Bull. Korean Math. Soc. **35** (1998), No. 3, pp. 397–408

HAUSDORFF DIMENSION OF SOME SUB-SIMILAR SETS

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ABSTRACT. We often use the Hausdorff dimension as a tool of measuring how complicate the fractal is. But it is usually very difficult to calculate that value. So there have been many tries to find the dimension of the given set and most of these are related to the density theorem of invariant measure. The aims of this paper are to introduce the k-irreducible subsimilar sets as a generalization of the set defined by V.Drobot and J.Turner in ([1]) and calculate their Hausdorff dimensions by using algebraic methods.

1. Introduction

Let X be a non-empty compact subset of the Euclidean space $\mathbb{R}^d$ such that the closure of its interior is itself, that is, $\overline{(X^\circ)} = X$. And let $T_i : X \to X$ for $i = 0, 1, \cdots, N-1$ be similarity maps with common contraction ratio r, $(0 < r < 1)$, such that $T_i(X^\circ) \cap T_j(X^\circ) = \emptyset$ for $i \neq j$. Then these similiarity maps $T_0, T_1, \cdots, T_{N-1}$ form a self-similar set

$$F := \bigcap_{n=1}^{\infty} \Big(\bigcup_{(i_1 i_2 \cdots i_n)} T_{i_1 i_2 \cdots i_n}(X) \Big),$$

in which $T_{i_1 i_2 \cdots i_n}$ denotes the composition map $T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}$. For the details, see [**3**].

For $S \subset \{0, 1, \cdots, N-1\}$, we now define

$$\Omega_S^\ell \equiv \Omega_{N,S}^\ell = \{(i_1, i_2, \cdots, i_\ell) : i_k \in S \text{ for } k = 1, 2, \cdots, \ell\},$$

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Received June 18, 1997. Revised November 4, 1997.
1991 Mathematics Subject Classification: 28A78, 28C20.
Key words and phrases: k-irreducible sub-similar set, Hausdorff dimension.

and when $S = \{0, 1, \cdots, N-1\}$, we simply write Ω_N^ℓ and Ω_N for above sets.

For each number x, let $[x]$ denote the largest integer not greater than x and $\mathcal{K}(X)$ be the space of all compact subsets of X. For given non-negative integers k, ℓ with $1 \le k \le \ell$ and for each non-negative integer m, we define

(1) $\pi_\ell : \Omega_S^\ell \to \mathcal{K}(X)$ by $\pi_\ell\{(i_1, i_2, \cdots, i_\ell)\} = T_{i_1 i_2 \cdots i_\ell}(X)$,
(2) $\pi : \Omega_S \to \mathcal{K}(X)$ by $\pi\{(i_1, i_2, \cdots\cdots)\} = \bigcap_{n=1}^{\infty} T_{i_1 i_2 \cdots i_n}(X)$,
(3) $\mathrm{proj}_\ell : \Omega_S$ (or Ω_S^n for $n \ge \ell$) $\to \Omega_S^\ell$ by

$$\mathrm{proj}_\ell\{(i_1, i_2, \cdots,)\} \text{ (or } \mathrm{proj}_\ell\{(i_1, i_2, \cdots, i_n)\}\ ) = (i_1, i_2, \cdots, i_\ell),$$

(4) $\mathrm{s}_{k,\ell}^m : \Omega_S$ (or Ω_S^n for $n \ge mk + \ell$) $\to \Omega_S^\ell$ by

$$\mathrm{s}_{k,\ell}^m\{(i_1, i_2, \cdots)\}(\text{or } \mathrm{s}_{k,\ell}^m\{(i_1, i_2, \cdots, i_n)\}\ ) = (i_{mk+1}, i_{mk+2}, \cdots, i_{mk+\ell}).$$

Let $A \subset \mathbb{R}$ and $P_\ell : \Omega_S^\ell \to \mathbb{R}$ be a given real valued function. Then for these P_ℓ and A, a non-empty set $I = I(P_\ell; k; A) \subset \Omega_S^\ell$ is called the *k-irreducible precoding space generated by* (P_ℓ, A) if it satisfies

(1) $P_\ell(\sigma) \in A$ for every $\sigma \in I$
(2) $\{\mathrm{s}_{k,\ell}^1[\mathrm{proj}_\ell^{-1}(\sigma)]\} \bigcap I \ne \emptyset$ for every $\sigma \in I$
(3) $\mathrm{s}_{k,\ell}^1[\mathrm{proj}_\ell^{-1}(\sigma)] \ni \sigma$ for some $\sigma \in I$,

and $\Delta(I) = \bigcap_{m=0}^{\infty} (\mathrm{s}_{k,\ell}^m)^{-1}(I)$ is called the *k-irreducible coding space generated by* (P_ℓ, A).

For the self-similar set F defined previously and $I = I(P_\ell; k; A)$, the *k-irreducible sub-similar set* $F(I) = F\{I(P_\ell; k; A)\}$ is defined by

$$\begin{aligned} F(I) &:= \{\mathrm{x} \in F : \pi^{-1}(\{\mathrm{x}\}) \in \Delta(I)\} \\ &= \{\mathrm{x} \in F : P_\ell[\mathrm{s}_{k,\ell}^m\{\pi^{-1}(\{\mathrm{x}\})\}] \in A \text{ for each integer } m > 0\}. \end{aligned}$$

Each τ (or $\tau(n)$) $\in \Omega_S^n$ is called the *admissible code of rank n in* $\Delta(I)$ if there exists $\varrho \in \Delta(I)$ such that $\varrho|_n = \mathrm{proj}_n(\varrho) = \tau$, and denoted by $\tau \prec \Delta(I)$. If $\tau(n) \prec \Delta(I)$ for $n \ge \ell$ and $\mathrm{s}_{k,\ell}^{[(n-\ell)/k]}(\tau) = \sigma$ for $\sigma \in I$, then τ is called the *admissible code of rank n and of type* σ and denoted by $\tau \prec \Delta(I; \sigma)$.

Throughout this paper, let $|A|$ denote the cardinal number of the set A and define followings:

(1) $\Delta^n(I) := \{\tau \in \Omega_S^n \ : \tau \prec \Delta(I)\}$,
(2) $\Delta^n(I;\sigma_i) := \{\tau \in \Omega_S^n \ : \tau \prec \Delta(I;\sigma_i)\}$,
(3) $\phi_i(n) := \mid \Delta^n(I;\sigma_i) \mid$,
(4) $\Phi(n) := (\phi_1(n), \phi_2(n), \cdots, \phi_s(n))$,
(5) $\mid \Delta^n(I) \mid := \sum_{i=1}^s \phi_i(n)$,
(6) $\varphi^m(\tau(n)) := \mid \{\varrho(n+m) : \varrho \prec \Delta^{n+m}(I) \text{ and } \varrho|_n = \tau\} \mid$.

For the k-irreducible precoding space I, say, $= \{\sigma_1, \sigma_2, \cdots, \sigma_s\}$, define $M = M(I) = (\sigma_{ij})_{s\times s}$ by

$$\sigma_{ij} = \begin{cases} 1 & \text{if } \mathrm{s}_{k,\ell}^1\{\mathrm{proj}_\ell^{-1}(\sigma_i)\} \ni \sigma_j \quad \text{for } \sigma_i, \sigma_j \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then M is a strongly irreducible matrix. Thus we have the following lemma for this matrix. For its proof, see [**4**].

LEMMA 1.1. *Let M be the $s \times s$ matrix defined as above. Then M has a unique positive eigenvalue Λ_P which is the spectral radius of M and is simple. Moreover $\mid \Lambda \mid < \Lambda_P$ for any other eigenvalue Λ of M and its corresponding eigenvector $\mathbf{v} = (v_1, v_2, \cdots, v_s)$ is positive, that is, each $v_i > 0$ for $i = 1, 2, \cdots, s$.*

Let E be a subset of $\mathbb{R}^d$ and $\delta > 0$. Then a family of subsets of $\mathbb{R}^d$, $C_\delta = \{U_i\}_{i=1}^\infty$, is called a *$\delta$-cover of E* if $E \subset \cup_{i=1}^\infty U_i$ and $\mathrm{diam}(U_i) \le \delta$ for each i. For $\alpha > 0$, the *α-dimensional Hausdorff measure* $H^\alpha(E)$ is defined by $H^\alpha(E) = \lim_{\delta\to 0} \inf \sum_{C_\delta} \mathrm{diam}(U_i)^\alpha$ where the infimum is taken over all $\{C_\delta\}$, and the Hausdorff dimension of E is defined by the value $\dim_H(E)$ which satisfies $H^\alpha(E) = \infty$ if $\alpha < \dim_H(E)$ and $H^\alpha(E) = 0$ if $\alpha > \dim_H(E)$. From this, we have the following two lemmas ([**2**]).

LEMMA 1.2. *Let $E \subset \mathbb{R}^d$ and $\alpha > 0$. For given $\epsilon > 0$ and $\delta > 0$, if there exists a δ-cover $C = \{U_i\}$ of E such that $\sum_C \mathrm{diam}(U_i)^\alpha < \epsilon$, then $\dim_H(E) \le \alpha$.*

LEMMA 1.3. *Let $E \subset \mathbb{R}^d$ be a compact set and $\alpha > 0$. If there exists $\epsilon > 0$ and $\delta > 0$ such that any finite collection $\mathcal{C}$ of closed non-overlapping sets $U_1, U_2, \cdots, U_n$ with $\operatorname{diam}(U_i) < \delta$ and $\sum_{\mathcal{C}} \operatorname{diam}(U_i)^\alpha \leq \delta$ can not cover E, then $\dim_H(E) \geq \delta$.*

2. Main Results

Let N, k, ℓ and s be the positive integers and M be the matrix as in section 1.

THEOREM 2.1. *Let $\mathbf{u} = (1, 1, \cdots, 1)$ and $\mathbf{v}_i = (\overbrace{0, \cdots, 0, 1}^{(i)}, 0, \cdots, 0)$ be vectors in $\mathbb{R}^s$. Then for each $n \geq \ell$, we have*

(1) $\Phi(n) = N^r(\mathbf{u}M^{[(n-\ell)/k]})$, $|\Delta^n(I)| = N^r(\mathbf{u}M^{[(n-\ell)/k]} \cdot \mathbf{u})$ *and* $\phi_i(n) = (\Phi(n) \cdot \mathbf{v}_i)$ *where* $n - \ell \equiv r \pmod{k}$.

(2) *For each $\tau \in \Delta^n(I; \sigma_i)$, we have*

$$\varphi^{n'}(\tau(n)) \begin{cases} = N^{r'}(\mathbf{v}_i M^{[n'/k]} \cdot \mathbf{u}) & \text{for } r = 0, \\ \leq N^{k+r'} \max\limits_{1 \leq i \leq s} (\mathbf{v}_i M^{[n'/k]-1} \cdot \mathbf{u}) & \text{for } r \neq 0,\ 0 < r + r' < k, \\ \leq N^{r'} \max\limits_{1 \leq i \leq s} (\mathbf{v}_i M^{[n\prime/k]} \cdot \mathbf{u}) & \text{for } r \neq 0,\ k \leq r + r', \end{cases}$$

where $n - \ell \equiv r \pmod{k}$ and $n' \equiv r' \pmod{k}$.
In particular, $\varphi^{n'}(\sigma_i) = N^{r'}(\mathbf{v}_i M^{[n'/k]} \cdot \mathbf{u})$ for $\sigma_i \in I$.

Proof. (1) For each $\sigma_j = (j_1, j_2, \cdots, j_\ell) \in I$, there exist $\sum_{i=1}^{s} \sigma_{ij}$ $(i_1, i_2, \cdots, i_k)$'s such that $(i_1, i_2, \cdots, i_k, j_1, j_2, \cdots, j_\ell) \in \Delta^{k+\ell}(I; \sigma_j)$. Thus we have $\phi_j(k + \ell) = \sum_{i=1}^{s} \sigma_{ij}$ and $\Phi(k + \ell) = \mathbf{u}M$. By induction, $\Phi(mk + \ell) = \mathbf{u}M^m$. Since each $\tau(mk + \ell) \in \Delta^{mk+\ell}(I)$ has N^r $\tau(mk + r + \ell)$'s for $0 \leq r < k$,

$$\Phi(n) = N^r(\mathbf{u}M^{[(n-\ell)/k]}),$$

for $n = mk + r + \ell$. The other parts are clear.

(2) **Case (i)** : Suppose that r is zero or $n = mk + \ell$ for some integer $m \geq 0$. For each $\tau(n) = (q_1, q_2, \cdots, q_\ell, i_1, i_2, \cdots, i_{mk}) \in \Delta^n(I;\sigma_i)$, there exist $\sum_{j=1}^{s} \sigma_{ij}$ $(j_1, j_2, \cdots, j_k)$'s such that each $(q_1, q_2, \cdots, q_\ell, i_1, i_2$ $\cdots, i_{mk}, j_1, j_2, \cdots, j_k) \in \Delta^{n+k}(I)$ and is descended from $\tau(n)$. Hence $\varphi^k(\tau(n)) = (\mathbf{v}_i M \cdot \mathbf{u})$. As above, we deduce $\varphi^{m'k}(\tau(n)) = (\mathbf{v}_i M^{m'} \cdot \mathbf{u})$ for each integer $m' \geq 0$. Since each $(q_1, q_2, \cdots, q_\ell, i_1, i_2, \cdots, i_{mk}, j_1, j_2, \cdots,$ $j_{m'k}) \in M^{n+m'k}$ descending from $\tau(n)$ has $N^{r'}$ subcodings in $\Delta^{n+m'k+r'}$ (I) for $0 \leq r' < k$,

$$\varphi^{n'}(\tau(n)) = N^{r'}(\mathbf{v}_i M^{[n'/k]} \cdot \mathbf{u}) \text{ for } n' = m'k + r'.$$

Case (ii) : Suppose $n = mk + \ell + r$ for some integer $m \geq 0$ and $0 < r < k$. Let $\tau(n) = (q_1, q_2, \cdots, q_\ell, i_1, i_2, \cdots, i_{mk}, i_{mk+1},$ $\cdots, i_{mk+r}) \in \Delta^n(I;\sigma_i)$. For this $\tau(n)$, there exist at most N^{k-r} $\tau(n+k-r)$'s in $\Delta^{n+k-r}(I)$, so

$$\varphi^{n'}(\tau(n)) \leq N^{k-r} \cdot \max\{\varphi^{n'-k+r}(\tau(n+k-r)) : \tau(n+k-r) \in \Delta^{n+k-r}(I)\}.$$

By the case (i),

$$\varphi^{n'}(\tau(n)) \leq N^{k-r} \cdot N^{r+r'} \max_{1 \leq i \leq s}(\mathbf{v}_i M^{[n'/k]-1} \cdot \mathbf{u}) \text{ for } 0 < r + r' < k,$$

and

$$\varphi^{n'}(\tau(n)) \leq N^{k-r} \cdot N^{r+r'-k} \max_{1 \leq i \leq s}(\mathbf{v}_i M^{[n'/k]} \cdot \mathbf{u}) \text{ for } k \leq r + r' < 2k.$$

Therefore

$$\varphi^{n'}(\tau(n)) \leq \begin{cases} N^{k+r'} \max_{1 \leq i \leq s}(\mathbf{v}_i M^{[n'/k]-1} \cdot \mathbf{u}) & \text{for } r \neq 0,\ 0 < r + r' < k, \\ N^{r'} \max_{1 \leq i \leq s}(\mathbf{v}_i M^{[n'/k]} \cdot \mathbf{u}) & \text{for } r \neq 0, k \leq r + r' < 2k. \end{cases} \quad \square$$

THEOREM 2.2. *Let Λ_P be as in* Lemma 1.1. *Then there exist two positive constants c_1, c_2 with $c_1 < c_2$ such that*

$$c_1 {\Lambda_P}^{[(n-\ell)/k]} \leq |\ \Delta^n(I)\ | \leq c_2 {\Lambda_P}^{[(n-\ell)/k]}$$

for sufficiently large n.

Proof. Let $n \geq \ell$ and $\mathbf{v} = (v_1, v_2, \cdots, v_s)$ be the eigenvector corresponding to Λ_P. And let's λ be given such that $0 < v_i < \lambda$ for all $i = 1, 2, \cdots, s$. Since $\mathbf{v}$ is the eigenvector of M, we have $\mathbb{R}^s = \langle \mathbf{v} \rangle \oplus Y$ where $\langle \mathbf{v} \rangle$ is the linear span of $\mathbf{v}$ and Y is the invariant subspace under M. So the absolute values of all eigenvalues of $M|_Y$ are less than that of Λ_P by Lemma 1.1. From Theorem 2.1, we have $| \Delta^n(I) | = N^r(\mathbf{u}M^{[(n-\ell)/k]} \cdot \mathbf{u})$.

Now we claim that $\mathbf{u}$ is not in Y and so there exists a non-zero real number κ and $\mathbf{w}$ in Y such that $\mathbf{u} = \kappa\mathbf{v} + \mathbf{w}$. In fact, if $\mathbf{u}$ is in Y, then we have following contradiction.

$$\begin{aligned}
\Lambda_P &= \lim_{n\to\infty} \| \mathbf{v}M^n \|^{1/n} \leq \limsup_{n\to\infty} \| \lambda\mathbf{u}M^n \|^{1/n} \\
&= \limsup_{n\to\infty} \| \lambda\mathbf{u}M|_Y^n \|^{1/n} \leq \limsup_{n\to\infty} \| \lambda\mathbf{u} \|^{1/n} \| M|_Y^n \|^{1/n} \\
&= \lim_{n\to\infty} \| M|_Y^n \|^{1/n} < \Lambda_P.
\end{aligned}$$

Then

$$\begin{aligned}
| \Delta^n(I) | &= N^r(\mathbf{u}M^{[(n-\ell)/k]} \cdot \mathbf{u} \\
&= N^r\kappa\Lambda_P^{[(n-\ell)/k]}(\mathbf{v} \cdot \mathbf{u}) + N^r(\mathbf{w}M^{[(n-\ell)/k]} \cdot \mathbf{u}).
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{n\to\infty} N^r(\mathbf{w}M^{[(n-\ell)/k]} \cdot \mathbf{u})/\Lambda_P^{[(n-\ell)/k]} \\
&\qquad \leq \lim_{n\to\infty} N^k \| \mathbf{u} \| \| \mathbf{w} \| \| M|_Y^{[(n-\ell)/k]} \| / \Lambda_P^{[(n-\ell)/k]} = 0
\end{aligned}$$

and

$$\kappa(\mathbf{v} \cdot \mathbf{u}) \leq N^r\kappa\Lambda_P^{[(n-\ell)/k]}(\mathbf{v} \cdot \mathbf{u})/\Lambda_P^{[(n-\ell)/k]} \leq N^k\kappa(\mathbf{v} \cdot \mathbf{u}),$$

we have

$$\kappa(\mathbf{v} \cdot \mathbf{u}) \leq \lim_{n\to\infty} | \Delta^n(I) | / \Lambda_P^{[(n-\ell)/k]} \leq N^k\kappa(\mathbf{v} \cdot \mathbf{u}). \qquad \square$$

THEOREM 2.3. *Let Λ_P be as in* Lemma 1.1. *Then there exist a constant $c_3 > 0$ such that for all $n \geq \ell$ and for all $\tau \in \Delta^n(I)$, we have*

$$\varphi^n(\tau) \leq c_3\Lambda_P.$$

Here the constant c_3 is independent of the rank and of the type of τ but depends only on the eigenvector corresponding to Λ_P.

Proof. We may suppose that $\tau \in \Delta^n(I;\sigma_i)$. Then by the Theorem 2.1,

$$\varphi^{n'}(\tau) \leq N^{2k} \max_{1\leq i\leq s}(\mathbf{v}_i M^{[n'/k]} \cdot \mathbf{u}).$$

Let $\mathbf{v} = (v_1, v_2, \cdots, v_s)$ be the eigenvector corresponding to the eigenvalue Λ_P and let $\mu := \min_{1\leq i\leq s}\{v_i\} > 0$. Then for any non-negative integer q and for $i = 1, 2, \cdots, s$

$$\mu(\mathbf{v}_i M^q \cdot \mathbf{u}) = (\mu \mathbf{v}_i M^q \cdot \mathbf{u}) \leq (\mathbf{v} M^q \cdot \mathbf{u}) = \Lambda_P^q(\mathbf{v} \cdot \mathbf{u})$$

and so

$$\max_{1\leq i\leq s}(\mathbf{v}_i M^q \cdot \mathbf{u}) \leq \mu^{-1}(\mathbf{v} \cdot \mathbf{u})\Lambda_P^q.$$

Now put $c_3 := N^{2k}\mu^{-1}(\mathbf{v} \cdot \mathbf{u})$. Then we have

$$\varphi^{n'}(\tau) \leq N^{2k}\mu^{-1}(\mathbf{v} \cdot \mathbf{u})\Lambda_P^{[n'/k]} = c_3\Lambda_P^{[n'/k]}. \qquad \square$$

THEOREM 2.4. *Let $F(I) = F\{I(P_\ell; k; A)\}$ be any k-irreducible set determined by (P_ℓ, A) and Λ_P be the value as in* Lemma 1.1. *Then*

$$\dim_H(F(I)) = -\log \Lambda_P / k \log r.$$

Proof. Let take α with $\alpha > -\log \Lambda_P / k \log r$. Since $r^{k\alpha}\Lambda_P < 1$, for given $\epsilon > 0$ we can take n large enough so that

$$c_2(r^{k\alpha}\Lambda_P)^{[(n-\ell)/k]} < \epsilon\{\mathrm{diam}(X)\}^{-\alpha} \quad \text{and } r^{k[(n-\ell)/k]}\mathrm{diam(X)} < \epsilon$$

where c_2 is the constant given in Theorem 2.2. Since diam(B) $\leq r^{k[(n-\ell)/k]}\text{diam}(X)$, $F(n) = \{B \mid B = \pi_n(\tau),\ \tau \in \Delta^n(I)\}$ is an ϵ-cover of $F(I)$. Since π_n is an injection on $\Delta^n(I)$, by Theorem 2.2 we have

$$c_1 \Lambda_P^{[(n-\ell)/k]} \leq \mid F(n) \mid \leq c_2 \Lambda_P^{[(n-\ell)/k]}.$$

So

$$\sum_{F(n)} \text{diam}(B)^\alpha \leq c_2 (r^{k\alpha})^{[(n-\ell)/k]} \Lambda_P^{[(n-\ell)/k]} \text{diam}(X)^\alpha$$

$$= c_2 \text{diam}(X)^\alpha (r^{k\alpha}\Lambda_P)^{[(n-\ell)/k]} < \epsilon.$$

Therefore $\dim_H\{F(I)\} \leq \alpha$ by Lemma 1.2. Now to prove the converse inequality, let $\alpha < -\log \Lambda_P/(k \log r)$. By Lemma 1.3, it suffices to show that we can take $\epsilon > 0$ so that any finite family $\mathcal{U}$ of subsets U's in X satisfying $\text{diam}(U) < \epsilon$ and $\sum_{\mathcal{U}} \text{diam}(U)^\alpha < 1$ cannot cover $F(I)$. Since $\sum_{n=1}^{\infty} (r^{-k\alpha} \Lambda_P{}^{-1})^n$ converges, take a positive integer M such that $M - \ell = 0 \pmod{k}$ and

$$\sum_{m \geq [(M-\ell)/k]} (r^{-k\alpha}\Lambda_P{}^{-1})^m \leq c_1 r^{\ell\alpha}/2 \cdot 3^d c_3 N^k,$$

where c_1 and c_3 are the constants given in Theorem 2.2 and Theorem 2.3. Then the number $\epsilon := r^M$ will be the one what we want. To certify this fact, let $\mathcal{U}$ be any finite family of subsets U's in X with $\text{diam}(U) < \epsilon$ and $\sum_{\mathcal{U}} \text{diam}(U)^\alpha < 1$. Since $F(n)$ defined above is a cover of $F(I)$ for each n and each B in $F(n)$ has to meet $F(I)$, for $\mathcal{U}$ not to cover $F(I)$ it suffices to prove that $F(n)$ must contain some element which is disjoint from any U's in $\mathcal{U}$ for large n. For each positive integer p, let

$$\mathcal{U}(p) := \{U \in \mathcal{U} : r^{M+pk} < \text{diam}(U) \leq r^{M+(p-1)k}\}.$$

Then each $U \in \mathcal{U}(p)$ meets at most 3^d elements in $F(M + (p-1))$, and so at most $3^d N^k$ elements in $F(M + pk)$. Let $\gamma_p = \mid \mathcal{U}(p) \mid$ and δ_p

be the number of sets in $F(M+pk)$ which meet some element in $\mathcal{U}(p)$. Then

$$\gamma_p \cdot r^{\ell\alpha+k\alpha[(M-\ell)/k+p]} \le \sum_{\mathcal{U}(p)} \operatorname{diam}(U)^\alpha \le \sum_{\mathcal{U}} \operatorname{diam}(U)^\alpha < 1,$$

so we have

$$\delta_p \le 3^d N^k \gamma_p \le 3^d N^k r^{-k\alpha[(M-\ell)/k+p]-\ell\alpha}.$$

Since $\mathcal{U}$ is finite, we can take a positive integer p_0 such that

$$\mathcal{U}(p) = \emptyset \text{ for any } p > p_0.$$

Now let $m' > p_0$ be given. Then for each B in $F(M+m'k)$ which meets some element in $\mathcal{U}$, there exists some p' with $1 \le p' \le p_0$ such that B must be contained in a $B' \in F(M+p'k)$. By Theorem 2.3, this specific B' contains at most $c_3\Lambda_P^{[(M+n'k)-(M+p'k)/k]} = c_3\Lambda_P^{n'-p'}$ such B's in $F(M+n'k)$. Therefore

$$\begin{aligned} &|\{B \in F(M+n'k) : B \text{ meets some element in } \mathcal{U}(p')\}| \\ &\le 3^d N^k r^{-k\alpha[(M-\ell)/k+p']-\ell\alpha} c_3 \Lambda_P^{n'-p'} \\ &= 3^d N^k c_3 (r^{-k\alpha}\Lambda_P{}^{-1})^{[(M-\ell)/k+p']} \Lambda_P{}^{[M-\ell/k]+n'} r^{-\ell\alpha}. \end{aligned}$$

So

$$\begin{aligned} &|\{B \in F(M+n'k) : B \text{ meets some element in } \mathcal{U}\}| \\ &\le 3^d N^k c_3 r^{-\ell\alpha} \Lambda_P^{[(M-\ell)/k]+n'} \sum_{p=1}^{p_0} (r^{-k\alpha}\Lambda_P{}^{-1})^{[(M-\ell)/k+p]} \\ &\le 3^d N^k c_3 r^{-\ell\alpha} \Lambda_P{}^{[(M-\ell)/k]+n'} \sum_{m>[(M-\ell)/k]} (r^{-k\alpha}\Lambda_P{}^{-1})^m \\ &< 1/2c_1 \Lambda_P{}^{[(M-\ell)/k]+n'}. \end{aligned}$$

However, from Theorem 2.2 we have

$$| F(M + n'k) | \geq c_1 {\Lambda_P}^{[(M+n'k-\ell)/k]} = c_1 {\Lambda_P}^{[(M-\ell)/k]+n'}.$$

So we find that there must exist some element B in $F(M + n'k)$ which does not meet any element of $\mathcal{U}$. That is, for large n, $F(n)$ contains some element B which is disjoint from any $U \in \mathcal{U}$ and so $\mathcal{U}$ cannot cover $F(I)$. □

COROLLARY 2.5. *Let F be the self-similar set defined in section 1. Then*

$$\dim_H(F) = -\log N / \log r.$$

Proof. Take $A = \mathbb{R}$. Then for each $1 \leq k \leq \ell$, $I = I(P_\ell; k; \mathbb{R}) = \Omega_N^\ell$ and is independent of the real valued map P_ℓ. And $F = F(I(P_\ell; k; \mathbb{R}))$ and M has $\Lambda_P = N^k$. By Theorem 2.4, the Corollary holds. □

COROLLARY 2.6. *For any pair $(P_\ell; A)$ with $| I(P_\ell; \ell; A) | = n > 0$, we have*

$$\dim_H(F(I)) = -\log n / \ell \log r.$$

3. Examples

Let F be a self-similar set generated by two similarity maps T_0 and T_1 on $X = [0,1] \times [0,1]$ with common simility ratio $1/2$ and satisfying open set condition, that is, there exists non-empty open set V such that $T_0(V)$ and $T_1(V)$ are disjoint. For this F, consider following two sub-similar sets.

EXAMPLE 3.1. For $Q = \{(0,0,1), (0,1,1), (1,0,1), (1,1,0)\}$, define $P_3 : \{0,1\}^3 \to \mathbb{R}$ by

$$P_3(\tau) = \begin{cases} 1 & \text{if } \tau \in Q \\ 0 & \text{if } \tau \notin Q. \end{cases}$$

Then $I(P_3; 1; \{1\}) = Q$ is not 1-irreducible, but $I(P_3; 2; \{1\}) = Q$ is 2-irreducible precoding space with

$$M(I) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ and } \Lambda_P = 2.$$

So by Theorem 2.4, we have

$$\dim_H\{F(I)\} = -\log 2/\{2\log(1/2)\} = 1/2.$$

EXAMPLE 3.2. In digital communication, we usually use binary digital codes. Then to check if an error occurred during transmission, we encode the given messages by adding parity checking bits as follows: For given $\tau = (i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, \cdots)$, encode it by $\rho = (i_1, i_2, i_3, i_4, j_1, i_5, i_6, i_7, i_8, j_2, i_9, \cdots)$, where

$$j_k = \begin{cases} 1 & \text{if } \sum_{q=1}^{4} i_{4(k-1)+q} \text{ is odd} \\ 0 & \text{if } \sum_{q=1}^{4} i_{4(k-1)+q} \text{ is even.} \end{cases}$$

Now for the map $P_5 : \{0,1\}^5 \to \mathbb{R}$ defined by

$$P_5((i_1, i_2, i_3, i_4, i_5)) = \sum_{k=1}^{5} i_k,$$

$I = I(P_5; 5; \{0,2,4\}) = \{\sigma \in \{0,1\}^5 : P_5(\sigma) = 0, 2, 4\}$ is 5-irreducible with 16 precodes and by Corollary 2.6, the encoding image $F(I) = F\{I(P_5; 5; \{0,2,4\})\}$ has

$$\dim_H\{F(I)\} = -\log 16/\{5\log(1/2)\} = 4/5.$$

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