Bull. Korean Math. Soc. **35** (1998), No. 3, pp. 409–420

BANACH SUBSPACES AND ENVELOPE NORM OF $wL_{\hat{1}}$

JEONGHEUNG KANG

ABSTRACT. In this paper as a universal Banach space of the separable Banach spaces we investigate the complemented Banach subspaces of $wL_{\hat{1}}$. Also, using Peck's theorem and the properties of the envelope norm of $wL_{\hat{1}}$ we will find a canonical basis of l_1^n, l_∞^n for each n.

1. Introduction

The space $weakL_1$ was introduced in analysis because key operators of harmonic analysis do not map L_1 into L_1. Examples of such operators are the Hardy-Littlewood maximal function and the Hilbert transform. In this viewpoint, it became natural to investigate $weakL_1$, the space of measurable functions f satisfying $\mu(\{x \in \Omega : |f(x)| > y\}) \leq \frac{c}{y}$.

It is known that (except for some trivial measure space), $weakL_1$ is not normable (see [3]). The question therefore arise as to whether any nontrivial continuous linear functionals on $weakL_1$ exists. Also, one can ask the structures of $weakL_1$. In [3], the answer for this question was considered. This implies $weakL_1$ has nontrivial dual space. In [4], J. Kupka and T. Peck studied the structure of $weakL_1$ and showed that the space L_∞ is dense in the dual of $weakL_1$ endowed with $weak^*$-topology.

As a Lorentz space, we will study the space $L(1,\infty)$ which is called $weakL_1$ denoted by wL_1:

$$(1.1) \qquad wL_1 = \{f \in L_0 : \mu(\{x \in \Omega : |f(x)| > y\}) < \frac{c}{y}\},$$

Received September 11, 1997.
1991 Mathematics Subject Classification: 46B03.
Key words and phrases: Banach envelope of $wL_{\hat{1}}$, complemented subspaces, basis of l_1^n.

where $c > 0$ is independent of $y > 0$. As we mentioned in the above, wL_1 is not normable, but we can find nontrivial linear functionals on wL_1. This was first observed by M. Cwikel and Y. Sagher in [3].

In [1], if μ is nonatomic, then we can get an equivalent integral-like seminorm

$$\|f\|_{wL_{\hat{1}}} = \lim_{n\to\infty} \sup_{\frac{q}{p}\ge n} \frac{1}{\log \frac{q}{p}} \int_{\{p\le|f|\le q\}} |f| d\mu. \tag{1.2}$$

Later on, in [2] actually the Banach envelope seminorm on $wL_{\hat{1}}$ was calculated to be exactly as above. Note that the seminorm on $wL_{\hat{1}}$ defined in (1.2) is a lattice seminorm. Even though wL_1 is complete with respect to the quasinorm $q_1(f) = \sup_{a>0} \mu(\{x \in \Omega : |f(x)| > a\})$, it is not complete with respect to the seminorm $\|\cdot\|_{wL_{\hat{1}}}$. This is due to M. Cwikel and C. Feffreman([1] and [4]). Let $\mathcal{N} = \{f \in wL_1 : \|f\|_{wL_{\hat{1}}} = 0\}$. Then we obtain the quotient space $wL_1/\mathcal{N}$. We define $wL_{\hat{1}}$ as the *normed envelope* (and its completion as the *Banach envelope*) of wL_1.

To study this subject, we need some basic facts about the dual of $wL_{\hat{1}}$. We would like to convert the nonlinear limit superior expression (1.2) for $\|\cdot\|_{wL_{\hat{1}}}$ into a linear limit expression by assigning the numbers $I_a^b(f) = \frac{1}{\log \frac{b}{a}} \int_{\{a\le|f|\le b\}} |f| d\mu$ in some fashion. For this, we introduce an ultrafilter $\mathcal{U}$ so that the limit of the I_a^b along $\mathcal{U}$ determines a canonical integral-like linear functional $I_{\mathcal{U}} \in wL_{\hat{1}}^*$. We now begin with the discussion of $\mathcal{U}$. For $n = 1, 2, 3, \ldots$, let

$$F_n = \{(a, b) : 1 \le a < b, \frac{b}{a} \ge n\}. \tag{1.3}$$

and then define $\mathcal{F} = \{F_n : n \ge 1\}$. Treating $\mathcal{F}$ as a filter of subsets of the set $S = [1, \infty) \times [1, \infty)$, we obtain from Zorn's lemma an ultrafilter $\mathcal{U}$ of subsets of S such that $\mathcal{F} \subset \mathcal{U}$.

From now on, we will fix the ultrafilter $\mathcal{F} \subset \mathcal{U}$. Define the "ersatz integral" $I_{\mathcal{U}}$ for every nonnegative function $f \in wL_{\hat{1}}$ by

$$I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} I_a^b(f) = \lim_{\mathcal{U}} \frac{1}{\log \frac{b}{a}} \int_{\{a\le f\le b\}} f d\mu. \tag{1.4}$$

J. Kupka and T. Peck gave the properties of ersatz integral $I_{\mathcal{U}}$ as following ; see [4].

THEOREM 1.1 (J. KUPKA AND T. PECK). *Let $f, g \in wL_{\hat{1}}$ be nonnegative and let $r > 0$. Then we have*

i) $I_{\mathcal{U}}(rf) = rI_{\mathcal{U}}(f)$,
ii) $I_{\mathcal{U}}(f+g) = I_{\mathcal{U}}(f) + I_{\mathcal{U}}(g)$,
iii) *If* $f \leq g$, *then* $I_{\mathcal{U}}(f) \leq I_{\mathcal{U}}(g)$,
iv) $I_{\mathcal{U}}(f) \leq \|f\|_{wL_{\hat{1}}}$.

From these properties, we define $I_{\mathcal{U}}(f)$ for an arbitrary function $f \in wL_{\hat{1}}$ by $I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} \frac{1}{\log \frac{b}{a}} \int_{\{a \leq |f| \leq b\}} (f^+ - f^-) d\mu$;

i) $I_{\mathcal{U}}$ is linear.
ii) $|I_{\mathcal{U}}(f)| \leq \|f\|_{wL_{\hat{1}}}$ for all $f \in wL_{\hat{1}}$.
iii) $I_{\mathcal{U}}$ vanishes on $\mathcal{N} = \{f \in wL_{\hat{1}} : \|f\|_{wL_{\hat{1}}} = 0\}$ and hence determines a well defined, bounded linear functional on $wL_{\hat{1}}$.

Define $wL_1(\mathcal{U}) = \{f \in wL_1 : \|f\|_{\mathcal{U}} < \infty\}$ where $\mathcal{U}$ is the ultrafilter defined in (1.3). Then we have $\|f\|_{\mathcal{U}} \leq \|f\|_{wL_{\hat{1}}}$ where

$$\|f\|_{\mathcal{U}} = I_{\mathcal{U}}(|f|) = \lim_{\mathcal{U}} \frac{1}{\log \frac{b}{a}} \int_{\{a \leq |f| \leq b\}} |f| d\mu. \tag{1.5}$$

Hence we have $wL_{\hat{1}} \subset wL_1(\mathcal{U})$. Moreover $\|f\|_{\mathcal{U}} = I_{\mathcal{U}}(f)$ has the following properties:

i) $\|\cdot\|_{\mathcal{U}}$ is a lattice seminorm on wL_1,
ii) $\|f+g\|_{\mathcal{U}} = \|f\|_{\mathcal{U}} + \|g\|_{\mathcal{U}}$ whenever f and g are nonnegative,
iii) $\|f\|_{wL_{\hat{1}}} = \sup\{\|f\|_{\mathcal{U}} : \mathcal{U}$ is an ultrafilter, $\mathcal{F} \subset \mathcal{U}\}$ for all $f \in wL_1$.

Again, we convert $\|\cdot\|_{\mathcal{U}}$ into a norm by forming the ideal

$$\mathcal{N}_{\mathcal{U}} = \{f \in wL_1 : \|f\|_{\mathcal{U}} = 0\}. \tag{1.6}$$

and then the quotient vector lattice $wL_1(\mathcal{U}) = wL_1/\mathcal{N}(\mathcal{U})$ on which $\|\cdot\|_{\mathcal{U}}$ acts as a lattice norm.

We now have an information on the dual of wL_1:

THEOREM 1.2 (J. KUPKA AND T. PECK). *Define a linear operator* $T_{\mathcal{U}} : L_\infty \longrightarrow wL_{\hat{1}}^*$ *by* $T_{\mathcal{U}}(m) : f \longmapsto I_{\mathcal{U}}(mf)$ *for all* $m \in L_\infty(\mu)$, *and for all* $f \in wL_{\hat{1}}$. *Then* $T_{\mathcal{U}}$ *constitutes an isometric, order isomorphism of* $L_\infty(\mu)$ *into* $wL_{\hat{1}}^*$. *Moreover, the linear span of the subspace* $T_{\mathcal{U}}(L_\infty(\mu))$, *as* $\mathcal{U}$ *ranges over the collection of ultrafilters which contains* $\mathcal{F}$, *constitutes a norming and hence a weak* dense subspace of* $wL_{\hat{1}}^*$.

This theorem gives some favorable information for $wL_1(\mathcal{U})^*$ and the very last part of theorem says

$$\overline{T_{\mathcal{U}}(B_{L_\infty})}^{wL_1(\mathcal{U})^*} = B_{wL_1(\mathcal{U})^*}. \tag{1.7}$$

From this theorem, for any $m \in L_\infty(\mu)$, we have

$$\widehat{m} = T_{\mathcal{U}}(m) \in wL_1(\mathcal{U})^*. \tag{1.8}$$

Clearly, every linear functional $\varphi \in wL_1(\mathcal{U})^*$ is a linear functional on $wL_{\hat{1}}$ (see more detail in [4, 2.20]).

We now give a lemma about linear functionals on $wL_{\hat{1}}$ which is actually due to J. Kupka and T. Peck (see [4, 2.20]).

LEMMA 1.3. *For an ultrafilter* $\mathcal{U}$ *containing* $\mathcal{F}$ *in* (1.3), *let* $f \in wL_{\hat{1}}$ *be a nonnegative function with* $\|f\|_{\mathcal{U}} = 1$. *Then for any* $g \in wL_{\hat{1}}$, *disjointly supported from* f, *there exists a* $\phi \in wL_{\hat{1}}^*$ *such that* $\|\phi\| = 1$, $\phi(f) = 1$, *and* $\phi(g) = 0$.

We can now generalize this lemma for arbitrary pairwise disjointly supported elements in $wL_{\hat{1}}$.

COROLLARY 1.4. *Let* $(f_n)_{n=1}^\infty$ *be a sequence of nonnegative elements in* $wL_{\hat{1}}$ *with* $\|f_n\|_{wL_{\hat{1}}} = 1$, *for all* $n = 1, 2, 3, \ldots.$ *and such that the* f_n *have pairwise disjoint supports. Then for each* n, *there exists a linear functional* ϕ_n *on* $wL_{\hat{1}}$ *such that* $\phi_n(f) = 1, \quad \|\phi_n\| = 1$ *and* $\phi_n(f_m) = 0 \quad if \quad n \neq m$.

Proof. We can show this by induction with lemma 1.3 for each f_n. For given $f_1 \in wL_{\hat{1}}$, by lemma 1.3, we can choose ϕ_1 with $\phi_1(f_1) = \|\phi\| = 1$ and $\phi_1(f_j) = 0$, for all $j = 2, 3, \ldots$. If we selected $\phi_1, \phi_2, \ldots, \phi_n$ satisfying all the conclusions of corollary, then ϕ_{n+1} can be selected by applying lemma 1.3 again. This proves the corollary. □

We need one technical lemma whose proof can be seen in [7], namely;

LEMMA 1.5. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative elements in $wL_{\hat{1}}$ such that the f_n have pairwise disjoint supports with $\|f_n\|_{wL_{\hat{1}}} = 1$, for all $n = 1, 2, 3, \ldots$ and let $(\phi_n)_{n=1}^{\infty}$ be a sequence of linear functionals on $wL_{\hat{1}}$ selected as in* Corollary 1.4. *Then for any $f \in wL_{\hat{1}}$, we have $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_{\hat{1}}}$.*

Proof. See the lemma 1.6 in [7] □

2. Complemented Banach subspaces in $wL_{\hat{1}}$.

We study some relations between a Banach space E and $wL_{\hat{1}}$. As an universal Banach lattice $wL_{\hat{1}}$ for the separable Banach lattices, one can observe that $wL_{\hat{1}}$ is a "big" Banach space. One can obviously ask the following questions:

What kind of Banach spaces can be embedded into $wL_{\hat{1}}$?

Moreover, is the range of the embedding map a complemented subspace of $wL_{\hat{1}}$?

We now start with the following proposition about the universality of $wL_{\hat{1}}$.

PROPOSITION 2.1. *Every separable Banach space E is isometric to a Banach subspace of $wL_{\hat{1}}$.*

Proof. Let E be a separable Banach space. Then E is isometric to a subspace of $C([0,1])$. Since $C([0.1])$ is a separable Banach lattice with weak unit, by Lotz- Peck's theorem in [6], one can find a lattice isometry $T : C[0,1] \longrightarrow wL_{\hat{1}}$ so that $C[0,1]$ is isometric and order isomorphic to a sublattice of $wL_{\hat{1}}$. Then the restriction to E is the desired isometric embedding map. This proves the proposition. □

REMARK 2.2. Is every separable Banach subspace of $wL_{\hat{1}}$ complemented? Unfortunately, the answer for this question is negative, by the theorem of Lindenstrauss and Tzafiri which states that if every closed subspace of given Banach space E is complemented in E, then E is isomorphic to Hilbert space. (As we know $wL_{\hat{1}}$ is not a Hilbert space, moreover it is not a reflexive space.)

Nonetheless, one can find a lot of complemented Banach subspaces of $wL_{\hat{1}}$. As one easy example, we give the following theorem.

THEOREM 2.3. *Let $(f_n)_{n=1}^{\infty}$ be a sequence in $wL_{\hat{1}}$ such that the f_n are nonnegative pairwise disjointly supported. If $E = \overline{span}(f_n)$, then E is a complemented Banach subspace of $wL_{\hat{1}}$.*

Proof. Without loss of generality, we can assume that $\|f_n\|_{wL_{\hat{1}}} = 1$, for all n. Hence by corollary 1.4, one can find linear functionals ϕ_n on $wL_{\hat{1}}$ with $\phi_n(f_m) = \delta_{n,m}$, and $\|\phi_n\| = 1$ for all n. Now for arbitrary $f \in wL_{\hat{1}}$, the number $\phi_n(f)$ is the limit of the subnet $\{I_{\mathcal{U}}(\chi_{D_{n,k}} \cdot f)\}$ where $(D_{n,k})_{k=1}^{\infty}$ is a sequence of subsets of $D_n = supp(f_n)$.

Define $P : wL_{\hat{1}} \longrightarrow E$ by

$$P(f) = \sum_{n=1}^{\infty} \phi_n(f) f_n. \tag{2.1}$$

Moreover by the lemma 1.5, we have $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_{\hat{1}}}$. Hence $\|P\| \leq 1$.

Finally, we need to show that $P^2 = P$. For $f \in wL_{\hat{1}}$,

$$\begin{aligned} P^2(f) &= P(\sum_{n=1}^{\infty} \phi_n(f) f_n) \\ &= \sum_{j=1}^{\infty} \phi_j(\sum_{i=1}^{\infty} \phi_i(f) f_i) f_j \quad \text{by} \quad \phi_i(f_j) = \delta_{ij} \\ &= \sum_{j=1}^{\infty} \phi_j(f) f_j \\ &= P(f) \end{aligned} \tag{2.2}$$

Then P is the desired projection on $wL_{\hat{1}}$ with $\|P\| = 1$. This proves the theorem. □

3. An application of Peck's theorem

In this section, we give an easy application of Peck's theorem, in [9], combined with well known Banach space results. Peck's theorem gives some information about the decomposition of finite dimensional Banach subspaces of $wL_{\hat{1}}$. (Especially, decompositions into l_∞^n and some other Banach space.) Considering [9], we state the theorem with no proof for our purpose.

THEOREM 3.1 (T. PECK). *For $1 \leq i \leq n$, let $f_i > 0$ be elements of $wL_{\hat{1}}$ such that the f_i have pairwise disjoint supports, and such that $\|f_i\|_{wL_{\hat{1}}} = 1$ for each i. Then there exist functions e_i, $0 \leq e_i \leq f_i$ so that*

i) $\|\sum_{i=1}^n a_i e_i\|_{wL_{\hat{1}}} = \sup_{1\leq i\leq n} |a_i|$, *for all* $(a_i)_{i=1}^n$,

ii) *if we set $g_i = f_i - e_i$, then $\|\sum_{i=1}^n a_i g_i\|_{wL_{\hat{1}}} = \|\sum_{i=1}^n a_i f_i\|_{wL_{\hat{1}}}$ for all $(a_i)_{i=1}^n \in \mathbf{R}^n$.*

REMARK 3.2. From this theorem, we can ask one obvious question. Namely, if we take away much more from each f_i, can we make the span of (e_i) different from l_∞^n, something like l_p^n ? But these questions are still open.

Now fix an ultrafilter $\mathcal{F} \subset \mathcal{U}$ defined in (1.3). Define $L(\mathcal{U}) = \{f \in wL_{\hat{1}} : \|f\|_{wL_{\hat{1}}} = \|f\|_{\mathcal{U}}\}$. In [4], one can see several properties of $L(\mathcal{U})$. Note that $\|f\|_{\mathcal{U}} = I_{\mathcal{U}}(|f|)$ in (1.5). We now are in a position to give some properties for $L(\mathcal{U})$.

PROPOSITION 3.3. *The set $L(\mathcal{U})$ is a norm closed subset of $wL_{\hat{1}}$.*

Proof. Let (f_n) be a sequence such that f_n converges to f in $wL_{\hat{1}}$. Then for all $\epsilon > 0$, there exists N_0 such that if $n \geq N_0$, we have $\|f_n - f\|_{wL_{\hat{1}}} < \frac{\epsilon}{2}$. Now,

$$\begin{aligned}
\|f\|_{\mathcal{U}} &\leq \|f\|_{wL_{\hat{1}}} \quad \text{by the theorem 1.1} \\
&\leq \|f - f_{N_0}\|_{wL_{\hat{1}}} + \|f_{N_0}\|_{wL_{\hat{1}}} \\
&\leq \frac{\epsilon}{2} + \|f_{N_0}\|_{\mathcal{U}} \quad \text{by the definition of } L(\mathcal{U}) \\
&= \frac{\epsilon}{2} + \|f_{N_0}\|_{\mathcal{U}} - \|f\|_{\mathcal{U}} + \|f\|_{\mathcal{U}} \\
&\leq \frac{\epsilon}{2} + |\|f_{N_0}\|_{\mathcal{U}} - \|f\|_{\mathcal{U}}| + \|f\|_{\mathcal{U}} \\
&\leq \frac{\epsilon}{2} + \|f - f_{N_0}\|_{\mathcal{U}} + \|f\|_{\mathcal{U}} \\
&\leq \frac{\epsilon}{2} + \|f - f_{N_0}\|_{wL_{\hat{1}}} + \|f\|_{\mathcal{U}} \\
&\leq \epsilon + \|f\|_{\mathcal{U}}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\|f\|_{\mathcal{U}} \leq \|f\|_{wL_{\hat{1}}} \leq \|f\|_{\mathcal{U}}$. This implies $f \in L(\mathcal{U})$. This proves the proposition. □

EXAMPLES. We can give a lot of elements in $L(\mathcal{U})$ as following. For each $0 \leq \alpha < 1$, $f(x) = \frac{1}{x-\alpha}$ is an element of $L(\mathcal{U})$.

$$\begin{aligned}
1 &= \|f\|_{wL_{\hat{1}}} \\
&= \|f\|_{\mathcal{U}}.
\end{aligned}$$

Hence these examples and the proposition 3.3 give that $L(\mathcal{U})$ contains infinite number of elements. Next we give one more property of $L(\mathcal{U})$. That is, every finite sequence of pairwise disjointly supported nonnegative elements in $L(\mathcal{U})$ form a basis of l_1.

THEOREM 3.4. *Let $f_1, f_2, \dots, f_n$ be pairwise disjointly supported nonnegative elements in $L(\mathcal{U})$ with $\|f_i\|_{wL_{\hat{1}}} = 1$, for all $i = 1, 2, 3, \dots, n$. Then the span of $(f_i)_{i=1}^n$ in $wL_{\hat{1}}$ is isometrically isomorphic to l_1^n.*

Proof. Define $T : l_1^n \longrightarrow \overline{span}(f_i)$, by $Te_i = f_i$ where $(e_i)_{i=1}^n$ is the usual basis of l_1^n. Then $T(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i f_i$ is a well defined continuous linear operator.

We need to show that $\| \sum_{i=1}^n a_i f_i\|_{wL_{\hat{1}}} = \sum_{i=1}^n |a_i|$. Since $\| \cdot \|_{wL_{\hat{1}}}$ is a lattice norm, one can assume with no loss of generality $a_i > 0$ and

$a_i \neq 0$, for all i. Now,

$$\begin{aligned}
\sum_{i=1}^{n} a_i &= \sum_{i=1}^{n} a_i \|f_i\|_{wL_{\hat{1}}} \\
&= \sum_{i=1}^{n} \|a_i f_i\|_{wL_{\hat{1}}} \\
&= \sum_{i=1}^{n} a_i \|f_i\|_{\mathcal{U}} \quad (\|f_i\|_{wL_{\mathcal{U}}} = \|f_i\|_{wL_{\hat{1}}}, \forall i) \\
&= \sum_{i=1}^{n} \|a_i f_i\|_{\mathcal{U}} \\
&= \|\sum_{i=1}^{n} a_i f_i\|_{\mathcal{U}} \quad (\text{by the property of } \|\cdot\|_{\mathcal{U}}) \\
&\leq \|\sum_{i=1}^{n} a_i f_i\|_{wL_{\hat{1}}} \quad \text{by the theorem 1.1} \\
&\leq \sum_{i=1}^{n} a_i \|f_i\|_{wL_{\hat{1}}} = \sum_{i=1}^{n} a_i .
\end{aligned}$$

Therefore T is linear isometry from l_1 onto $\overline{span}(f_i)$. This proves the theorem. □

COROLLARY 3.5. *Let $(f_i)_{i=1}^{\infty}$ be a sequence of nonnegative elements in $L(\mathcal{U})$ which have pairwise disjoint supports, with each $\|f_i\|_{wL_{\hat{1}}} = 1$. Then $\overline{span}(f_i)_{i=1}^{\infty}$ is isometrically isomorphic to l_1.*

Proof. Define $T : l_1 \longrightarrow \overline{span}(f_i)_{i=1}^{\infty}$ by $Te_i = f_i$, for all i. Then T is linear and $T(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} a_i f_i$. Exactly the same argument in the theorem 3.4 gives an isometric isomorphism of T. This proves the corollary. □

Next, using Peck's theorem we can give one more property of the envelope norm in $wL_{\hat{1}}$. For this, we need a technical lemma.

LEMMA 3.6. *For $h \in wL_{\hat{1}}$, $h \geq 0$, h can be written as a strictly increasing nonnegative function with no atoms.*

Proof. Let $h \in wL_{\hat{1}}$, $\|h\|_{wL_{\hat{1}}} = 1$ and $h \geq 0$. If the function h satisfies $\mu(h = r) > 0$ for some r, then we embed the measure algebra of Lebesgue measure λ on $[0, 1]$ into the measure algebra of normalized μ-measure on the measurable set $\{h = r\}$. We replace h on this set by the image of the function $\phi(t) = t + r$. Since there are at most countably many points $r \geq 0$ for which $\mu\{h = r\} > 0$, the performance of such replacement for each of these points will change h by at most a bounded measurable function. This means that h can be everywhere strictly positive. This proves the lemma. □

Now, if we use one element $f \in wL_{\hat{1}}$, then we can get the following theorem.

THEOREM 3.7. *Let $f \in wL_{\hat{1}}$ be a nonnegative element with $\|f\|_{wL_{\hat{1}}} = 1$. Then there exists a sequence $(E_n)_{n=1}^{\infty}$ of subspaces of $wL_{\hat{1}}$ such that*
i) $E_1 = span(f)$,
ii) $\dim(E_n) = n$, *for all* $n = 1, 2, 3, \ldots$,
iii) E_n *is isometrically isomorphic to* l_{∞}^n, *for all* $n = 1, 2, 3, \ldots$.

Proof. Assume $\|f\|_{wL_{\hat{1}}} = 1$. Then there exist disjoint intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ such that $b_n/a_n \to \infty$ and $a_n \to \infty$,

$$\text{(3.1)} \qquad \frac{1}{\log \frac{b_n}{a_n}} \int_{\{a_n \leq f \leq b_n\}} f(x) d\mu \geq 1 - \frac{1}{n}, \quad \text{for all } n = 1, 2, 3, \ldots$$

Without loss of generality, we can assume f is everywhere strict by the lemma 3.6. The construction of the subspaces E_n follow an inductive argument by using Peck's theorem. Let $E_1 = span(f)$. Then $dim(E_1) = 1$.

We now construct a 2-dimensional subspace E_2 of $wL_{\hat{1}}$ satisfying the conclusion of the theorem. Let

$$\text{(3.2)} \qquad e_1 = \sum_{n=1}^{\infty} f \chi_{(a_{2n} \leq f \leq b_{2n})}.$$

Then clearly $\|e_1\|_{wL_{\hat{1}}} = 1$. And let $g_1 = f - e_1$. Then by theorem 3.1, we have $\|g_1\|_{wL_{\hat{1}}} = \|f - e_1\|_{wL_{\hat{1}}} = \|f\|_{wL_{\hat{1}}} = 1$. Hence we have g_1 and e_1 which are nonnegative and disjointly supported with $\|e_1\|_{wL_{\hat{1}}} = \|g_1\|_{wL_{\hat{1}}} = 1$. Let E_2 be the subspace of $wL_{\hat{1}}$ generated by $\{g_1, e_1\}$. Then $dim(E_2) = 2$ and E_2 is constructed from E_1 in this sense. Define $T_2 : l^2_\infty \longrightarrow E_2$ by $T_2(a_i) = a_1 g_1 + a_2 e_1$. Then T_2 is an isomorphism from l^2_∞ onto E_2. To show T_2 is an isometry, it suffices to show that $\|g_1 + e_1\|_{wL_{\hat{1}}} = 1$, since $\|\cdot\|_{wL_{\hat{1}}}$ is a lattice norm on $wL_{\hat{1}}$. But this holds, since $\|g_1 + e_1\|_{wL_{\hat{1}}} = \|f\|_{wL_{\hat{1}}} = 1$. Hence we have

$$\begin{aligned} \|T_2(a_i)\|_{wL_{\hat{1}}} &= \|a_1 g_1 + a_2 e_1\|_{wL_{\hat{1}}} \\ &= \|(a_i)\|_\infty. \end{aligned} \tag{3.3}$$

Next, we give the argument for constructing E_3 from E_2 satisfying the conclusion of theorem. Then step by step, by applying the same argument, we can construct E_n from E_{n-1}. Fix e_1 in E_2. Decompose g_1 as g_2 and e_2 using (3.1) and Peck's theorem. Then we have three pairwise disjointly supported elements $\{g_2, e_1, e_2\}$ with $\|e_1\|_{wL_{\hat{1}}} = \|g_2\|_{wL_{\hat{1}}} = \|e_2\|_{wL_{\hat{1}}} = 1$. Then $span(g_2, e_1, e_2) = E_3$. Then $dim(E_3) = 3$. The rest of the argument about isometry exactly follows the previous one. This proves the theorem. □

ACKNOWLEDGEMENTS. The author thanks the referees for careful reading of original draft and for their valuable comments and suggestions.

References

[1] M. Cwikel and C. Fefferman, *Maximal seminorm on wL_1,*, Studia Math. **69** (1980), 149-154.

[2] ———, *The canonical seminorm on wL_1*, Studia Math. **78** (1984), 275-278.

[3] M. Cwikel and Y. Sagher, $L(p,\infty)^*$, Indiana Univ. Math. J. **21** (1972), 782-786.

[4] J. Kupka and T. Peck, *The L_1-structure of wL_1*, Math. Ann. **269** (1984), 235-262, 583-586.

[5] Lindenstrauss and Tzafriri, *Classical Banach spaces II*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[6] H. P. Lotz and T. Peck, *Sublattices of the Banach envelope of weakL_1*, Proc. Amer. Math. Soc. (to appear).

[7] JeongH. Kang, *Complemented sublattices of* wL_1 *isomorphics to classical Banach lattices.*, Comm. of Kor. Math. Soc. **11** (1996), 1015-1030.
[8] T. Peck and M. Talagrand, *Banach sublattices of* $weakL_1$, Israel. J. Math. **59** (1987), 257-271.
[9] T. Peck, *A decomposition property of* $weakL_1$, Pro. Edinburgh Math. Soc. **35** (1992), 109-114.

DEPARTMENT OF MATHEMATICS, KOREA MILITARY ACADEMY, SEOUL 139-799, KOREA