rho_q)_j}{h^3}.\end{aligned}$$

This proves $(\phi_p)_{ij}\psi^i\psi^j \geq 0$ for all functions $\psi = (\psi^i) : N \to \mathbb{R}^n$, which is the proof of convexity of ϕ_p.

Now for the convenience of notation, let $g = g \circ u$ and $h = h \circ u$. Then since

$$(\Delta - \frac{\partial}{\partial t})(g \circ u)(x,t) \geq \cos \rho_p(u(x,t))e(u(x,t))$$
$$\text{and } (\Delta - \frac{\partial}{\partial t})(h \circ u)(x,t) \leq -\cos \rho_q(u(x,t))e(u(x,t)),$$

we can get

$$\begin{aligned}
&(\Delta - \frac{\partial}{\partial t})(\phi_p \circ u) \\
&= \frac{1}{h^2}\{h(\Delta - \frac{\partial}{\partial t})g - g(\Delta - \frac{\partial}{\partial t})h\} - \sum_\alpha \frac{g_\alpha^2}{2gh} + \sum_\alpha \frac{(hg_\alpha - 2gh_\alpha)^2}{2h^3 g} \\
&\geq \frac{1}{h^2}\{h(\Delta - \frac{\partial}{\partial t})g - g(\Delta - \frac{\partial}{\partial t})h\} - \frac{\sin^2 \rho_p(u)}{2hg} e(u) \\
&\geq \frac{h \cos \rho_p(u) + g \cos \rho_q(u)}{h^2} e(u) - \frac{\sin^2 \rho_p(u)}{2hg} e(u) \\
&\geq \frac{2g - \sin^2 \rho_p(u)}{2hg} e(u) \\
&= \frac{g}{2h} e(u) \geq \frac{1}{2} h e(u).
\end{aligned}$$

□

Before we state the main theorem, let us introduce the following notations. Let $y = (y^1, \ldots, y^n)$ be normal coordinates at q of $B_r(q)$. If $u : M \times [0, \infty) \to B_r(q) \subset N$, u can be written as $u = (u^1, \ldots, u^n)$ with respect to this normal coordinates. Then the norm $|u(x,t)|$ in $\mathbb{R}^n$ is the same as the distance $\rho(u(x,t), q)$ from q in N. We shall use the following two kinds of norms

$$\begin{aligned}
||u||_{C_T^1} &= \sup_{M\times[0,T]} |u(x,t)| + \sup_{M\times[0,T]} |D_x u(x,t)|, \\
||u||_{C_T^{1+c}} &= ||u||_{C_T^1} + \sup_{M\times[0,T]} \frac{|D_x u(x,t) - D_x u(x',t)|}{\gamma(x,x')^c} \\
&\quad + \sup_{M\times[0,T]} \frac{|u(x,t) - u(x,t')|}{|t-t'|^c},
\end{aligned}$$

where $\gamma(x,x')$ is the distance between x and x' on M, for $0<c<1$ and $0<T<\infty$. These are defined in the usual manner, using an arbitrary, but fixed, finite atlas of M. These two different atlases yield equivalent norms.

3. Gradient estimate and existence theorem

For any given $C^{1+\alpha}$ function $\psi:\Lambda\to B_r(q)\subset N$, we consider the following system:

$$(\Delta-\frac{\partial}{\partial t})u^i(x,t)=f^{\alpha\beta}(x)\Gamma^i_{jk}(u(x,t))\frac{\partial u^j}{\partial x^\alpha}(x,t)\frac{\partial u^k}{\partial x^\beta}(x,t),$$
$$u|_\Lambda(x,t)=\psi(x,t),$$

for $i=1,\ldots,n$.

First, we have to get the C^1- estimate of the solution of (1.1) in order to use the Leray-Schauder degree theory. C^0- estimate of the solution of (1.1) and interior estimate of energy of the solution of (1.1) can be easily obtained by the same method as that in [2]. The boundary estimate of energy is obtained by a modification of the proof in [8].

THEOREM 3.1. *Suppose $\psi:\Lambda\to B_r(q)$ is of class C^{1+c} and u is a solution of* (1.1), *where $c>0$. Then for all $T>0$,*

$$||u||_{C_T^{1+c}}\le C,$$

where C depends only on $||\psi||_{C_T^{1+c}}$ and the geometries of M and N.

Proof. First, we have to claim that $u(M\times[0,\infty))\subset B_\tau(p)$, that is a C^0-estimate of u.

As the same in Lemma 2.1, define $g(x,t):M\times[0,\infty)\to\mathbb{R}$ by

$$g(x,t)=1-\cos(\rho(u(x,t),p)).$$

Then

$$(\Delta-\frac{\partial}{\partial t})g(x,t)\ge\cos\rho_p(u(x,t))e(u(x,t))\ge 0.$$

Since $u(\Lambda) = \psi(\Lambda) \subset B_\tau(p)$, we can get $g|_\Lambda(x,t) < 1 - \cos\tau$. Then by the maximum principle, we have $g(x,t) \leq 1 - \cos\tau$. Therefore $\cos(\rho(u(x,t),p)) \geq \cos\tau$, which implies that $u(x,t) \in B_\tau(p)$ for all $(x,t) \in M \times [0,\infty)$.

Let $x_0 \in M$ be any point and $a > 0$. Let γ be the distance function from x_0 in M and let $B_a(x_0)$ be the closed geodesic ball of radius a and center x_0 in M.Take any $T > 0$. Let $\sup_\Lambda g(x,t) = b_1$. We can choose a constant $b > 0$ such that $\sup_{M\times[0,\infty)} g(x,t) < b_1 < b$.

Let us consider the function

$$\Phi = \left\{ \frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2} \right\},$$

which is defined on $(B_a(x_0) \cap M) \times [0,T]$.

Since $\Phi|\partial B_a(x_0) = 0$, Φ attains its maximum on $(B_a(x_0) \cap M) \times [0,T]$. Let

$$\Phi(x_1,t_1) = \max_{(B_a(x_0)\cap M)\times[0,T]} \left\{ \frac{(a^2-\gamma^2)^2 e(u)}{(b-g^2)^2} \right\}.$$

Then we can have the three cases : $(x_1,t_1) \in B_a(x_0) \times \{0\}$, $(B_a(x_0) - \partial M) \times (0,T]$ or $(B_a(x_0) \cap \partial M) \times (0,T]$.

In the first case, i.e.$(x_1,t_1) \in B_a(x_0) \times \{0\}$,

$$\begin{aligned} \frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2}(x,t) &\leq \frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2}(x_1,t_1) \\ &\leq \frac{a^4}{(b-g)^2} \sup_{\Lambda_T} e(\psi), \end{aligned}$$

for $(x,t) \in B_a(x_0) \times [0,T]$. Then we have, for $(x,t) \in B_{\frac{a}{2}}(x_0) \times [0,T]$,

$$e(u)(x,t) < \frac{16}{9} \frac{b^4}{(b-b_1)^2} \sup_{\Lambda_T} e(\psi). \tag{3.1}$$

In the second case, i.e. when $(x_1,t_1) \in B_a(x_0) \times (0,T]$, by a similar computations as in [1], we have

$$\begin{aligned} &e(u)(x_1,t_1) \\ \leq& 4\max\left\{ \frac{128\gamma^2}{(a^2-\gamma^2)^2},\quad (b-g) + \frac{C_1(1+\gamma)(b-g)}{(a^2-\gamma^2)} + \frac{8\gamma^2(b-g)}{(a^2-\gamma^2)^2} \right\}. \end{aligned}$$

For any $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0,T]$,

$$\left\{\frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2}\right\}(x,t) \le \left\{\frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2}\right\}(x_1,t_1)$$
$$\le 4\max\left\{\frac{16a^2}{(b-g(x_1,t_1))^2},\ \frac{a^2}{(b-g(x_1,t_1))} + \frac{C_1(1+a)a^2}{(b-g(x_1,t_1))} + \frac{8a^2}{(b-g(x_1,t_1))^2}\right\}.$$

Therefore

$$(3.2)\qquad \begin{aligned} &e(u)(x,t)\\ \le&4\max\left\{\frac{256b^2}{9(b-b_1)^2a^2},\ \frac{16Kb^2}{9(b-b_1)} + \frac{16C_1(1+\sqrt{K}a)b^4}{9a^2(b-b_1)} + \frac{128b^2}{9a^2(b-b_1)}\right\},\end{aligned}$$

for $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0.T]$.

We consider the last case $(x_1,t_1) \in \partial M \times [0,T]$. Let n be the outer normal vector of ∂M at (x_1,t_1), and $p = u(x_1,t_1)$. Since $u(x,t) = \psi(x,t)$ for all $(x,t) \in \Lambda_T$, $e(u)(x_1,t_1) \le C_1(||\psi||^2_{C^1_T} + ||\frac{\partial u}{\partial n}||^2_{(x_1,t_1)})$, for the same constant C_1 depending only on the geometries of M and N. Hence it suffices to get the estimate of $||\frac{\partial u}{\partial n}||_{(x_1,t_1)}$.

One can choose a sufficiently small $\delta > 0$ such that $1 - 2\sin\frac{\delta}{2} > 0$, and let $p_0 \in N$ be a point on the geodesic in the direction $\frac{\partial u}{\partial n}|_{(x_1,t_1)}$ with $\rho_p(p_0) = \delta$. Define

$$w(x,t) = \phi_p(u(x,t)) + \frac{1}{2}\{1 - \cos\rho(u(x,t),p_0)\} - \eta,$$

where ϕ_p is as defined in Section 2 and η is the solution of the following linear heat equation with boundary condition:

$$(\Delta - \frac{\partial}{\partial t})\eta = 0,$$
$$\eta|_{\Lambda_T} = \frac{1-\cos\rho_p(\psi)}{\cos\rho_q(\psi)} + \frac{1}{2}\{1-\cos\rho(\psi,p_0)\}.$$

The well-known Schauder estimate for the partial differential equations of parabolic type implies $||\eta||_{C^1_T} \le C(||\psi||_{C^{1+c}_T})$. Applying Lemma 2.1, we can easily get

$$(\Delta - \frac{\partial}{\partial t})w \ge \frac{1}{2}\{1 - \cos\rho_p(u) + \cos\rho(u, p_0)\}e(u)$$
$$= \frac{1}{2}\left\{1 - 2\sin\frac{\rho_p(u) + \rho(u, p_0)}{2}\sin\frac{\rho_p(u) - \rho(u, p_0)}{2}\right\}e(u)$$
$$\ge \frac{1}{2}(1 - 2\sin\frac{\delta}{2})e(u) > 0,$$

where the last inequality comes from the fact that $\rho_p(u) - \rho(u, p_0) \le \rho_p(p_0) \le \delta$ i.e. w is a subsolution of linear heat equation. By the easy computation, $\frac{\partial}{\partial n}|_{(x_1,t_1)}\phi_p(u(x,t)) = 0$.

Since $w|_{\Lambda_T} = 0$ and

$$\frac{\partial}{\partial n}|_{(x_1,t_1)}(1 - \cos\rho(u(x,t), p_0)) = -\sin\delta||\frac{\partial u}{\partial n}||_{(x_1,t_1)},$$

we get

$$0 \le \frac{\partial w}{\partial n}|_{(x_1,t_1)} = -\sin\delta||\frac{\partial u}{\partial n}||_{(x_1,t_1)} - \frac{\partial \eta}{\partial n}|_{(x_1,t_1)}$$
$$||\frac{\partial u}{\partial n}||_{(x_1,t_1)} \le \frac{1}{\sin\delta}||\frac{\partial \eta}{\partial n}||_{(x_1,t_1)} \le \frac{1}{\sin\delta}||\eta||_{C^1} \le C_2||\psi||_{C^{1+c}_T},$$

for some constant C_2 depending only on δ and the geometries of M and N.

We have by the above computation (3.1) and (3.2),

$$e(u)(x,t) \le 4\max\left\{\frac{b^2}{(b^2 - b_1^2)^2}\sup_{\Lambda_T} e(\psi), \frac{256b^4}{9(b^2 - b_1^2)^2a^2},\right.$$
$$\frac{16b^4}{9(b^2 - b_1^2)} + \frac{16C_1(1+a)b^4}{9a^2(b^2 - b_1^2)}$$
$$\left. + \frac{128b^4}{9a^2(b^2 - b_1^2)}, C_2||\psi||_{C^{1+c}_T}\right\},$$

for $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0.T]$.

Since a is arbitrary, as a goes to infinity, we have

$$\sup_{M\times[0,T)} e(u) \le C_3,$$

for C_3 depends only on $||\psi||_{C_T^{1+c}}$ and the geometries of M and N. □

Let $y = (y^1, \ldots, y^n)$ be normal coordinates at q of $B_r(q)$, and let h_{ij} be the metric of N with respect to this normal coordinates. For $0 \le s \le 1$, let us define a new metric ${}^s\gamma_{ij}(y) = \gamma_{ij}(sy)$. Note that ${}^1\gamma_{ij} = \gamma_{ij}$ and ${}^0\gamma_{ij}$ is a flat metric on $B_r(q)$. Furthermore, since there is no change of metric in the radial direction, y is still a normal coordinates for the metric ${}^sh_{ij}$. The Christoffel symbol ${}^s\Gamma^i_{ik}(y)$ with respect to the metric ${}^s\gamma$ is $s\Gamma^i_{ik}(sy)$, and the heat equation for harmonic map with ${}^sh_{ij}$ on the target becomes

(H_s)

$$\begin{aligned} &(\Delta - \frac{\partial}{\partial t})u^i_s(x,t) + f^{\alpha\beta}(x,t)s\Gamma^i_{jk}(su_s(x,t))\frac{\partial u^j_s}{\partial x^\alpha}(x,t)\frac{\partial u^k_s}{\partial x^\beta}(x,t) = 0 \\ &u_s(x,t) = \psi(x,t), \qquad\qquad (x,t) \in \Lambda, \end{aligned}$$

Note that the solution of (H_1) is the solution of (1.1). To prove the main theorem, it is important to get the energy estimate of u_s independent of s. It is easy to check that the upper bound of the sectional curvaturre with repect to ${}^sh$ is the same as that with repect to h. We can get the following theorem.

THEOREM 3.2. *Suppose for all $0 \le s \le 1$, $\psi : \Lambda \to B_r(q)$ is of class C^{1+c} and u_s is a solution of (H_s). Then for all $T > 0$,*

$$||u||_{C_T^{1+c}} \le C,$$

where C depends only on $||\psi||_{C_T^{1+c}}$ and the geometries of M and N.

Now we prove the existence of solutions of the heat equation for harmonic map using Leray-Schauder degree theory.

THEOREM 3.3. *Let (M, f) be a Riemannian manifold with boundary ∂M and let (N, γ) be a Riemannian manifold with the sectional curvature bounded above by $K > 0$. Let $\Lambda = (M \times \{0\}) \cup (\partial M \times [0, \infty))$. To a given C^{1+c} function $\psi : \Lambda \to B_r(q)$, there exists the solution $u : M \times [0, \infty) \to B_r(q)$ of (1.1) in class C^3 on $M \times [0, \infty)$ and C^1 on Λ.*

Proof. Without loss generality, we may assume $K = 1$. To apply the Leray-Schauder degree theory, we need an appropriate Banach space, a bounded domain of the Banach space and a homotopy of maps. Let $T > 0$ be fixed.

First let us define the space B by the set of all C^1 maps from $M \times [0, T)$ to $\mathbb{R}^n$. Then clearly $(B, ||\cdot||)$ becomes a Banach space, where $||\cdot||$ is the C^1_T–norm.

Now, we define a homotopy of maps. Let $0 \leq s \leq 1$. For $u = (u^1, \ldots, u^n) \in B$, define

$$^sF^i(u) = \sum_{j,k,\alpha,\beta} s\Gamma^i_{jk}(su) f^{\alpha\beta} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta},$$

for all $i = 1, \ldots, n$.

Define $\Psi_s : B \to B$ by $\Psi_s(u) := v = (v^1, \ldots, v^n)$, where v is the solution of the following linear heat equation with boundary condition:

$$(\Delta - \frac{\partial}{\partial t}) v^i(x, t) = \; ^sF^i(u)(x, t) \qquad \text{on } M \times [0, \infty), \qquad v|_\Lambda = 0,$$

for all $i = 1, \ldots, n$.

For $u \in B$, $\Psi(u)$ is of class $C^{1+\beta}$ for some $0 < \beta < 1$ (see [6]), and Arzela-Ascoli theorem implies that Ψ is a compact mapping from B into B. Now let $h = h(\phi)$ be the uniquely determined solution of the boundary value problem of the linear equation

$$(\Delta - \frac{\partial}{\partial t}) h(x, t) = 0 \qquad \text{on } M \times [0, \infty), \qquad h|_\Lambda = \psi.$$

Let us define a homotopy $H_s : B \to B$ as follows,

$$H_s(u) = u - \Psi_s(u) - h.$$

By Theorem 3.2, there is a constant C_4 depending only on $||\psi||_{C_T^{1+c}}$ and the geometries of M and N such that $||u_s||_{C_T^1} \leq C_4$, for all the solution u of (H_s), where C_4 is independent of s. Let $D = \{u \in B \mid ||u|| \leq 2C_4\}$. Here the degree of H_s is calculated with respect to the set D and the element $0 \in B$. Note that for all $0 \leq s \leq 1$, any solution u_s of $H_s(u) = 0$ is the solution of (H_s) on $M \times [0, T]$, which is in D and

$$\sup_{M\times[0,T)} e(u_s) \leq C_4,$$

as above. And for all $0 \leq s \leq 1$, $u_s \notin \partial D$, from which $\deg(H_s, D, 0)$ is well-defined and is finite. Since $\Psi_0 = 0$, a solution of $H_0(u) = 0$ is a solution of linear heat equation with ψ on the bondary Λ, $\deg(H_0, D, 0) \neq 0$. Then the homotopy invariance of degree implies that $\deg(H_1, D, 0) \neq 0$. Since $H_1^{-1}(0)$ is not nonempty set, $H_1(u) = 0$ has the solution $u \in D$ that is the harmonic map for heat equation on $M \times [0, T)$.

Since $T > 0$ is arbitrary and the solution of (1.1) on $M \times [0, T]$ is unique (see Section 4 of IV in [7]), we can obtain a unique solution of (1.1) on $M \times [0, \infty)$. □

References

[1] H. Choi, *On the Liouville theorem for harmonic maps*, Proc. Amer. Math. Soc. **85** (1982), 91–94.

[2] D. Chi, H. Choi, and H. Kim, *Heat equation for harmonic maps of the compactification of complete manifolds*, to appear in J. Geo. Anal. (1998).

[3] J. Cheeger and D. Ebin, *Comparison Theorems in Geometry*, North-Holland, Amsterdam, 1975.

[4] S. Cheng, *Liouville theorem for harmonic maps*, Proc. Sym. Pure Math. **36** (1980), 147–151.

[5] J. Eells and J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[6] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964.

[7] R. Hamilton, *On homotopic harmonic maps*, Canadian J. Math. **19** (1967), 673–687.

[8] S. Hilderbrandt, H. Kaul, and K. Widman, *Harmonic mappings into Riemannian manifold with nonpositive sectional curvature*, Math. Scand. **37** (1975), 257–263.

[9] W. Jager and H. Kaul, *Uniqueness and stability of harmonic maps and their Jacobi fields*, Manuscripta Math. **28** (1979), 269–291.

[10] W. Kendall, *Probability, convexity, and harmonic maps with small image I: uniqueness and fine existence*, Proc. London Math. Soc. **61** (1990), 371–406.

[11] P. Li and L. F. Tam, *The heat equation and harmonic maps of complete manifolds*, Invent. Math. **105** (1991), 1–46.

Dong Pyo Chi, Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail: dpchi@math.snu.ac.kr

Hyun Jung Kim, Department of Mathematics, Hoseo University, Baebang Myun, Asan 337-795, Korea
E-mail: hjkim @math.hoseo.ac.kr

Won Kuk Kim, Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794-3651, U.S.A.