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THE PALEY–WIENER THEOREM BY THE HEAT KERNEL METHOD

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ABSTRACT. We use the heat kernel method to prove newly the Paley-Wiener theorem for the distributions with compact support.

1. Introduction

Concerning the Schwartz distribution theories, one of the most famous theorems is the Paley - Wiener theorem for the distributions with compact support which characterizes the functions or generalized functions via the growth at infinity and the regularity of their Fourier transform.

Generally speaking, it is easy to pass from the support information about a distribution f to the analyticity condition on their Fourier transform $\hat{f}$, but it is rather tricky to proceed in the reverse direction.

The first step to this was introduced by N. Wiener and R. Paley[12] for the first time as for the case of functions, and extended to the distributions[10], to the ultradistributions by Komatsu[8] under some conditions.

Recently, T. Matsuzawa[9] introduced the heat kernel method to characterize hyperfunctions with compact support and Gevrey ultradistributions with compact support. This method has been developed very much to study various generalized functions[2, 3, 4].

The purpose of this paper is to apply the heat kernel method to prove newly the Paley - Wiener Theorem for distributions with compact support.

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2. Preliminaries

Throughout this paper we use a conventional multi-index notations such as; $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, $i = \sqrt{-1}$, and

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

for an n - tuple $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ where $\mathbb{N}_0$ is the set of nonnegative integers.

First we introduce the distributions and the distributions with compact support.

DEFINITION 2.1. (i) For an open set X in $\mathbb{R}^n$ a distribution T on X is a continuous linear functional on $C_0^\infty(X)$ i.e. for every compact subset $K \subset X$, there exist a nonnegative integer N and a positive constant M such that

$$|\langle T, \phi \rangle| \le M \sup_{\substack{|\alpha| \le N \\ x \in K}} |\partial^\alpha \phi(x)|, \quad \phi \in C_0^\infty(K).$$

By $\mathcal{D}'(X)$ we denote the set of all distributions on X.
(ii) A distribution T with support in a compact set K is a continuous linear functional on $C^\infty(\mathbb{R}^n)$ i.e there exists a positive number N such that for every $\epsilon > 0$ we have

$$|\langle T, \phi \rangle| \le C_\epsilon \sup_{\substack{|\alpha| \le N \\ x \in K_\epsilon}} |\partial^\alpha \phi(x)|, \quad \phi \in C^\infty(\mathbb{R}^n) \tag{2.1}$$

where $K_\epsilon = \{x \in \mathbb{R}^n; \mathrm{dist}(x, K) \le \epsilon\}$ and C_ϵ is a positive constant depending only on $\epsilon > 0$. By $\mathcal{E}'(K)$ we denote the set of all distributions with support in a compact subset K.

DEFINITION 2.2. We say that a function $\phi \in C^\infty(\mathbb{R}^n)$ is rapidly decreasing at infinity if

$$\lim_{|x| \to \infty} |x^\alpha \partial^\beta \phi(x)| = 0$$

for all pairs of multi-indices α and β.

We shall use $\mathcal{S}$ to denote the set of all rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^n)$ to denote the set of all continuous linear functionals T on the space $\mathcal{S}$ with respect to semi–norm $P_{\alpha,\beta}(\phi) = \sup_{x\in\mathbf{R}^n} |x^\alpha \partial^\beta \phi(x)|$, or equivalently $P_{N,\beta}(\phi) = \sup(1+|x|^2)^N |\partial^\beta \phi(x)|$ i.e. there are a constant $C \geq 0$ and a nonnegative integer N such that

$$|\langle T, \phi\rangle| \leq C \sum_{|\alpha|,|\beta|\leq N} \sup |x^\alpha \partial^\beta \phi|, \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

The members of $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions.*

For a subset E of $\mathbb{R}^n$ we define

$$H_E(\xi) = \sup_{x\in E} \langle x, \xi\rangle, \quad \xi \in \mathbb{R}^n .$$

One calls H_E the supporting function of E. It is easy to see that

$$H_E(\xi + \eta) \leq H_E(\xi) + H_E(\eta)$$

and

$$H_E(t\xi) = tH_E(\xi); \quad t \geq 0$$

for any ξ and η in $\mathbb{R}^n$ (see [7] for this).

3. The heat equation and the heat kernel

In this section we introduce the heat kernel and some basic properties of the solutions of the heat equation. It is well known that the locally integrable function

$$E(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} \exp[-\frac{|x|^2}{4t}] & t > 0, \\ 0 & t < 0, \end{cases}$$

satisfies $(\partial_t - \triangle)E = 0$ in $\mathbb{R}^n \times (\mathbb{R} \setminus \{0\})$, where $|x|^2 = x_1^2 + \cdots + x_n^2$ for $x \in \mathbb{R}^n$. In fact, $E(x,t)$ is a fundamental solution of heat operator $\partial_t - \triangle$, that is,

$$(\partial_t - \triangle)E(x,t) = \delta(x,t) .$$

For the later use, we recall some fundamental properties of the heat kernel.

LEMMA 3.1. *$E(\cdot, t)$ is an entire function in $\mathbb{C}^n$ of order 2 for each $t > 0$. We have the following properties on E:*

(i) *We have*

$$\int_{\mathbb{R}^n} E(x,t)\,dx = 1, \quad t>0. \tag{3.1}$$

(ii) *We have that for each* $\delta>0$ *and* $m>0$

$$\int_{|y|\ge\delta} (1+|y|^2)^m E(y,t)\,dy \to 0, \quad \text{as } t\to 0^+. \tag{3.2}$$

(iii) *There are positive constants* C *and* a *such that for each* $t>0$

$$|D_x^\alpha E(x,t)| \le C^{|\alpha|+2} t^{-\frac{(n+|\alpha|)}{2}} \alpha!^{\frac{1}{2}} \exp[-\frac{a|x|^2}{4t}], \tag{3.3}$$

where a *can be taken as close as desired to* 1 *and* $0<a<1$;

LEMMA 3.2. *For every* $\phi \in \mathcal{S}(\mathbb{R}^n)$, *let*

$$U_\phi(x,t) = \int_{\mathbb{R}^n} E(x-y,t)\,\phi(y)\,dy\ ,\ t\ >\ 0.$$

Then $U_\phi(x,t) \to \phi(x)$ *in* $\mathcal{S}(\mathbb{R}^n)$ *as* $t\to 0^+$.

Proof. First, it is easy to see that $U_\phi(\cdot,t)$ is infinitely differentiable in $\mathbb{R}^n$ for each $t>0$. Now to show the convergence let $\phi\in\mathcal{S}(\mathbb{R}^n)$. In fact, it suffices to show that for every $N>0$ and multi–index β

$$\sup_{x\in\mathbb{R}^n}(1+|x|^2)^N\left|\partial^\beta U_\phi(x,t)-\partial^\beta\phi(x)\right|\to 0$$

as $t\to 0^+$.

Applying the mean value theorem and Peetre's inequality

$$(1+|\xi|^2)^s(1+|\eta|^2)^{-s}\le 2^{|s|}(1+|\xi-\eta|^2)^{|s|}$$

for $s\in\mathbb{R}$ and $\xi,\eta\in\mathbb{R}^n$ we obtain

$$\begin{aligned}(1+|x|^2)^N|\partial^\beta\phi(x-y)-\partial^\beta\phi(x)| &\le (1+|x|^2)^N\left|\nabla\partial^\beta\phi(x-\theta y)\right|\cdot|y|\\ &\le C(1+|x|^2)^N(1+|x-\theta y|^2)^{-N}\cdot|y|\\ &\le C2^N(1+|y|^2)^N\cdot|y|,\quad x,y\in\mathbb{R}^n\end{aligned} \tag{3.4}$$

for a constant $C = C(N, \beta)$ and $0 < \theta < 1$. Then for each $N > 0$ and β we can write

$$(3.5) \quad \begin{aligned} &(1+|x|^2)^N |\partial^\beta U_\phi(x,t) - \partial^\beta \phi(x)| \\ &= (1+|x|^2)^N \left| \int_{\mathbf{R}^n} E(y,t) \left[\partial^\beta \phi(x-y) - \partial^\beta \phi(x)\right] dy \right| \\ &\leq \int_{\mathbf{R}^n} E(y,t)(1+|x|^2)^N \left|\partial^\beta \phi(x-y) - \partial^\beta \phi(x)\right| dy \\ &\leq C 2^N \int_{\mathbf{R}^n} E(y,t)(1+|y|^2)^N \cdot |y| dy \end{aligned}$$

Now let δ be a small positive number. The last integral in (3.5) can be written as

$$(3.6) \quad \begin{aligned} &\int_{\mathbf{R}^n} E(y,t)(1+|y|^2)^N \cdot |y| dy \\ = &\int_{|y|\leq\delta} E(y,t)(1+|y|^2)^N \cdot |y| dy + \int_{|y|>\delta} E(y,t)(1+|y|^2)^N \cdot |y| dy \\ \leq &\ \delta(1+\delta^2)^N + \int_{|y|\geq\delta} (1+|y|^2)^{N+1} E(y,t) dy \end{aligned}$$

In view of Lemma 3.1.(ii) the last integral becomes arbitrarily small as $t \to 0^+$. This completes the proof. □

Let $u \in \mathcal{E}'(K)$. Then the function

$$(3.7) \qquad U(x,t) = u_y(E(x-y,t)) \quad x \in \mathbb{R}^n \ , \ t > 0,$$

is well defined since $E(x-\cdot,t) \in \mathcal{S}$ for each $x \in \mathbb{R}^n$ and $t > 0$.

THEOREM 3.3. *Let $u \in \mathcal{E}'(K)$. Then $U(x,t) = u_y(E(x-y,t))$ is an infinitely differentiable function in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ satisfying the following condition:*

$$(3.8) \qquad (\partial_t - \triangle)\, U(x,t) = 0 \ \ in \ \ \mathbb{R}^{n+1}_+$$

and for every $\epsilon > 0$ there are positive constants C_ϵ and N such that

$$(3.9) \qquad |U(x,t)| \leq C_\epsilon\, t^{-\frac{(n+N)}{2}} \exp[-\frac{\mathrm{dist}(x,K_\epsilon)^2}{8t}], \quad (x,t) \in \mathbb{R}^{n+1}_+ .$$

Also, $U(x,t) \to u$ as $t \to 0^+$ in the following sense that

$$u(\phi) = \lim_{t\to 0^+} \int_{\mathbb{R}^n} U(x,t)\,\phi(x)\,dx\,, \quad \phi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{3.10}$$

Conversely, every C^∞ function $U(x,t)$ defined in $\mathbb{R}^{n+1}_+$ satisfying conditions (3.8) and (3.9) can be expressed in the form

$$U(x,t) = u_y(E(x-y,t)), \quad x \in \mathbb{R}^n, t > 0.$$

with a unique element $u \in \mathcal{E}'(K)$.

Proof. Let $u \in \mathcal{E}'(K)$ and $U(x,t) = u_y(E(x-y,t))$. Then $U(x,t)$ is well defined and infinitely differentiable for each $t > 0$ and also satisfies the heat equation (3.8). The fact that $u \in \mathcal{E}'(K)$ means that for any $\epsilon > 0$ there exist positive constants C_ϵ and N such that

$$|U(x,t)| \le C_\epsilon \sup_{\substack{|\alpha|\le N \\ y\in K_\epsilon}} |\partial_x^\alpha E(x-y,t)|\,.$$

Thus from (3.3) we have

$$\begin{aligned} |U(x,t)| &\le C_\epsilon C_a^{|N|+2} N!^{\frac{1}{2}} t^{-\frac{(n+N)}{2}} \exp[-\frac{a\,\mathrm{dist}(x,K_\epsilon)^2}{4t}] \\ &\le C_{\epsilon,N,a}\, t^{-\frac{(n+N)}{2}} \exp[-\frac{a\,\mathrm{dist}(x,K_\epsilon)^2}{4t}] \end{aligned}$$

This gives (3.9) by taking $a = \frac{1}{2}$ for convenience. Now to prove (3.10) let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then we have for each $t > 0$,

$$\int U(x,t)\,\phi(x)\,dx = u_y(\int E(x-y,t)\,\phi(x)\,dx) = u(U_\phi(y,t))$$

by taking the limit of the Riemann sum of the first integral. Then it follows from Lemma 3.2 that

$$u(\phi) = \lim_{t\to 0^+} \int_{\mathbb{R}^n} U(x,t)\,\phi(x)\,dx\,, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

which proves (3.10).

To prove the converse we assume that $U(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$ satisfies the conditions (3.8) and (3.9). For a positive integer m we define a function

$f(t)$ as

$$f(t) = \begin{cases} t^{m-1}/(m-1)! & ,t \geq 0 \\ 0 & ,t < 0 \,. \end{cases}$$

Multiplying f with a suitable C^∞ function with compact support it is possible to get the following relation

$$\left(\frac{d}{dt}\right)^m v(t) \;=\; \delta(t) \,+\, \omega(t) \tag{3.11}$$

for every $t \in \mathbb{R}$ for suitable functions $v(t)$ and $w(t)$ such that $v(t) \;=\; f(t)$ for $-\infty < t \leq 1\,,\, v(t) \;=\; 0$ for $2 \leq t < \infty\;,\; \omega(t) \in C_0^\infty(\mathbb{R})\,,\,$ supp $\omega \subset [1,2]$ and $\delta(t)$ is the Dirac delta function. Take $m \;=\; [\frac{(N+n)}{2}] + 2$ where N is a constant given in (3.9) and consider the following function

$$G(x,t) \;=\; \int_0^\infty U(x, t+s)\, v(s)\, ds \;.$$

Then it is easily seen that $G(x,t)$ is a bounded and continuous function on $\bar{\mathbb{R}}_+^{n+1} = \{(x,t)|x \in \mathbb{R}^n,\ t \geq 0\}$ satisfying the heat equation

$$(\frac{\partial}{\partial t} - \triangle)\, G(x,t) = 0 \;\text{ in }\; \mathbb{R}_+^{n+1} \;. \tag{3.12}$$

Therefore, it follows from (3.11) and (3.12) that in $\mathbb{R}_+^{n+1}$

$$\begin{aligned} (-\triangle)^m G(x,t) \;&=\; (-\frac{d}{dt})^m G(x,t) \\ &=\; U(x,t) \,+\, \int_0^\infty U(x,t+s)\,\omega(s)\, ds \;. \end{aligned} \tag{3.13}$$

If we put

$$H(x,t) \;=\; -\int_0^\infty U(x,t+s)\,\omega(s)\, ds\,,$$

then $H(x,t)$ is also a C^∞ solution of heat equation in $\mathbb{R}_+^{n+1}$ which is continuously extended to $\bar{\mathbb{R}}_+^{n+1}$. Furthermore, if we define $g(x) \;=\; G(x,0)$ and $h(x) \;=\; H(x,0)$ then $g(x)$ and $h(x)$ are continuous and bounded on $\mathbb{R}^n$. In view of the well known uniqueness theorem of the solutions of the heat equation we have

$$(3.14)\qquad \begin{aligned} G(x,t) &= \int E(x-y,t)\,g(y)\,dy = (g*E)(x,t),\\ H(x,t) &= \int E(x-y,t)\,h(y)\,dy = (h*E)(x,t) \end{aligned}$$

where $*$ denotes the convolution with respect to the x variable.

Now we define u as

$$(3.15)\qquad u = (-\triangle)^m g(x) + h(x)\ .$$

Since g and h are continuous it is easy to see that u is a distributions on $\mathbb{R}^n$. Then (3.14) implies that

$$\begin{aligned} (u*E)(x,t) &= (-\triangle)^m(g*E)(x,t) + (h*E)(x,t)\\ &= (-\triangle)^m G(x,t) + H(x,t)\\ &= U(x,t)\ . \end{aligned}$$

Thus in view of (3.9) and (3.10) we can see that u is a distribution with compact support. Hence it remains to show that $\operatorname{supp} u$ is contained in K. In fact, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\phi \cap K = \emptyset$, the conditions (3.9) and (3.10) enable us to have

$$|\langle u,\phi\rangle| = 0\ .$$

This implies that u belongs to $\mathcal{E}'(K)$. Since the uniqueness of such $u \in \mathcal{E}'(K)$ is obvious the proof is completed. □

REMARK 3.1. The condition (3.9) in Theorem 3.3 can be somewhat weakened as follows: For every $\epsilon > 0$

$$|U(x,t)| \le C_\epsilon\, t^{-\frac{(n+N)}{2}} \quad \text{in } \mathbb{R}^{n+1}_+\ .$$

and for every $\delta > 0$

$$U(x,t) \to 0$$

uniformly in $\{x \in \mathbb{R}^n : \operatorname{dist}(x,K) \ge \delta\}$ as $t \to 0^+$.

In fact, in the proof above we have proved the following structure theorem for the distributions with compact support:

COROLLARY 3.4. *Let K be a compact subset in $\mathbb{R}^n$. If $u \in \mathcal{E}'(K)$ then there is a number $m > 0$ and there are bounded continuous functions $g(x)$ and $h(x)$ such that*

$$u = \Delta^m g(x) + h(x) ,$$

where $g(x) \in C^\infty(\mathbb{R}^n \setminus K)$, $h(x) \in C^\infty(\mathbb{R}^n)$ and $\Delta^m g(x) + h(x) = 0$ in $\mathbb{R}^n \setminus K$.

4. A new proof of the Paley - Wiener theorem

For $u \in \mathcal{E}'(K)$ its Fourier transform $\hat{u}$ is defined by $\hat{u}(\xi) = \langle u_x, e^{-ix\xi}\rangle$, $\xi \in \mathbb{R}^n$. Then it is easy to see that $\hat{u}(\xi)$ is an infinitely differentiable function which can be extended to an entire function in $\mathbb{C}^n$ (See [1, 5, 7, 10]).

We are now in a position to prove the Paley–Wiener theorem in a new method depending on the idea of Theorem 3.3, which is a main theorem of this paper.

THEOREM 4.1. *Let K be a convex compact subset of $\mathbb{R}^n$ with supporting function H_K. If $F(\zeta)$ is an entire function satisfying*

$$|F(\zeta)| \leq C(1 + |\zeta|)^N \exp[H_K(Im\zeta)], \tag{4.1}$$

for all $\zeta = \xi + i\eta \in \mathbb{C}^n$ and for some $N \geq 0$, then $F(\zeta)$ is the Fourier–Laplace transform of a unique element in $\mathcal{E}'(K)$.

Proof. Let $F(\zeta)$ be an entire function in $\mathbb{C}^n$ satisfying property (4.1). Now, we define a function $\tilde{U}(\zeta, t)$ on $\mathbb{C}^n \times (0, \infty)$ as follows:

$$\tilde{U}(\zeta, t) = F(\zeta)\exp(-t\zeta^2). \tag{4.2}$$

Then for each $t > 0$ we have $\tilde{U}(\xi, t) = F(\xi)\exp(-t|\xi|^2) \in \mathcal{S}(\mathbb{R}^n_\xi)$. If we define

$$U(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi}\tilde{U}(\xi, t)\, d\xi \ , \ x \in \mathbb{R}^n, \ t > 0 \tag{4.3}$$

then $U(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$ and satisfies

$$(\frac{\partial}{\partial t} - \Delta)U(x,t) = 0 \ \text{in} \ x \in \mathbb{R}^n_+ \ t > 0. \tag{4.4}$$

In view of (4.1) we have

$$\begin{aligned}
|U(x,t)| &\le (2\pi)^{-n}\int_{\mathbb{R}^n} |e^{ix\xi}\tilde{U}(\xi,t)|\,d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} |F(\xi)|e^{-t|\xi|^2}\,d\xi \\
&\le C\int_{\mathbb{R}^n} (1+|\xi|)^N e^{-t|\xi|^2}\,d\xi \\
&= C\int_{\mathbb{R}^n}\sum_{k=0}^{N}\binom{N}{k}|\xi|^k e^{-t|\xi|^2}\,d\xi \\
&= C\sum_{k=0}^{N}\binom{N}{k}\int_{\mathbb{R}^n}|\xi|^k e^{-t|\xi|^2}\,d\xi \\
&= C\sum_{k=0}^{N}\binom{N}{k}\int_0^\infty\int_{S^{n-1}} r^{n+k-1}e^{-tr^2}\,d\omega\,dr \\
&= C\sum_{k=0}^{N}\binom{N}{k}\omega_{n-1}\int_0^\infty r^{n+k-1}e^{-tr^2}\,dr
\end{aligned}$$

where $\xi = r\omega$, $r > 0$, $|\omega| = 1$ and ω_{n-1} denotes the surface area of S^{n-1}.

An elementary calculation gives

$$\int_0^\infty r^{n+k-1}e^{-tr^2}\,dr = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^{m+1}}\sqrt{\pi}t^{-\frac{2m+1}{2}}$$

when $n+k-1 = 2m \ge 0$ and

$$\int_0^\infty r^{n+k-1}e^{-tr^2}\,dr = \frac{m!}{2t^{m+1}},$$

when $n+k-1 = 2m+1$.

Hence we have the estimate

$$|U(x,t)| \le C_N t^{-N}, \quad x\in\mathbb{R}^n,\ t>0 \tag{4.5}$$

for some integer $N \ge 0$. From this we can see that the initial value $U(x,0+)$ defines a distribution in $\mathbb{R}^n$.

Now we are going to find more detailed estimate which gives an information about support. By shifting the integration in (4.3) into the complex domain, we have

$$U(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix(\xi+i\eta)} \tilde{U}(\xi + i\eta, t)\, d\xi$$

for an arbitrary fixed vector $\eta \in \mathbb{R}^n$. Estimating the integral by using (4.1) and (4.5), we have for every $\epsilon > 0$

$$\begin{aligned} (4.6) \qquad & |U(x,t)| \\ \leq\ & C \exp[H_K(\mathrm{Im}\zeta) - x\eta + t|\eta|^2] \int_{\mathbb{R}^n} (1 + |\xi|)^N (1 + |\eta|)^N e^{-t|\xi|^2}\, d\xi \\ \leq\ & C'_N t^{-N} \exp[H_K(\mathrm{Im}\zeta) - x\eta + t|\eta|^2](1 + |\eta|)^N \end{aligned}$$

For $x_0 \notin K_\epsilon$ and $y_0 \in \partial K$ choose a vector $\eta_0 = \frac{x_0 - y_0}{|x_0 - y_0|} \in \mathbb{R}^n$ such that

$$H_0 = \{x | \langle x, \eta_0 \rangle = \langle y_0, \eta_0 \rangle\}\ , H_1 = \{x | \langle x, \eta_0 \rangle = \langle y_0, \eta_0 \rangle + \epsilon\}.$$

Then it follows that

$$\langle x_0, \eta_0 \rangle - \langle y, \eta_0 \rangle \geq \epsilon\ .$$

for every $y \in K$. Namely, for every $x \notin K_\epsilon$ there is $\eta_0 \in \mathbb{R}^n$ such that $\langle x, \eta_0 \rangle \geq \sup_{y \in K} \langle y, \eta_0 \rangle + \epsilon$.

Taking $\eta = \frac{1}{\sqrt{t}} \eta_0$ in (4.6) it follows that

$$|U(x,t)| \leq C'_N t^{-N'} \exp[-\frac{1}{\sqrt{t}}(x\eta_0 - H_K(\eta_0))]$$

which is reduced to the condition (3.9). By Theorem 3.3 , there exists a unique element $u \in \mathcal{E}'(K)$ such that $U(x,t) = u_y(E(x-y,t))$ and $U(x,t) \to u$ in the sense of (3.10). Hence it remains to show that $F(\zeta)$ is the Fourier – Laplace transform of u as an element of $\mathcal{E}'(K)$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform and the Fourier inverse transform respectively. Then for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\langle U(x,t), \phi(x)\rangle &= \langle \mathcal{F}^{-1}(\tilde{U}(\xi,t)), \phi(x)\rangle = \langle \tilde{U}(x,t), \mathcal{F}^{-1}(\phi)\rangle \\
&= \langle F(x)\exp(-t|x|^2), \mathcal{F}^{-1}(\phi)\rangle \\
&= \langle F(x), \exp(-t|x|^2)\mathcal{F}^{-1}(\phi)\rangle \\
&= \langle F(x), \mathcal{F}(E(\xi,t))\mathcal{F}^{-1}(\phi)\rangle \\
&= \langle F(x), \mathcal{F}(E(\xi,t))(2\pi)^{-n}\mathcal{F}(\phi(-\xi))\rangle \\
&= \langle F(x), (2\pi)^{-n}\mathcal{F}(E(\xi,t) * \phi(-\xi))\rangle \\
&= \langle \mathcal{F}(F(\xi)), (2\pi)^{-n}E(x,t) * \phi(-x)\rangle
\end{aligned}$$

Therefore, in view of Lemma 3.2 we have

$$\begin{aligned}
\langle u, \phi\rangle &= \lim_{t\to 0+}\langle U(x,t), \phi(x)\rangle \\
&= \lim_{t\to 0+}\langle \mathcal{F}(F(\xi)), (2\pi)^{-n}U_\phi(-x,t)\rangle \\
&= \langle (2\pi)^{-n}\mathcal{F}(F), \phi(-x)\rangle \\
&= \langle \mathcal{F}^{-1}(F), \phi\rangle,
\end{aligned}$$

which implies that $\mathcal{F}(u) = F$. This completes the proof. □

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