한국퍼지 및 지능 시스템학회 논문지 1998, Vol. 8, No. 6. (1998) 53~57

A Note on Fuzzy Strong Continuities and Fuzzy Midly Normal Spaces

K. Hur, J. R. Moon, J. H. Ryou and M. J. Ryou

Department of Mathematics, Wonkwang University

ABSTRACT

We study some properties of fuzzy strong continuities. And we introduce the concept of a fuzzy midly normal space and investigate its properties.

1. Introduction

In 1960, N. Levine introduced the concept of strong continuity in ordinary topological spaces and investigated their some properties. Arya and Gupta studied various properties about strong continuities in [2]. In 1961, H.F. Cullen introduced the concept of complete continuity on ordinary topological spaces and investigated their some properties.

The study of fuzzy sets was initiated with the famous paper of Zadeh [15] in 1965, and many authors [1,3,4,6,7,9-14] investigated topological structures in the view of fuzzy set theory.

In particular, Mukherjea and Ghosh introduced the concepts of strong continuity and complete continuity in the view of fuzzy set theory, and investigated their some properties. We study another properties about them. Also, we introduce the concept of fuzzy midly normal space and inverstigate some properties.

2. Preliminaries

Throughout this paper X, Y, Z $\cdots$ etc. will denote fuzzy topological spaces and for a set X, I^X the collection of all fuzzy sets in X, where I=[0, 1]. The definition of fuzzy sets, fuzzy topological spaces and other concepts can be found in [3,4,6,7,9-14]. In particular the standard definition of a fuzzy topological space (fts) is defined by Chang [4].

Now we will list a concept, some notations, and some results used next sections.

Lemma 1.A[6]. Let X and Y be fts's and let f: $X \longrightarrow Y$ be any mapping. Then f is f-open if and only if $f(int\,A) \subset int f(A)$ for each $A \in I^X$

Lemma 1.B[6]. Let f: $X \longrightarrow Y$ be any mapping. Then f is f-closed if and only if $cl f(A) \subset f(clA)$ for each $A \in I^X$.

Definition 1.1[12]. Let X be a fuzzy topological space and let $A \in I^X$. Then

(1) A is said to be a *fuzzy regular open set* in X if $A = int(clA)$.

(2) A is said to be a *fuzzy regular closed set* in X if $A = cl(intA)$.

Notation. For a fts X, we denote the family of all fuzzy open (resp. closed, regular open, regular closed) sets in X as $FO(X)$ (resp. $FC(X)$, $FRO(X)$, $FRC(X)$.

Lemma 1.C[13]. Let $(X, \mathfrak{T})$ be the product fts of the family of fts's $\{(X_\alpha \mathfrak{T}_\alpha): \alpha \in \Lambda\}$.

(1) For each $\alpha \in \Lambda$, the projection π_α is f-continuous.

(2) The product fuzzy topology is the smallest topology for X such that (1) is true.

(3) Let (Y, U) be a fts and let f be a function from Y to X. Then f is f-continuous iff for every $\alpha \in \Lambda$, $\pi_\alpha \circ f$ is f-continuous.

2. Fuzzy Strongly Continuous Mappings

Definition 2.1[9]. A mapping f: $X \longrightarrow Y$ is said to be *fuzzy strongly continuous* if for each $A \in I^X$, $f(clA)$

*This paper was supported by Wonkwang University in 1997.

$\subset f(A)$ i.e., $f(clA)=f(A)$.

It is clear that every fuzzy strongly continuous mapping is f-continuous. However the converse is not true, as shown by Example 2.2.

Example 2.2. Let X be a set and let $\mathfrak{T}$ be the fuzzy indiscrete topology on X. Consider the identity mapping id_X: $(X, \mathfrak{T}) \rightarrow (X, \mathfrak{T})$. Then id_X is f-continuous but not fuzzy strongly continuous.

Remark 2.3. Example 2.2 provides also that even a f-homeomorphism may fail to be fuzzy strongly continuous.

Theorem 2.4. A mapping $f:X \rightarrow Y$ is fuzzy strongly continuous if and only if for each $B\in I^X$, $f^{-1}(B)\in FC(X)\cap FO(X)$.

Proof.$(\Rightarrow)$ Suppose f is fuzzy strongly continuous and let $B\in I^Y$. Then, by the hypothesis, $f(clf^{-1}(B))\subset f(f^{-1}(B))\subset B$. Thus $clf^{-1}(B)\subset f^{-1}(B)$, and hence $f^{-1}(B)\in FC(X)$. By the hypothesis, $f(clf^{-1}\subset(B^c))\subset f(f^{-1}(B^c))\subset B^c$. Thus $clf^{-1}(B^c)\subset f^{-1}(B^c)$, and hence $f^{-1}(B^c)\in FC(X)$. But $f^{-1}(B^c)=[f^{-1}(B)]^c$. So $[f^{-1}(B)]^c\in FC(X)$ and thus $f^{-1}(B)\in FO(X)$. Therefore $f^{-1}(B)\in FC(X)\cap FO(X)$.

$(\Leftarrow)$: Suppose $f^{-1}(B)\in FC(X)\cap FO(X)$ for each $B\in I^Y$. Let $A\in I^X$. Then clearly $f^{-1}(f(A))\in FC(X)$. But $A\subset f^{-1}(f(A))$. So $clA\subset f^{-1}(f(A))$ and thus $f(clA)\subset f(A)$. Hence f is fuzzy strongly continuous.

Corollary 2.4.1. A mapping $f:X \rightarrow Y$ is fuzzy strongly continuous if and only if $f^{-1}(y_\lambda)\in FO(X)\cap FC(X)$ for each $y_\lambda\in F_p(Y)$.

Corollary 2.4.2. For any fts $(X, \mathfrak{T})$, the identity mapping $id:(X, \mathfrak{T}) \rightarrow (X, \mathfrak{T})$ is fuzzy strongly continuous if and only if $\mathfrak{T}$ is the fuzzy discrete topology on X.

Theorem 2.5. Restriction of a fuzzy strongly continuous mapping $f:X \rightarrow Y$ to any subset of X is fuzzy strongly continuous.

Proof. Let A be any subset of X and let $y_\lambda\in F_p(Y)$. Since f is fuzzy strongly continuous, by Corollary 2.5.1, $f^{-1}(y_\lambda)\in FC(X)\cap FO(X)$. But $(f|A)^{-1}(y_\lambda)=f^{-1}(y_\lambda)\cap A$. So $(f|A)^{-1}(y_\lambda)\in FC(A)\cap FO(A)$. Hence $f|A$ is fuzzy strongly continuous.

Theorem 2.A[9]. If $f:X \rightarrow Y$ is fuzzy strongly continuous and $g:Y \rightarrow Z$ any mapping, then $g\circ f:X \rightarrow Z$ is fuzzy strongly continuous and hence the composite of two fuzzy strongly continuous mappings is fuzzy strongly continuous.

Corollary 2.A.1. Let f: $X \rightarrow \Pi_{\alpha\in\Lambda}X_\alpha$ be fuzzy strongly continuous. For each $\alpha\in\Lambda$, let f_α: $X \rightarrow X_\alpha$ be defined as $f_\alpha(x)=x_\alpha$ if $f(x)=(x_\alpha)$. Then f_α is fuzzy strongly continuous for each $\alpha\in\Lambda$.

Proof. Let π_α denoted the projection of $\Pi_{\alpha\in\Lambda}X_\alpha$ onto X_α. Then, obviously $f_\alpha=\pi_\alpha\circ f$ for each $\alpha\in\Lambda$, and, by Lemma 1.C.(1), π_α is f-continuous for each $\alpha\in\Lambda$. Therefore, by Theorem 2.A, f_α is fuzzy strongly continuous for each $\alpha\in\Lambda$.

Corollary 2.A.2. Let $f_1:X_1 \rightarrow Y_1$ and f_2: $X_2 \rightarrow Y_2$ be fuzzy strongly continuous mappings. Let $X=X_1\times X_2$ and $Y=Y_1\times Y_2$. Let $f:X \rightarrow Y$ be defined as $f(x_1, x_2)=(f_1(x_1), f_2(x_2))$. Then f is fuzzy strongly continuous.

The following example shows that the result of Theorem 2.A is not necessarily true for f-continuous mappings.

Example 2.6. Let X be any set and let $\mathfrak{T}$ and $\mathfrak{T}^*$ be the fuzzy indiscrete topology, and the fuzzy discrete topology on X, respectively. Consider the identity mapping $f:(X, \mathfrak{T}) \rightarrow (X, \mathfrak{T})$ and g: $(X, \mathfrak{T}) \rightarrow (X, \mathfrak{T}^*)$. Then clearly f is f-continuous. But $g\circ f$ is not f-continuous.

Definition 2.7[1,3]. A mapping f: $X \rightarrow Y$ is said to be ***fuzzy weakly continuous*** if for each $x_\lambda\in F_p(X)$ and each $H\in FO(X)$ containing $f(x_\lambda)$, there exists a $G\in FO(X)$ containing x_λ such that $f(G)\subset clH$.

Lemma 2.B[1]. A mapping $f:X \rightarrow Y$ is fuzzy weakly continuous if and only if for each $V\in FO(Y)$, $f^{-1}(V)\subset intf^{-1}(clV)$.

Theorem 2.8. Every fuzzy weakly continuous mapping into a fuzzy discrete space is fuzzy strongly continuous.

Proof. Let $f:X \rightarrow Y$ be fuzzy weakly continuous and let Y a fuzzy discrete space. Let $B\in I^Y$. Since Y is a fuzzy discrete space, $B\in FC(Y)\cap FO(Y)$. Since f

is fuzzy weakly continuous, by Lemma 2.B, $f^{-1}(B)\subset intf^{-1}(clB)$. So $f^{-1}(B)\subset intf^{-1}(B)$, and hence $f^{-1}(B)\in FO(X)$. Similarly, we have $f^{-1}(B)\in FC(X)$. Therefore f is fuzzy strongly continuous.

Theorem 2.9. If $f:X\rightarrow Y$ is fuzzy weakly continuous and $g:Y\rightarrow Z$ is fuzzy strongly continuous, then $g\circ f$ is fuzzy strongly continuous.

Proof. Let $A\in I^Z$. Since g is fuzzy strongly continuous, by Theorem 2.5, $g^{-1}(A)\in FO(Y)\cap FC(Y)$. Since f is fuzzy weakly continuous, by Lemma 2. B, $f^{-1}(g^{-1}(A))\subset intf^{-1}(clg^{-1}(A))=intf^{-1}(g^{-1}(A))$. So $f^{-1}(g^{-1}(A))\in FO(X)$ and thus $(g\circ f)^{-1}(A)\in FO(X)$. Similarly, by Lemma 2.B, we have $(g\circ f)^{-1}(A)\in FC(X)$. Hence $g\circ f$ is fuzzy strongly continuous.

Corollary 2.9.1. If $f:X\rightarrow Y$ is f-continuous and g: $Y\rightarrow Z$ fuzzy strongly continuous, then $g\circ f:X\rightarrow Z$ is fuzzy strongly continuous.

Definition 2.10[9]. A mapping $f:X\rightarrow Y$ is said to be *fuzzy completely continuous* if for each $O\in FO(Y)$, $f^{-1}(O)\in FRO(X)$.

Obviously, every fuzzy strongly continuous mapping is fuzzy completely continuous and every fuzzy completely continuous mapping is f-continuous. The converse implications do not hold (See Example 3.16 and Example 3.3 in [9]). We can see some properties of fuzzy complete continuities in [9].

The restriction of a fuzzy completely continuous mapping may fail to be fuzzy completely continuous as shown by the following example.

Example 2.11. Let $(X, \mathfrak{T})$ be the fuzzy topological space. Let $X=\{a, b, c, d\}$, $Y=\{p, q, r\}$ and let $\mathfrak{T}=\{X, \emptyset, O_1, O_2, O_3, O_4\}$ and let $U=\{Y, \emptyset, O_5\}$, where $O_1=\{(a, 0.3), (b, 0.6), (c, 0.2), (d, 0.6)\}$, $O_2=\{(a, 0.6), (b, 0.5), (c, 0.7), (d, 0.5)\}$, $O_3=\{(a, 0.3), (b, 0.5), (c, 0.2), (d, 0.5)\}$, $O_4=\{(a, 0.6), (b, 0.6), (c, 0.7), (d, 0.6)\}$, $O_5=\{(p, 0.6), (q, 0.5), (r, 0.7)\}$.

Let $f:(X, \mathfrak{T})\rightarrow(Y, u)$ be the mapping defined by $f(a)=p$, $f(b)=f(d)=q$, $f(c)=r$. Then f is fuzzy completely continuous. But the restriction of f to the set $\{b, c\}$ is not fuzzy completely continuous.

3. Fuzzy Midly Normal Spaces

We introduce the concept of a fuzzy midly normal space and study its properties.

Definition 3.1[10]. A mapping $f:X\rightarrow Y$ is said to be *fuzzy almost open* if for each $A\in FRO(X)$, $f(A)\in FO(Y)$.

Theorem 3.2. Let a mapping $f:X\rightarrow Y$ be fuzzy almost open, fuzzy completely continuous and surjective. If G : Y → Z is the mapping such that $g\circ f$ is fuzzy completely continuous, then g is f-continuous.

Proof. Let $G\in FO(Z)$. Since $g\circ f$ is fuzzy completely continuous, $(g\circ f)^{-1}(G)\in FRO(X)$. Since f is fuzzy almost open, $f((g\circ f)^{-1}(G))\in FO(Y)$. But $(g\circ f)^{-1}(G)=f^{-1}(g^{-1}(G))$. Since f is surjective, $f((g\circ f)^{-1}(G))=g^{-1}(G)$. So $g^{-1}(G)\in FO(Y)$. Hence g is f-continuous.

Definition 3.3. A fts X is said to be *fuzzy mildly normal* if for each $F_1, F_2\in FRC(X)$ such that $F_1\odot F_2=\emptyset$, there exist $U, V\in FO(X)$ such that $U\odot V=\emptyset$, $F_1\subset U$ and $F_2\subset V$, where $(A\odot B)(x)=\max[0, A(x)+B(x)-1]$ for each $x\in X$.

It is clear that $A\odot B=\emptyset$ if and only if $A\subset B^c$. Hence Definition 3.8 is equivalent to Definition 6.1 in [12].

Definition 3.4[7]. A fts X is said to be *fuzzy normal* if for each $F\in FC(X)$ and each $U\in FO(X)$ such that $F\subset U$, there exists a $V\in I^X$ such that $F\subset intV\subset clV\subset U$.

Theorem 3.5. Let X be a fuzzy mildly normal space and let Y be a fts. If $f:X\rightarrow Y$ is fuzzy completely continuous, fuzzy closed, open and surjective, then Y is fuzzy normal.

Proof. Let $F\in FC(Y)$ and each $U\in FO(Y)$ such that $F\subset U$. Then, by the hypothesis, $f^{-1}(F)\in FRC(X)$, $f^{-1}(U)\in FRO(X)$ and $f^{-1}(F)\subset f^{-1}(U)$. Also, there exist $V\in FRO(X)$ such that $f^{-1}(F)\subset V\subset clV\subset f^{-1}(U)$. But $F\subset f(V)\subset f(clV)\subset U$. By Lemma 1.A and Lemma 1.B, $F\subset intf(V)\subset clf(V)\subset U$. Then Y is fuzzy normal.

Definition 3.6. A mapping $f:X\rightarrow Y$ is said to be *fuzzy almost continuous at a fuzzy point* x_λ in X if for each fuzzy open set V in Y containing $f(x_\lambda)$, there exists a fuzzy open set U in X containing x_λ such that $f(U) \subset int(clV)$. The mapping f is said to be *fuzzy almost continuous* on X if it is fuzzy almost continuous at each fuzzy point x_λ in X.

It is clear that every f-continuous mapping is fuzzy almost continuous. But the converse is not true (See [6]).

Lemma 3.A[6]. For a fsts X, the followings are equivalent;

(1) f is fuzzy almost continuous.

(2) For each $V \in FRO(Y)$, $f^{-1}(V) \in FO(X)$.

(3) For each $C \in FRC(Y)$, $f^{-1}(C) \in FC(X)$.

Theorem 3.7. Let $f: X \to Y$ be fuzzy almost continuous, f-closed, f-open and surjective. If X is fuzzy normal, then Y is fuzzy midly normal.

Proof. Let $A \in FRC(Y)$ and let $B \in FRO(Y)$ containing A. Since f is fuzzy almost continuous, by lemma 3.A, $f^{-1}(A) \in FC(X)$, $f^{-1}(B) \in FO(X)$. Moreover $f^{-1}(A) \subset f^{-1}(B)$. By the hypothesis, there exists $V \in I^X$ such that $f^{-1}(A) \subset intV \subset clV \subset f^{-1}(B)$. Thus $A \subset f(intV) \subset f(clV) \subset B$. Since f is F-closed and F-open. $f(intV) \in FO(Y)$ and $f(clV) \in FC(Y)$. Let $U = f(intV)$. Then clearly $clU = clf(clV) = f(clV)$. So $A \subset U \subset clU \subset B$ and $U \in FO(Y)$. Hence Y is fuzzy midly normal.

From the fact that if $f: X \to Y$ is f-continuous, then f is fuzzy almost continuous, we obtain the following result:

Corollary 3.7.1. Let $f: X \to Y$ be f-continuous, f-closed, f-open and surjective. If X is fuzzy normal, then Y is fuzzy midly normal.

Reference

[1] Y. S. Ahn, Various weaker forms of fuzzy continuous mappings, Dr. Thesis (1996).

[2] S. P. Arya and R. Gupta, On strongly continuous mappings, Kyungpook math. J., **14**(2), pp. 131-141, 1974.

[3] K. K. Azad, On fuzzy semi-continuity, fuzzy almost-continuity and weakly continuity, *J. Math. Anal. Appl.*, pp. 14-32, 1981.

[4] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, **24**, pp. 182-190, 1968.

[5] H. F. Cullen, Complete continuity for functions, Monthly **68**, pp. 165-168, 1961.

[6] K. Hur, J. R. Moon and J. H. Ryou, A note on fuzzy continuities and fuzzy mildly normal space, To appear.

[7] T. Hutton, Normality in fuzzy topological spaces, *J. Math. Anal. Appl.* **50**, pp. 74-79, 1975.

[8] N. Levine, Strong continuity in topological spaces, Monthly **67**, p. 269, 1960.

[9] M. N. Mukherjee and B. Ghosh, Some stronger forms of fuzzy continuous mappings on fuzzy topological spaces, Fuzzy Sets and Systems **38**, pp. 375-387, 1990.

[10] M. N. Mukherjee and S. P. Sinha, Irresolute and almost open functions between fuzzy topological spaces, Fuzzy Sets and Systems **29**, pp. 381-388, 1989.

[11] Pao-Ming Pu and Ying-Ming Liu, Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.* **76**, pp. 571-599, 1977.

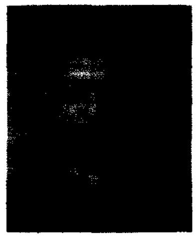
[12] M. K. Singal and Niti Rajvanshi, Regularly open sets in fuzzy topological spaces, Fuzzy Sets and Systems **50**, pp. 343-353, 1992.

[13] C. K. Wong, Fuzzy topology: Product and quotient theorems, *J. Math. Anal. Appl.* **45**, pp. 512-521, 1974.

[14] C. K. Wong, Fuzzy points and local properties of fuzzy topology, *J. Math. Anal. Appl.* **46**, pp. 16-328, 1974.

[15] L. A. Zadeh, Fuzzy Sets, Inform and Control **8**, pp. 316-328, 1965.

허 길(Kul Hur)
1972년 : 연세대학교 수학과 졸업 (학사)
1975년 : 전북대학교 대학원 수학과 졸업(석사)
1988년 : 연세대학교 대학원 수학과 졸업(박사)
1988년~현재 : 원광대학교 수학과 교수
주관심분야 : 위상수학, 초공간론



유 장 현(Ryou Jang Hyun)
1994년 : 대전대학교 수학과 졸업
1996년 : 원광대학교 교육대학원 수학과 졸업
1996년~현재 : 원광대학교 대학원 수학과 박사과정
주관심분야 : 퍼지위상

문 주 란(Joo Ran Moon)
1989년 : 원광대학교 수학과 졸업(이학사)
1991년 : 원광대학교 대학원 수학과 졸업(이학석사)
1995년 : 원광대학교 대학원 수학과 졸업(이학박사)
1993년~현재 : 호원대학교 시간강사
주관심분야 : 퍼지위상, 초공간론

유 미 정(Ryou Mi Jung)
1994년 : 원광대학교 수학과 졸업
1997년 : 원광대학교 교육대학원 수학과 졸업