經營科學 第15卷 第1號
1998年 5月

A Face Optimization Algorithm for Optimizing over the Efficient Set

Dong-Yeop Kim* · Taeho Ahn**

ABSTRACT

In this paper a face optimization algorithm is developed for solving the problem (P) of optimizing a linear function over the set of efficient solutions of a multiple objective linear program. Since the efficient set is in general a nonconvex set, problem (P) can be classified as a global optimization problem. Perhaps due to its inherent difficulty, relatively few attempts have been made to solve problem (P) in spite of the potential benefits which can be obtained by solving problem (P). The algorithm for solving problem (P) is guaranteed to find an exact optimal or almost exact optimal solution for the problem in a finite number of iterations.

1. Introduction

One of the more popular and practical models has been used to help make decisions involving multiple criteria is the multiple objective linear programming problem (MOLP) model. This model can be written

(MOLP) "max" Cx, subject to $x \in X$,

where $X = \{x \mid Ax \leq b\}$ and X is a nonempty, compact polyhedron, C and A are $k \times n$ and $m \times n$ matrices, respectively, and $b \in R^m$.

Usually the most preferred compromise solution in the multiple criteria decision making (MCDM) problem is required to be an efficient (nondominated, Pareto-optimal) solution.

Definition 1

A point $x^0 \in X$ is an efficient solution of problem MOLP if and only if there exists no $x \in X$ such that $Cx \geq Cx^0$ and $Cx \neq Cx^0$.

Let X_E denote the set of all efficient

* 대전전문대 비서행정과
** 숭실대학교 경영학부

solutions of problem MOLP. This problem may be written

(P) max <d,x>, subject to $x \in X_E$, where $d \in R^n$.

In general, when X is nonempty, X_E is a nonconvex set. Therefore problem P is generally a nonconvex programming problem. Such problems generally possess large number of local optima which need not be globally optimal.

Problem P serves several usuful purposes in multiple criteria decision making. First, by optimizing a linear function over the efficient solutions for the problem MOLP, the computational burden of generating the entire efficient set is avoided. Second, the decision maker is not required to choose a preferred solution from a potentially overwhelmingly-large set of efficient solutions. Finally, since problem P can be specialized to solve the problem of minimizing an objective function of problem MOLP over X_E , it aids greatly in several ways. For instance, it helps decision makers to set goals and to rank objective functions, and it yields important improvements in the performance of certain interactive algorithms for problem MOLP [7], [8], [10], [17].

In spite of the potential benefits which can be obtained by optimizing a linear function over the efficient set, relatively few attempts have been made to solve problem P. This is probably at least partially due to the inherent difficulties involved in solving this global optimization problem.

Problem P was first proposed by Philip [13]. He presented an outline of a procedure which uses cutting planes to solve the problem P. This procedure attempts to find a globally optimal solution for problem P. It is based on the fact that the set of efficient points of a polyhedron is connected [14]. However, there are considerable difficulties in implementing this procedure. These difficulties are mainly due to the following two weakness in the algorithm. First, whenever a cutting plane restriction is added, this algorithm requires searching for all extreme points created by the original feasible set X and the added hyperplane. In the algorithm, it is not clearly stated how to implement this search. Second, even for a small problem, we may need to add many cutting plane restrictions. Later, Dessouky, Ghiassi and Davis [8] developed an algorithm for minimizing any criterion of a multiple objective linear program over its set of efficient solutions. Their problem, which is a special case of problem P, is motivated by the desire to determine the ranges of values the criteria take over the efficient set. In a more theoretical vein, Benson [3] has studied properties of the problem P and has examined the nature of an optimal solution and of the optimal solution set. Recently, Benson and Sayin [5] presented a heuristic algorithm for problem P, which sequentially identified individual efficient faces of X and maximizes <d,x> over each face found. They showed that their algorithm gave good estimates of the global optimum of problem P. More recently, Benson and Lee [6] developed an algorithm for optimizing a lower semicontinuous function over the efficient set of a bicriteria linear programming problem.

In this article we present a face optimization algorithm for problem P. The goal of the algorithm is to find an exact optimal or an almost exact optimal solution for problem P with a relatively small computational effort. To accomplish this, the algorithm generates an improved efficient extreme point at each iteration. If the algorithm does not find such an efficient extreme point, it tells this and terminates. To find if the incumbent solution is an ε-optimal, a new result derived from reformulating a bilinear programming problem as a concave minimization problem is introduced in this paper and is used in the algorithm.

This paper is organized as follows. In Section 2 we present the necessary theoretical prerequisites for developing our algorithm for solving problem P. In Section 3 the algorithm for solving problem P is presented. In Section 4 a small example is solved to illustrate the face optimization algorithm and its implementation. Concluding remarks are given in Section 5.

2. Theoretical Background

The algorithm presented in this paper consists of two steps at each iteration. The goal of step 1 is to find an improved efficient extreme point, which maximizes $\langle d,x \rangle$ over the new efficient face found at previous iteration. At step 2, a new efficient point which belongs to a new efficient face is found. When no more new efficient point is found, the algorithm tells this and terminates.

2.1 Theoretical Background of Step 1

One of the more attractive features of problem P is described in the following theorem, which follows immediatedly from [3].

Theorem 1: Problem P has an optimal solution which is an extreme point of X.

The algorithm we shall present for solving problem P will find an improved efficient extreme point at each iteration and terminate with an efficient extreme point.

Let $x^{\circ} \in X$. Consider the following linear program $P_{x^{\circ},\lambda}$.

$$(P_{x^{\circ},\lambda}) \quad \max \ \langle \lambda^T C, x \rangle$$
$$\text{subject to}$$
$$Cx \geq Cx^{\circ} \qquad (1)$$
$$x \in X$$

Step 1 in our algorithm is based on the following two results, theorem 2 and theorem 3. The first result follows easily from [1]. The second result is derived from the first result by using duality theory [5].

THEOREM 2: Let $x^\circ \in X$. Then $x^\circ \in X_E$ if and only if for any $\lambda > 0$, x° is an optimal solution of the linear program $(P_{x^\circ,\lambda})$.

THEOREM 3: Assume that $\lambda^\circ > 0$ and $x^\circ \in X_E$. Let $(u^{\circ T}, w^{\circ T})$ be any optimal solution to the linear programming dual $D_{x^\circ,\lambda^\circ}$ of problem $P_{x^\circ,\lambda^\circ}$, where u° represents the dual variables corresponding to the constraints (1). Then x° belongs to the efficient face $X_{\bar{\lambda}^\circ}$ of X, where $\bar{\lambda}^\circ = u^\circ + \lambda^\circ$ and $X_{\bar{\lambda}^\circ}$ denotes the optimal solution set of the weighted sum problem (P_λ) with $\lambda = \bar{\lambda}^\circ$: $\max < \lambda^T C, x >$ subject to $x \in X$.

The following corollary of Theorem 3 is immediate.

Corollary 1: Assume that $\lambda^\circ > 0$ and $x^\circ \in X_E$. Let $(u^{\circ T}, w^{\circ T})$ be any optimal solution to the linear programming dual $D_{x^\circ,\lambda^\circ}$ of problem $P_{x^\circ,\lambda^\circ}$, where u° represents the dual variables corresponding to the constraints (1). Let $\bar{\lambda}^\circ = u^\circ + \lambda^\circ$, and let $v_o = (\bar{\lambda}^\circ)^T C x^\circ$. Then the efficient face $X_{\bar{\lambda}^\circ}$ of X can be represented as

$$X_{\bar{\lambda}^\circ} = \{x \in X \mid (\bar{\lambda}^\circ)^T Cx = v_o\}$$

From corollary 1, it can be easily seen that the following linear programming $F_{x^\circ,\lambda^\circ}$ finds an efficient extreme solution which maximizes <d,x> over the set of the efficient face $X_{\bar{\lambda}^\circ}$ which contains x°

$$(F_{x^\circ,\lambda^\circ}) \quad \max \quad \langle d,x \rangle$$
$$\text{subject to}$$
$$(\bar{\lambda}^\circ)^T Cx = (\bar{\lambda}^\circ)^T Cx^\circ$$
$$x \in X \ .$$

2.2 Theoretical Background of Step 2

Let v denote the optimal objective function value for problem P.

Consider the function $g: G \to \mathbf{R}$ given by

$$g(x) = \max\ e^T Cy - e^T Cx$$
$$\text{subject to}$$
$$Cy \geq Cx$$
$$y \in X,$$

where $G = \{x \in \mathbf{R}^n \mid Cy \geq Cx \text{ for some } y \in X\}$. Notice that $X \subseteq G$. It can be shown that since X is a compact polyhedron, g is a continuous, concave function on X (see Lemma 6.1 in [2]).

Now let $t \in \mathbf{R}$ and consider the problem (U_t) given by

$$(U_t) \quad \min\ g(x)$$
$$\text{subject to}$$
$$\langle d,x \rangle \geq t$$
$$x \in X.$$

Let v_t denote the optimal objective function value for problem U_t. When the feasible set of the problem U_t is nonempty, since X is compact and g is continuous on X, an optimal solution for U_t exists. Benson [3] derived the following result, which uses problem U_t to characterize optimal solutions for problem P.

Theorem 4: Let $\bar{t}$ denote the largest value of t in problem U_t for which v_t equals zero. Then (a) $v = \bar{t}$, and (b) x° is an optimal solution for problem P if and only if x° is an optimal solution for problem $U_{\bar{t}}$.

Theorem 4 shows that problem U_t and problem P are closely related to one another. By considering the dual linear programming problem to the linear program which defines g, the following corollary is immediate [3].

Corollary 2: $v=\bar{t}$, where $\bar{t}$ is the largest value of t in the problem (V_t) given by

$$(V_t)\ \psi_t = \min -p^TCx + \langle b,u\rangle - \langle e^TC,x\rangle$$

subject to

$$\langle d,x\rangle \geq t$$
$$Ax \leq b$$
$$-p^TC + u^TA \geq e^TC$$
$$x,p,u \geq 0$$

such that $\psi_t = 0$. Furthermore, x° is an optimal solution for problem P if and only if for some $p^\circ \in \mathbf{R}^k$ and $u^\circ \in \mathbf{R}^m$, $(x^\circ, p^\circ, u^\circ)$ is an optimal solution for problem ($V_{\bar{t}}$).

For each t, problem V_t is a nonconvex quadratic programming problem called a bilinear programming problem. It is known that any bilinear programming problem can be converted to a concave minimization problem [15].

To convert problem V_t into a concave minimization problem, first do the following. Let

$$y = \begin{matrix} k \\ m \end{matrix}\begin{bmatrix} p \\ u \end{bmatrix},\quad Q = \begin{matrix} k \\ m \end{matrix}\overset{n}{\begin{bmatrix} -c \\ o \end{bmatrix}},\quad l = \begin{matrix} k \\ m \end{matrix}\begin{bmatrix} o \\ b \end{bmatrix},\quad B = \begin{matrix} k \\ m \end{matrix}\overset{n}{\begin{bmatrix} c \\ -A \end{bmatrix}}.$$

Then, problem V_t can be rewritten as

$$\min\ y^TQx + \langle l,y\rangle - \langle e^TC, x\rangle$$
subject to
$$x \in X_t,\ \ y \in Y,$$

where $X_t=\{x\in X \mid \langle d,x\rangle \geq t\}$ and $Y=\{y\in\mathbf{R}^{k+m} \mid y^TB\leq -e^TC,\ y\geq 0\}$ are nonempty and compact sets.

By using the conversion procedure, the bilinear programming program V_t , for a fixed t, is equivalent to the following concave minimization problem, (P′).

$$(P')\quad \Pi_t = \min f(x) \text{ subject to } x\in X_t$$
$$= \min -\langle e^TC,x\rangle + h(x)$$
$$\text{subject to } x\in X_t,$$
$$\text{where } h(x) = \min \langle y, Qx+l\rangle$$
$$\text{subject to } y\in Y.$$

For problem P′, it can be shown that f and h are continuous, concave functions on R^n. With f, X and Y defined as above, if x° is an optimal solution for problem P′ and y° is an optimal solution for the linear program: $\min \langle y, Qx^\circ+l\rangle$ subject to $y\in Y$, then (x°, y°) is an optimal solution for the problem (V_t) from the equivalence of the two problems. This shows that, for any optimal solution x° for problem P′, there exists a $y^\circ\in Y$ such that (x°, y°) is an optimal solution for problem V_t and $f(x^\circ) = -\langle e^TC,x^\circ\rangle + y^{\circ T}Qx^\circ+\langle l,y^\circ\rangle$.

From this conversion, the following corollary, which would give descriptions of v and of optimal solutions to problem P, can be derived. This is key result used in our algorithm to implement Step 2.

Corollary 3: $v=\bar{t}$, where $\bar{t}$ is the largest value of t in the problem P′ such that $\Pi_t = 0$. Furthermore, x° is an optimal solution for problem P if and only if x° is an optimal solution for problem P′ with $t = \bar{t}$.

The result of Corollary 3 will be used in our algorithm to find a new efficient point which belongs to a new efficient face. Notice that if, for some t, $\Pi_t = 0$, then $v \geq t$. On the other hand, if $\Pi_t > 0$ for some t, then $v < t$.

3. The Algorithm

The algorithm uses a face optimization procedure to find an exact optimal or almost exact optimal solution for problem P in a finite number of iterations. Let $x \in X$ and $e \in \mathbf{R}^k$ is a vector whose entries each equal to one. The algorithm can be described as follows.

Iteration 0.

Choose arbitrary small positive number ε.

Find any efficient point x° by solving the linear program given by

$$\max \langle e^T C, x\rangle$$
$$\text{subject to } Cx \geq C\bar{x}$$
$$x \in X,$$

where $\bar{x}$ is an any feasible solution of MOLP and $e \in \mathbf{R}^k$ is a vector whose entries each equal to one.

At iteration k (k=1,2,..),

Step 1. (Face optimization)

Step 1.1

Solve the linear program $(D_{x^\circ,e})$ given by

$$D_{x^\circ,e} : \min -\langle x^{\circ T} C^T, u\rangle + \langle b, w\rangle$$
$$\text{subject to}$$
$$-C^T u + A^T w = C^T e$$
$$u, w \geq 0.$$

Let $(u^{\circ T}, w^{\circ T})$ be an optimal solution to problem $D_{x^\circ,e}$.

Step 1.2

Solve the linear program $(F_{x^\circ,e})$ given by

$$F_{x^\circ,e} : \max \langle d, x\rangle$$
$$\text{subject to}$$
$$(u^\circ + e)^T Cx = (u^\circ + e)^T Cx^\circ$$
$$x \in X$$

Let x^* be an optimal solution to problem $F_{x^\circ,e}$ and $\langle d, x^*\rangle = t^*$.

Step 2. (Finding a new efficient point which belongs to a new efficient face)

Find an optimal solution $\hat{x}$ for problem P′ with $t = t^* + \varepsilon$ and determine whether $\Pi_t = 0$ or $\Pi_t > 0$. If $\Pi_t = 0$, go to Step 1 by substituting $\hat{x}$ with x°.

If $\Pi_t > 0$, conclude that v is almost equal to t^* and that x^* is an almost exact optimal solution for problem P, and terminates.

Notice that, in Step 1, linear program $D_{x^\circ,e}$ is the dual linear program to $P_{x^\circ,e}$. At each iteration, a newly generated efficient extreme solutions x^*, which maximizes <d,x> over the efficient face containing the point x°, yields an improved value of t^*. Therefore, as iteration goes, the value of t^* would be improved until the algorithm terminates. From the fact that there exists a finite number of efficient extreme solutions, our algorithm must terminate in a finite number of iterations. Step 2 finds either a new efficient point with a new efficient face containing it or terminates the algorithm when no more new efficient point is found. If the algorithm terminates with an efficient extreme solution x^*, v is almost equal to t^* from the following two senses: (i) Problem P has an optimal solution which is an efficient extreme point. (ii) $0 \leq v-t \leq \varepsilon$. In Step 2, it is required to solve the problem P′ with a fixed value of t, which involves the minimization of a concave, continuous function over a polyhedral set. Several algorithms have been proposed for solving problems of this sort, including those of Benson [4], Majthay and Whinston [11], Falk and Hoffman [9], and Tuy [16]. The computational efficiency of our algorithm would heavily depend on the method used to solve Step 2.

4. A Small Example

To illustrate the suggested implementation of the face optimization algorithm, consider a realization of problem P in which

(a) $X = \{(x_1,x_2) \in \mathbf{R}^2 \mid x_2 \leq 4,\ x_1+2x_2 \leq 10,\ 2x_1+x_2 \leq 10,\ -x_1 \leq 0,\ -x_2 \leq 0\}$;

(b) $C=\begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}$;

(c) d = (-1, 2).

The sets X and X_E are shown in Figure 1, where $X_E = \delta(x^2,x^3) \cup \delta(x^3,x^4)$. The maximum value of <d,x> over X equals eight and is achieved at the extreme point x^1, which does not belong to X_E.

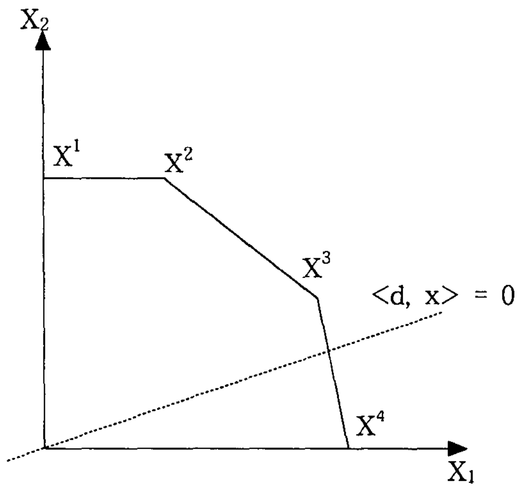

Figure 1. The Sets X and X_E

Iteration 0.

Let $\varepsilon = 0.006$.

Suppose we find an efficient point x° = (4, 2).

At iteration 1,

Step 1.

Step 1.1

$(u_1,u_2,w_1,w_2,w_3,w_4,w_5)$= (0,4/3,0,0,5/3,0,0) is found.

Step 1.2

$x^* = (10/3, 10/3)$ and $<d,x^*> = t^* = 10/3$ are found.

Step 2.

With $t = 3.34$, $\hat{x} = (3.33, 3.335)$ and $\Pi_t = 0$ are found.

Since $\Pi_t = 0$, go to Step 1 with $x^\circ = (3.33, 3.335)$.

At iteration 2,

Step 1.

Step 1.1

$(u_1,u_2,w_1,w_2,w_3,w_4,w_5) = (1/2,0,0,5/2,0,0,0)$ is found.

Step 1.2

$x^* = (2, 4)$ and $<d,x^*> = t^* = 6$ are found.

Step 2.

With $t = 6.006$, $\hat{x} = (1.994, 4)$ and $\Pi_t > 0$ are found. Since $\Pi_t > 0$, conclude that v is almost equal to $t^* = 6$ and that $x^* = (2, 4)$ is an almost exact optimal solution for problem P, and terminates.

5. Concluding Remarks

The problem P of optimizing a linear function over the set of efficient solutions of a multiple objective linear program arises in many practical situations. This paper has presented a face optimization algorithm for finding an optimal solution to problem P. As a special case, the algorithm can be applied to the problem of minimizing any individual criterion of the multiple objective linear program over X .

At each iteration, the algorithm generates an improved efficient extreme point and finds a new efficient point with a new efficient face containing it. When no more new efficient point is found, the algorithm tells this and terminates. The algorithm is guaranteed to find an exact optimal or almost exact optimal solution for problem in a finite number of iterations. Since problem P has many important applications in multiple criteria decision making, these properties of our presented algorithm imply that the algorithm represents a potentially valuable, practical tool for aiding decision makers faced with multiple objective problems.

References

[1] Benson, H. P., "Existence of Efficient Solutions for Vector Maximization Problems," Journal of Optimization Theory and Applications, 26, pp. 569-580, 1978.

[2] Benson, H. P., "Optimization over the Efficient Set," Discussion Paper No.35, Center for Econometrics and Decision Sciences, University of Florida, Gainesville, Fl, 1981.

[3] Benson, H. P., "Optimization over the Efficient Set," Journal of Mathematical

Analysis and Applications, 98, pp. 562-580, 1984.

[4] Benson, H. P., "A Finite Algorithm for Concave Minimization over a Polyhedron," Naval Research Logistics Quarterly, 32, pp. 165-177, 1985.

[5] Benson, H. P., and Sayin, S., "A Face Search Heuristic Algorithm for Optimizing over the Efficient Set," Naval Research Logistics Quarterly, 40, pp. 103-116, 1993.

[6] Benson, H. P., and Lee, D.Y.," Outcome-Based Algorithm for Optimizing over the Efficient Set of a Bicriteria Linear Programming Problem," Journal of Optimization Theory and Applications, 88, pp. 77-105, 1996.

[7] Benson, H. P., Lee, D.Y., and McClure, J.P.,"A Multiple-Objective Linear Programming Model for the Citrus Rootstock Selection Problem in Florida," Journal of Multi-Criteria Decision Analysis, 6, pp. 160-173, 1997.

[8] Dessouky, M.I., Ghiassi, M., and Davis, W.J., "Estimates of the Minimum Nondominated Criterion Values in Multiple-Criteria Decision-Making, Engineering Costs and Production Economics, 10, pp. 95-104, 1986.

[9] Falk, J. E., and Hoffman, K.R., "A Successive Underestimation Method for Concave Minimization Problems," Mathematics of Operations Research, 1, pp. 251-259, 1976.

[10] Isermann, H., and Steuer, R.E., "Computational Experience Concerning Payoff Tables and Minimum Criterion Values over the Efficient Set." European Journal of Operational Research, 33, pp. 91-97, 1987.

[11] Majthay, A., and Whinston, A., "Quasiconcave Minimization Subject to Linear Constraints," Discrete Mathematics, 9, pp. 35-59, 1974.

[12] Murty, K.G., Linear Programming, Wiley, New York, 1983.

[13] Philips, J., "Algorithms for the Vector Maximization Problem," Mathematical Programming, 2, pp. 207-229, 1972.

[14] Steuer, R.E., Multiple Criteria Optimization: Theory, Computation, and Application, Wiley, New York, 1986.

[15] Thieu, T. V., "Relationship between Bilinear Programming and Concave Minimization under Linear Constraints," Acta Mathematica Vietnamica, 5, pp. 106-113, 1978.

[16] Tuy, H., "Concave Programming Under Linear Constraints," Soviet Mathematics, 5, pp. 1437-1440, 1964.

[17] Weistroffer, H.R., "Careful Usage of Pessimistic Values is Needed in Multiple Objectives Optimization," Operations Research Letters, 4, pp. 23-25, 1985.