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CRITICAL KÄHLER SURFACES

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ABSTRACT. We characterize real 4-dimensional Kähler metrics which are critical for natural quadratic Riemannian functionals defined on the space of all Riemannian metrics. In particular we show that such critical Kähler surfaces are either Einstein or have zero scalar curvature. We also make some discussion on criticality in the space of Kähler metrics.

1. Introduction

There are several natural functionals defined on a space of Riemannian metrics on a smooth manifold. For analytic or geometric reasons one needs to study critical points of a functional, which we call *critical metrics* of the functional.

It is interesting to characterize the critical metrics of L^2 norm functionals of some curvature component of Riemannian metrics in dimension 4. Such L^2 norm functionals are particularly important in this dimension for several reasons; for instance one may recall the generalized Gauss-Bonnet theorem which expresses topological invariants in L^2 norms of curvature tensors. The critical metrics satisfy a bit complicated Euler-Lagrange equation and so it is not easy to characterize them. We are interested first of all in the critical metrics of $\mathcal{R}^2$, L^2 norm of the full curvature tensor, as this functional measures the full L^2 energy of Riemannian metrics.

Two families of metrics are known to be $\mathcal{R}^2$-critical; Einstein metrics and half-conformally flat metrics with scalar curvature zero. A number

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of new metrics in the latter family have been found recently [4,5,6,7]. Indeed one of the motivations for this paper was to understand the naturality of these two families of metrics on 4 manifolds.

Now one is led to ask if $\mathcal{R}^2$-critical metrics are precisely these two families of metrics. As this question seems hard in general, we want to start from testing a small class of manifolds. In this paper we ask: what are $\mathcal{R}^2$-critical *Kähler* metrics? We show that a $\mathcal{R}^2$-critical Kähler metric on a compact real 4-dimensional manifold is either Einstein or has zero scalar curvature. We further discuss another functional $\mathcal{R}ic^2$, the L^2 norm of the ricci tensor of a Riemannian metric and prove the same conclusion for $\mathcal{R}ic^2$-critical Kähler metrics. These seemingly natural results were perhaps known to experts as they are not distant from the works of Kähler and special metric geometers in the last decades. Indeed we proved them in the framework of [1] and [3]. However the author could not find a literature discussing these theorems.

We then discuss $\mathcal{S}^2$-criticality *among Kähler metrics*. The purpose is to understand a nature of the space of all Kähler metrics inside the space of all Riemannian metrics when a geometric bound is given. So we prove that there is an example of such a critical metric which is not Einstein nor has zero scalar curvature.

The paper is organized as follows: In section 2 we describe quadratic Riemannian functionals and some known facts regarding to the existence of their critical metrics. In section 3 we prove that a real 4-dimensional $\mathcal{R}^2$-critical Kähler metric is either Einstein or has zero scalar curvature. In section 4 we prove the characterization of 4-dimensional $\mathcal{R}ic^2$-critical Kähler metrics similarly to section 3. In section 5, we present examples of $\mathcal{R}^2$-critical metrics and prove a result on $\mathcal{S}^2$-criticality by an example.

In this paper we denote the scalar, ricci and full Riemannian curvature tensor by s, r and R respectively with such a sign convention that in local coordinates $r_{ij} = R_{ipj}{}^p = g^{pq}R_{ipjq}$ and $g^{pq}r_{pq} = s$.

2. Riemannian functionals in 4 dimension

We consider the following functionals defined on the space of all smooth Riemannian metrics on a compact smooth oriented manifold

M of dimension four.

$$
(2.1)\qquad
\begin{aligned}
\mathcal{S}^2(g) &= \int_M (|s_g|_g)^2 dv_g, & \mathcal{R}ic^2(g) &= \int_M (|r_g|_g)^2 dv_g,\\
\mathcal{Z}^2(g) &= \int_M (|z_g|_g)^2 dv_g & \mathcal{W}^2(g) &= \int_M (|W_g|_g)^2 dv_g,\\
\mathcal{R}^2(g) &= \int_M (|R_g|_g)^2 dv_g,
\end{aligned}
$$

where $z = r - \frac{1}{4}sg$, W and dv_g are respectively the trace-free ricci, Weyl curvature tensor and volume form of g. In this paper we call a smooth metric $\mathcal{F}$-critical if it is a critical point of $\mathcal{F}$-functional, i.e. if it satisfies the Euler-Lagrange equation of $\mathcal{F}$-functional, where $\mathcal{F}$ denotes one of the above functionals. Note that all these L^2 norm functionals are scalar invariant, i.e. $\mathcal{F}(g) = \mathcal{F}(cg)$ for a positive constant c and that $\mathcal{W}^2$ is conformally invariant, i.e. $\mathcal{W}^2(g) = \mathcal{W}^2(fg)$ for a positive function f.

We write the gradient of each functional in (2.1), [1, p.133];

(2.2)

$$
\begin{aligned}
(\mathrm{grad}\mathcal{S}^2)_g &= 2\nabla_g ds + 2(\Delta_g s_g)g + \frac{1}{2}{s_g}^2 g - 2s_g r_g,\\
(\mathrm{grad}\mathcal{R}ic^2)_g &= \nabla^*{}_g\nabla_g r_g + \nabla_g ds + \frac{1}{2}(\Delta_g s_g)g + \frac{1}{2}|r_g|^2 g - 2\breve{R}_g r_g,\\
(\mathrm{grad}\mathcal{Z}^2)_g &= \nabla^*{}_g\nabla_g r_g + \frac{1}{2}\nabla_g ds + \frac{1}{2}|z_g|^2 g + \frac{1}{2}s_g r_g - 2\breve{R}_g r_g,\\
(\mathrm{grad}\mathcal{W}^2)_g &= -4\delta^\nabla \delta W - 4\breve{W} r,\\
(\mathrm{grad}\mathcal{R}^2)_g &= 2\delta^\nabla d^\nabla r - 2(\widetilde{R} - \frac{1}{4}|R_g|^2 g),
\end{aligned}
$$

In the above the differential operator d^∇ acts on symmetric 2-tensors ψ in local coordinates by $(d^\nabla\psi)_{ijk} = \nabla_i\psi_{jk} - \nabla_j\psi_{ik}$. δ^∇ is the dual operator of d^∇ and so in local coordinates $(\delta^\nabla\phi)_{jk} = -2\nabla^i\phi_{ijk}$. $(\breve{W}r)_{ij} = W_{ipjq}r^{pq}$, $(\breve{R}r_{ij}) = R_{ipjq}r^{pq}$ and $\widetilde{R}_{ij} = R_{ipqs}{R_j}^{pqs}$.

The L^2 norm functionals are particularly interesting in 4 dimension due to the generalized Gauss-Bonnet theorem which expresses the Euler characteristic χ and signature σ of a 4-dimensional oriented compact Riemannian 4-manifold in terms of the L^2 norms ;

$$(2.3)\qquad \begin{aligned} \chi &= \frac{1}{8\pi^2}\int_M \left(\frac{1}{24}s^2 - \frac{1}{2}|z|^2 + |W^+|^2 + |W^-|^2\right)dv_g \\ \sigma &= \frac{1}{12\pi^2}\int_M (|W^+|^2 - |W^-|^2)dv_g \end{aligned}$$

where W^+ and W^- are the positive and negative (respectively) part of the Weyl tensor.

It is interesting to know what $\mathcal{F}$-critical metrics are and which manifolds admit such metrics if not all. But it is generally quite difficult because the Euler-Lagrange equations (2.2) are complicated. We just describe in a proposition some known facts, see [1, p.133];

PROPOSITION 2.1.

(1) *A metric g is $\mathcal{S}^2$-critical if and only if either it is an Einstein metric or has zero scalar curvature.*
(2) *If g is critical for either $\mathcal{S}^2$, $\mathcal{R}ic^2$, $\mathcal{Z}^2$, or $\mathcal{R}^2$, then the scalar curvature is constant.*
(3) *Einstein metrics are critical for all functionals in (2.1).*
(4) *Half-conformally flat metrics, i.e metrics with either $W^+ = 0$ or $W^- = 0$, with zero scalar curvature are critical for all functionals in (2.1).*

3. $\mathcal{R}^2$-critical Kähler metric

Concerning existence of $\mathcal{R}^2$-critical metrics, it is not known to the author's limited knowledge whether there are obstructions on a manifold to admit such metrics or whether there exist other examples than Einstein metrics or half-conformally-flat metrics with zero scalar curvature. In this section we characterize $\mathcal{R}^2$-critical Kähler metrics and find out that they are precisely either Einstein metrics or half-conformally-flat metrics with zero scalar curvature.

To highlight these two families of metrics we recall some immediate consequences of the χ and σ formulas (2.3).

PROPOSITION 3.1, [9]. *Einstein metrics and half-conformally-flat metrics with zero scalar curvature give absolute minima of the $\mathcal{R}^2$ functional.*

Proof.

$$\mathcal{R}^2(h) = \int_M (\frac{1}{24}s^2 + \frac{1}{2}|z|^2 + |W^+|^2 + |W^-|^2)dv_h$$
$$= 8\pi^2\chi + \int_M |z|^2 dv_h \geq 8\pi^2\chi$$

So Einstein metrics, i.e. metrics with $z = 0$, give absolute minima of $\mathcal{R}^2$. For half-conformally-flat metrics with zero scalar curvature one needs

$$\mathcal{R}^2(h) = -8\pi^2(\chi \pm 3\sigma) + \int_M \frac{1}{12}|s|^2 + 4|W^\pm|^2 dv_h$$
$$\geq -8\pi^2(\chi \pm 3\sigma).$$
□

PROPOSITION 3.2. *For a Riemannian metric g on a compact oriented 4-manifold the following statements are equivalent.*

(1) *g is $\mathcal{R}^2$-critical.*
(2) *g is $\mathcal{Z}^2$-critical.*
(3) *g is critical with respect to the functional $\mathcal{S}W^+(g) = \int_M(\frac{1}{48}s^2 + |W^+|^2)dv_g$.*
(4) *g is critical with respect to the functional $\mathcal{S}W^-(g) = \int_M(\frac{1}{48}s^2 + |W^-|^2)dv_g$.*

Proof. These equivalences follow from the proof of above proposition. □

Now we compute the gradient of $\mathcal{R}^2$ in local coordinates. Using an algebraic identity [1, (4.72)],

$$(\mathrm{grad}\mathcal{R}^2)_{jk} = 2(\delta^\nabla d^\nabla r)_{jk} - 2(\frac{sz}{3} + 2\check{W}r)_{jk} \tag{3.1}$$
$$= -4\nabla^p\nabla_p r_{jk} + 4\nabla^p\nabla_j r_{pk} - \frac{2}{3}(sz)_{jk} - 4(\check{W}r)_{jk}$$

The trace of this equation is equal to $-2\Delta s = 0$, so a $\mathcal{R}^2$-critical metric has constant scalar curvature as stated already in Proposition 2.1 (2).

As $R_{ijkl} = W_{ijkl} + \frac{1}{2}(g_{hj}r_{ik} + g_{ik}r_{hj} - g_{ij}r_{hk} - g_{hk}r_{ij}) - \frac{1}{6}s(g_{hj}g_{ik} - g_{ij}g_{hk})$, we compute

$$\breve{R}r = \breve{W}r - r \circ r + \frac{1}{2}|r|^2 g + \frac{2}{3}sz. \tag{3.2}$$

From the ricci identity $\nabla^k\nabla_j r_{ki} - \nabla_j\nabla^k r_{ki} = R^k{}_{jk}{}^p r_{pi} + R^k{}_{ji}{}^p r_{kp}$, we get

$$(\breve{R}r)_{ij} = (r \circ r)_{ij} + \frac{1}{2}(\nabla ds)_{ij} - \nabla^k\nabla_i r_{jk}. \tag{3.3}$$

From (3.2) and (3.3) we have

$$(\breve{W}r)_{jk} = -\nabla^p\nabla_j r_{kp} + (\frac{1}{2}\nabla ds + 2r \circ r - \frac{1}{2}|r|^2 g - \frac{2}{3}sz)_{jk}. \tag{3.4}$$

Now let ω be the Kähler form and ρ be the ricci form so that $\rho_{ji} = r_{jk}\omega^k{}_i$. Since $d\rho = 0$ by the Kähler condition, we have

$$-\nabla_p\rho_{ji} + \nabla_j\rho_{pi} = \nabla_i\rho_{pj}. \tag{3.5}$$

Then we put (3.4) and (3.5) into (3.1) to get

$$\begin{aligned}(\mathrm{grad}\mathcal{R}^2)_{jk}\omega^k{}_i =& -4\nabla^p\nabla_p\rho_{ji} + 4\nabla^p\nabla_j\rho_{pi} - \frac{2}{3}(sz)_{jk}\omega^k{}_i \\ & - 4(\breve{W}r)_{jk}\omega^k{}_i \\ =& 4\nabla^p\nabla_i\rho_{pj} - \frac{2}{3}(sz)_{jk}\omega^k{}_i + 4\nabla^p\nabla_j\rho_{pi} \\ & - 4(2r \circ r - \frac{1}{2}|r|^2 g - \frac{2}{3}sz)_{jk}\omega^k{}_i\end{aligned} \tag{3.6}$$

Now we use ricci identity

$$\nabla^p\nabla_i\rho_{pj} = \nabla_i\nabla^p\rho_{pj} + R_{pijq}\rho^{pq} + r_i{}^q\rho_{qj} \tag{3.7}$$

As the scalar curvature s is constant, $\nabla^p \rho_{jp} = -(\nabla^p r_{pk})\omega^k{}_j = 0$. Substituting (3.7) into (3.6) we get

$$\begin{aligned}(\mathrm{grad}\mathcal{R}^2)_{jk}\omega^k{}_i &= -\frac{2}{3}(sz)_{jk}\omega^k{}_i + (-8r \circ r + 2|r|^2 g + \frac{8}{3}sz)_{jk}\omega^k{}_i \\ &= (2sz - 8r \circ r + 2|r|^2 g)_{jk}\omega^k{}_i = 0\end{aligned} \tag{3.8}$$

Now we note a computation [3, Lemma 4 (i)] which holds for real 4 dimensional Kähler metrics;

$$-4r \circ r + |r|^2 g = -2sz \tag{3.9}$$

Putting (3.9) into (3.8), we get

$$\mathrm{grad}\mathcal{R}^2 = -2sz = 0 \tag{3.10}$$

So if the scalar curvature of a $\mathcal{R}^2$-critical Kähler metric is not identically zero, then equation $z = 0$ holds, i.e. g is Einstein. We proved

THEOREM 3.3. *A $\mathcal{R}^2$-critical Kähler metric on a compact 4-manifold is either Einstein or has zero scalar curvature.*

4. $\mathcal{R}ic^2$-critical Kähler metrics

In this section we discuss Kähler metrics which are critical for $\mathcal{R}ic^2$ functional. We prove the following

THEOREM 4.1. *A $\mathcal{R}ic^2$-critical Kähler metric on a compact 4-manifold is either Einstein or has zero scalar curvature.*

Proof. The computation follows in the same way as $\mathcal{R}^2$-critical Kähler case. Recall that $\mathcal{R}ic^2$-critical metric has constant scalar curvature. From (2.2),

(4.1)

$$\begin{aligned}(\text{grad}\mathcal{R}ic^2)_{jk}\omega^k{}_i &= (\nabla^\star\nabla r + \frac{1}{2}|r|^2 g - 2\check{R}r)_{jk}\omega^k{}_i \\ &= -\nabla^p\nabla_p\rho_{ji} + \frac{1}{2}|r|^2\omega_{ji} - 2R_{jpkq}r^{pq}\omega^k{}_i \\ &= 2\nabla^p\nabla_i\rho_{pj} + \frac{1}{2}|r|^2\omega_{ji} - 2R_{jpkq}r^{pq}\omega^k{}_i \\ &= 2R_{pijq}\rho^{pq} + 2r_i{}^q\rho_{qj} + \frac{1}{2}|r|^2\omega_{ji} - 2R_{jpkq}r^{pq}\omega^k{}_i \\ &= 2r_i{}^q\rho_{qj} + \frac{1}{2}|r|^2\omega_{ji}\end{aligned}$$

In the above we used (3.5) and (3.7) in the third equality and (3.7) in the fourth.
We get

$$\text{grad}\mathcal{R}ic^2 = 2r \circ r - \frac{1}{2}|r|^2 g$$

By (3.9), a $\mathcal{R}ic^2$-critical Kähler metric satisfies $sz = 0$. So the theorem is proved. □

By proposition 2.1 (1), 3.2 (2), theorem 3.3 and theorem 4.1, we now characterized $\mathcal{F}$-critical Kähler metrics for any $\mathcal{F}$ in (2.1) except $\mathcal{W}^2$. The study on $\mathcal{W}^2$-critical Kähler metrics has been successful though not complete and can be summarized as follows;

PROPOSITION 4.2 [3, 10]. *A $\mathcal{W}^2$-critical Kähler metric on a real 4-dimensional compact manifold is either conformal to an Einstein metric or has zero scalar curvature. Furthermore if it is not Einstein but conformal to Einstein, then the underlying manifold is biholomorphic possibly to $\mathbb{C}P^2$ blown up at 1, 2 or 3 points.*

Therefore we state as follows;

THEOREM 4.3. *A Kähler metric on a real 4-dimensional compact manifold which is critical with respect to one of the functionals in (2.1) is either conformal to an Einstein metric or has zero scalar curvature.*

5. Examples and further discussion

We have shown that $\mathcal{R}^2$-critical Kähler surfaces are precisely Einstein or has zero scalar curvature. We briefly review examples. First for Kähler Einstein surfaces, the existence question has been completely resolved by works of Yau [12] and Yau-Tian [11] etc.; the metrics exist on some blow ups of $\mathbb{C}P^2$ and minimal complex surfaces of Kodaira dimension 0 or 2. Next for Kähler surfaces of zero scalar curvature, some constructions are shown by works in [5,6,7]; these metrics can exist only on minimal and/or nonminimal ruled surfaces. But the constructions are far from completion and for instance the description of the moduli space of such metrics on blow-ups of $\mathbb{C}P^2$ is still not clear.

We have seen in Proposition 2.1 (1) that $\mathcal{S}^2$-critical metrics are relatively easier to characterize than others. Now we ask the following question; If we restrict the functional $\mathcal{S}^2$ to the subspace $\mathcal{K}$ of all smooth *Kähler* metrics on a compact Kählerian real 4-manifold, then are the critical metrics of the restricted functional $\mathcal{S}^2|_{\mathcal{K}}$ still Einstein or with zero scalar curvature?

On one hand this is a question to understand the geometric nature of the subspace of all smooth Kähler metrics in the whole space of Riemannian metrics. On the other hand this question is related to the so-called *extremal Kähler* metrics introduced by E. Calabi [2], which are critical metrics for $\mathcal{S}^2$ restricted to the subspace of all Kähler metrics with their Kähler forms in a fixed cohomology class $[\omega]$.

Now we perturb a fixed Kähler form ω by $\omega(t) = \omega + t(\alpha + i\partial\bar{\partial}\phi)$, where α is a harmonic (1,1) form and ϕ is a smooth function. We compute as in [8] that for the associated metric $g(t)$ to $\omega(t)$

$$\int_M s^2(t)dv_{g(t)} = \int_M [s^2 + t(-\frac{1}{2}s^2\Delta\phi + (\omega,\alpha)s^2 - s\Delta\Delta\phi + 2sr \cdot \nabla\nabla\phi + 2s\Delta(\omega,\alpha) - 4(\rho,\alpha)s) + O(t^2)]dv_g$$

so that any $\mathcal{S}^2|_{\mathcal{K}}$-critical metric should satisfy

$$\int_M [s^2(\omega,\alpha) + 2s\Delta(\omega,\alpha) - 4s(\rho,\alpha) - \frac{1}{2}s^2\Delta\phi - s\Delta\Delta\phi + 2sr \cdot \nabla\nabla\phi]dv_g = 0. \tag{5.1}$$

for any harmonic (1,1) form α and any smooth function ϕ.

We can split it into two equations;

$$\int_M [s^2(\omega,\alpha) + 2s\Delta(\omega,\alpha) - 4s(\rho,\alpha)]dv_g = 0, \tag{5.2}$$

for any harmonic (1,1) form α,

$$\int_M [-\frac{1}{2}s^2\Delta\phi - s\Delta\Delta\phi + 2sr\cdot\nabla\nabla\phi]dv_g = 0, \tag{5.3}$$

for any smooth function ϕ. We note from the definition that (5.3) is the condition for a Kähler metric to be extremal Kähler .

Let M be the complex surface $\mathbb{C}P^2$ blown up at one point. Here all the Kähler classes $[\omega]$ are in $H^{1,1}(M,\mathbb{R}) \cong \mathbb{R}^2$. It is known by Calabi [1, p.335] that on M there is an analytic family of extremal Kähler metrics in which there is a metric, to be denoted by $g(0)$ which has the minimum $\mathcal{S}^2$ value among the family, see [10].

We claim that $g(0)$ is $\mathcal{S}^2|_{\mathcal{K}}$-critical; Choose a nonzero harmonic (1,1) form α_0 which forms a basis of the vector space $H^{1,1}(M,\mathbb{R})$ with the Kähler form $\omega(0)$ and consider the Kähler classes $[\omega(0)+t\alpha_0]$ for small t. We denote by $g(t)$ the Calabi family of metrics associated to the classes $[\omega(0)+t\alpha_0]$. Then we have $\frac{d}{dt}|_{t=0}(\omega(t)) = \alpha_0 + i\partial\bar{\partial}\phi_0$ for a function ϕ_0. $(\alpha_0,\, \phi_0)$ satisfies (5.1) as $g(0)$ has the minimum $\mathcal{S}^2$ value. Let $\tilde{\omega}(t)$ be any perturbation with $\tilde{\omega}(0) = \omega(0)$ such that $\frac{d}{dt}|_{t=0}(\tilde{\omega}(t)) = \tilde{\alpha} + i\partial\bar{\partial}\tilde{\phi}$. We have $\tilde{\alpha} = c_1\cdot\alpha_0 + c_2\cdot\omega(0)$ for some constants c_1, c_2. Now $\tilde{\phi}$ satisfies (5.3) because $g(0)$ is extremal Kähler. ϕ_0 also satisfies (5.3) so α_0 satisfies (5.2). As $\omega(0)$ clearly satisfies (5.2), $\tilde{\alpha}$ satisfies (5.2). Therefore we have shown that any perturbation of $\omega(0)$ satisfies (5.1) and so $g(0)$ is $\mathcal{S}^2|_{\mathcal{K}}$-critical.

As it is well known that $g(0)$ is neither Einstein nor has zero scalar curvature, we showed

THEOREM 5.1. *There exist a $\mathcal{S}^2|_{\mathcal{K}}$-critical Kähler metric which is neither Einstein nor has zero scalar curvature.*

From this example, we ask what the $\mathcal{S}^2|_{\mathcal{K}}$-critical Kähler metrics are.

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