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An Improved Interactive Method for the Multi-Objective Linear Programming Problem Based on the Maximally Changeable Dominance Cone

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Abstract

This work has improved the method of Kim and Gal's [6] in that of requiring less response of the DM(decision maker) and ease of reply. The underlying notion is the MCDC(maximally changeable dominance cone) for describing all efficient solutions under the particular preference structure. According to the DM's partial preference expression, enlarging the MCDC is achieved, which results in reducing the solutions needed to take into consideration. The cone generators corresponding to the DM's response are added to the MCDC, which results the MCDC is enlarged. Adopting the scheme of pairwise comparison as a means of acquiring preference attitude, an improved interactive method is proposed. And also, a scheme of choosing a reference point is suggested to achieve the computational efficiency.

1. Introduction

The multi-objective linear programming problem(MOLPP) is of the type

$$\begin{aligned} \text{Max} \quad & z_1(x) = c^1 x \\ \text{Max} \quad & z_2(x) = c^2 x \\ & \cdots\cdots \\ \text{Max} \quad & z_k(x) = c^k x \\ \text{s.t.} \quad & x \in X \end{aligned} \tag{1}$$

where $X = \{x \in \mathcal{R}^n \mid Ax = b,\ x \geq 0\}$, $b \in \mathcal{R}^m$ and A is an $m \times n$ matrix. The problem (1) would be converted to a usual optimization problem as follows :

$$\text{Max}_{x \in X} \ U(z_1(x), \ldots, z_k(x)), \tag{2}$$

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where $U(\cdot)$ is a utility function. In real situations, major difficulties in solving (2) arise from the fact that the decision maker's (DM's) preference should be incorporated into the solution process, because the DM does not express his/her utility(preference) function exactly. But, the full information on the utility function is not necessary in order to solve the problem (2).

Many approaches and methods have been proposed to solve the MOLPP. Among them, several works have employed the interactive methods combining a mathematical programming technique of known efficiency with step-by-step questions to the DM [2, 6, 7, 15]. One of them is the work in [6] which applies the notion of maximally changeable dominance cone(MCDC) adopting the marginal rate of substitution(MRS) as a means of obtaining the partial preference information. But the work had a major shortcoming in interacting with the DM for assessing preference structure; the MRS has been turned to be a more or less impractical preference assessment measure.

In this work, it is suggested an improved method of the work[6] in view of requiring less responses from the DM and ease of reply. The underlying notion of the suggested method is the cone dominance, especially the maximally changeable dominance cone which properly describes all efficient solutions under the particular preference structure. Enhancing on enlargement of the MCDC will lighten the DM's burden based on the partial preference information. Through interactions with the DM, some cone generators corresponding to responses is added to the set of the dominance cone generators. For achieving the interaction efficiency, a suggestion is also made on how to choose the reference point.

2. Theoretical prerequisites

The objective space of the problem (1) is given by

$$Z=\{z(x) \mid x\in X\}$$

where $z(x)=(c^1x,\ c^2x,\ \ldots,\ c^kx)$. For two distinct $z^1,\ z^2\in Z$, z^2 is expressed as $z^2=z^1+d$ with $d=z^2-z^1$. If z^1 is preferred to z^2, written as $z^1 \succ z^2$, it is considered that this preference relation is occurred because of d. Assume the nonzero d has an additional property that, if $z=z^1+\lambda d$, $\lambda>0$, then $z^1 \succ z$. Then d is called a domination factor for z^1.

Let $D(z)$ be the set of all domination factors for z together with $0\in\mathcal{R}^k$. The family $\{D(z) \mid z\in Z\}$ is called the domination structure. In particular, if the set $D(z)$ is a convex cone, $D(z)$ is called a domination cone.

Definition

Given Z and $D(\cdot)$, a point $z^1 \in Z$ is an efficient solution with respect to $D(\cdot)$ if and only if there exists no $z^2 \in Z$ with $z^1 \neq z^2$ such that $z^1 \in z^2 + D(z^2)$. The set of all efficient solutions in the objective space under $D(\cdot)$ is denoted by $Eff[Z \mid D(\cdot)]$.

Pareto preference is based on the concept "more is better" for each objective, and that no other information about tradeoff between objectives is established or available. In other words, $z^1 > z^2$ if and only if $z^1 \geq z^2$ with $z^1 \neq z^2$. The domination cone with respect to Pareto preference is defined as

$$D(z) = \wedge^{\leq} = \{d \in \mathcal{R}^k \mid d \leq 0, \text{ and } d \neq 0\},$$

and so $z^1 > z^2$ if and only if $z^2 \in z^1 + \wedge^{\leq}$. The efficient solution under $\wedge^{\leq}$ is often called *a Pareto－efficient solution*. With respect to the Pareto cone, the set of efficient solutions is

$$Eff[Z \mid \wedge^{\leq}] = \{z \in \mathcal{R}^k \mid Z \cap (z + \wedge^{\leq}) = \phi, \text{ for } z \in Z\}.$$

In Kim and Gal's work [6], it has been proved that there exists at least one positive outer normal vector corresponding to a Pareto-efficient facet, moreover the set of Pareto-efficient solutions is defined by the union of all maximal Pareto-efficient facets. Consider a polyhedral cone

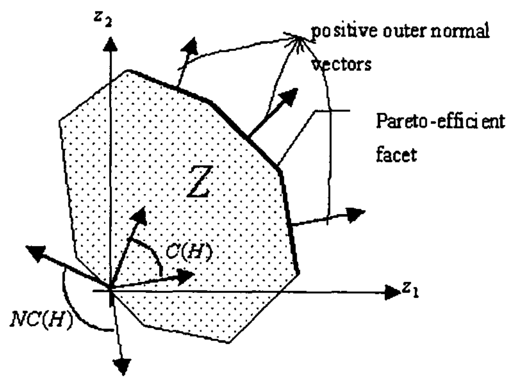
$$C(H) = \left\{ \sum_{i=1}^{l} \alpha_i \ h^i \mid \alpha_i \geq 0 \right\}$$

with a set $H = \{h^1, \ldots, h^l\}$ of essential generators of all maximal Pareto-efficient facets, then the domination cone $D(z) = NC(H)$ has the following relation (see [6] for proof) :

$$Eff[Z \mid \wedge^{\leq}] = Eff[Z \mid NC(H)] \quad \text{with } \wedge^{\leq} \subset NC(H),$$

wher $NC(H) = \{ d \in \mathcal{R}^k \mid dh^i < 0, \text{ for all } h^i \in C(H)/\{0\}\}$, the strictly negative polar cone of $C(H)$. Hence the $NC(H)$ is the greatest domination cone which does not affect the Pareto-efficiency (see Figure 1).

As the DM's preference information is obtained step by step, domination relation can be defined among Pareto-efficient solutions with respect to partial preference structure, which achieved by enlarging the cone $NC(H)$. In other words, the cone $NC(H)$ will be enlarged in order to encompass all



[Figure 1] The maximally changeable dominance cone partial information from the DM through adding some of cone generators corresponding to the DM's partial preference expression. Then the cone $NC(H)$ maximally describes the efficient solutions under the DM's partial preference structure, and hence called a maximally changeable dominance cone(MCDC). Gathering and synthesizing the partial preference information from the DM, the MCDC becomes larger whereas the set of efficient solutions gets smaller :

$$Eff[Z \mid NC(H_1)] \supset Eff[Z \mid NC(H_2)]$$

where $NC(H_1) \subset NC(H_2)$

With a proper scheme for describing a partial preference, one can gradually enlarge a MCDC and finally find out the most preferred extreme point(MPEP). For a distinction between a current and enlarged cone, $NC(H_1)$ and $NC(H_2)$ denote as the current and enlarged MCDC respectively, with relation $NC(H_1) \subset NC(H_2)$. An efficient solution z^1 under $NC(H_1)$ satisfying $(z^1 - NC(H_2)) \cap Z = \phi$ is called *refined-efficient* with respect to $NC(H_2)$. Then the following theorem holds.

Theorem 1

If an efficient extreme point z^1 under $NC(H_1)$ is not refined-efficient with respect to $NC(H_2)$, its positive outer normal vector(s) of the neighboring maximally efficient facets of z^1 is(are) not contained in $C(H_2)$.

Proof:

If z^1 is not a refined-efficient extreme point with respect to $NC(H_2)$, then $(z^1 - NC(H_2)) \cap Z \neq \phi$. Define a set S_0 as $S_0 = (z^1 - NC(H_2)) \cap Z$ and another set $S_0' = (z^1 - NC(H_2))/S_0$. Obviously,

$$(z^1 - NC(H_2)) \supset S_0', \text{ which induces}$$
$$-(z^1 - NC(H_2)) \subset -S_0'.$$

From [13, p.177],

$-(z^1 - NC(H_2)) \subset -S_0'$ implies

$C(H_2) \subset C(-S_0')$, where $C(-S_0')^*$

$= \{ d \in \mathcal{R}^k \mid dh \leq 0, \text{ for all } h \in (-S_0') \}$. (i)

With respect to all positive vectors belongs to S_0 at z^1 of the convex polyhedral set Z, the polar cone $C(S)^* = \{ d \in \mathcal{R}^k \mid dh \leq 0,$ for all $h \in S_0 \subset Z \}$. Therefore,

$C(-S_0)^* \cap C(-S_0')^* = \phi$, since $S_0 \cap S_0' = \phi$. (ii)

From (i) and (ii), $C(-S_0)^* \cap C(H_2) = \phi$. (Q.E.D.)

Corollary 1

If a refined-efficient extreme point with respect to $NC(H_2)$ has more than two positive outer normal vectors, then at least one of them is contained in $C(H_2)$.

Proof : It is obvious.

Denote the set of positive outer normal vectors of neighboring maximally efficient facets of an efficient extreme point $z^1 \in Z$ under $NC(H_1)$ by $\partial(z^1)=\{g^1, g^2, \ldots, g^s\}$, where $g^j \in \mathcal{R}^k$, and the essential generators of a new $C(H_2)$ by $H_2=\{h^1, h^2, \ldots, h^t\}$. Then the next corollary follows immediately.

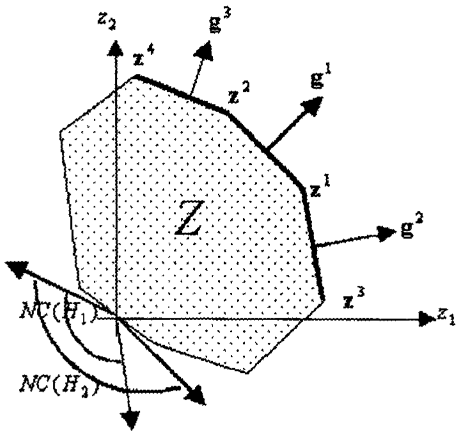
Corollary 2

If at least one g^j satisfies $g^j=\sum_i \lambda_i h_i$, $\lambda_i \ge 0$, for all $i=1, \ldots, t$ and $j=1, \ldots, s$, then an efficient extreme point z^1 is refined-efficient with respect to $NC(H_2)$.

3. Representation of partial preference using the maximally changeable dominance cone

An essential feature of the method described in this work is to be incorporated the DM's judgement on pairwise comparisons into the solution process. All inefficient solutions with respect to an MCDC are to be eliminated from consideration and never presented to the DM, because the MCDC reflects the DM's partial preference structure.

Consider a simple two objectives problem. Let $z^1 \in Eff[Z \mid NC(H_1)]$ be a point having $\partial(z^1)=\{g^1, g^2\}$ where $\partial(z^1)$ is the set of positive outer normal vectors neighboring maximally efficient facets of z^1, and z^2 be an adjacent efficient extreme point corresponding to one of $\partial(z^1)$(e.g. g^1 in Figure 2). And for pairwise inquiry between z^1 and z^2, suppose $z^1 > z^2$. Then we conclude that $z^2 > z^4$ for pseudo-concave utility function. The positive outer normal vectors of z^4 can be no longer the generator of $C(H_2)$ while it was a generator of $C(H_1)$. Moreover $d=z^2-z^1$ becomes one of essential generators of the new $C(H_2)$, having property $dg^{12}=0$.



[Figure 2] Enlargement of the maximally changeable dominance cone

Theorem 2

Suppose $U(z^1) > U(z^2)$ where $z^1 \in Z$ and

$z^2 \in Z$ are adjacent extreme points under $NC(H_1)$, and $U(\cdot)$ is differentiable and pseudo-concave defined on $\mathcal{R}^k$. Let $g^i \in \mathcal{R}^k$, be a positive outer normal vector of neighboring maximally efficient facets of z^2 from z^1. Then $d = z^2 - z^1$, having property $dg^i = 0$, becomes an essential generator of the new $NC(H_2)$.

Proof :

Since $U(\cdot)$ is pseudo-concave and differentiable,

$$\nabla U(z^1)d \le 0. \quad \text{(iii)}$$

To satisfy relation (iii), by definition of $C(H_2)$, g^i is to be an essential generator of $C(H_2)$. Then d becomes an essential generator of $NC(H_2)$ reflecting this partial preference information. (Q.E.D.)

With help of Theorem 2, an MCDC can be enlarged according to the DM's partial preference information. In addition, we state the condition of the MPEP in the next Theorem.

Theorem 3

Consider an efficient extreme point $z^* \in Z$ with its neighboring maximally efficient facets $\partial(z^*)$ and their corresponding efficient extreme points $\{z^1, z^2, \ldots, z^s\}$ under $NC(H_1)$. If z^* is preferred to all of its adjacent extreme points, then z^* becomes the MPEP.

Proof :

By assumption $\nabla_{z^i} U(z) \le 0$ where $\nabla_{z^i} U(z)$ is the directional derivative of $U(z)$ in direction of z^i. Consider a feasible direction d where $d = \sum_i \lambda_i z^i$, for $\lambda_i \ge 0$. Then

$$\nabla_{z^i} U(z) = \nabla U(z)d = \sum_i \lambda_i (\nabla U(z) z^i) \le 0.$$

Since $U(\cdot)$ is pseudo-concave and Z is a convex polyhedral cone, z^* is the MPEP. (Q.E.D.)

The pairwise comparison between an efficient point and its adjacent efficient extreme points leads to the MPEP. In pairwise comparison, selection of an appropriate reference point being efficient effectively enlarges the MCDC, and so help the DM find out the MPEP with less effort.

4. An interactive algorithm

Based on the developed theory, we develop an interactive algorithm in a step-by-step manner, interspersing comments and explanations between each steps.

Step 0 *Determine the Pareto-efficient facets of which outer normal vectors are positive. And build an initial maximally changeable dominance cone NC(H) using the minimal spanning system H of the Pareto-efficient* *facets.*

All the Pareto-efficient points can become the MPEP without preference information, so should be treated as a candidate at first. The Pareto-efficient points are described by the set of efficient facets of which outer normal vectors are positive.

Step 1 *Under NC(H), choose a reference point z^* for the purpose of pairwise comparisons*

It is helpful that a reference point is chosen such as pairwise comparison with it can eliminate the non-MPEPs as many as possible without losing the true MPEP. One possible way is to choose a point z^* which is near as possible from a so-called ideal point regarding $NC(H)$:

Find w such that

$\text{Min} \sum_i | w - h^i |$, *where* $h^i \in H$ for all i

and solve

Max wz

$$\text{s.t. } z \in Z, \text{ where } w \in \mathfrak{R}^k \quad (3)$$

The optimal solution of (3) which is efficient with respect to $NC(H)$ is served as a reference point $z^* \in Z$ for pairwise comparison.

Step 2 *Identify all adjacent efficient facets from z^* and corresponding adjacent efficient extreme points under NC(H).*

To find all adjacent efficient facets under $NC(H)$, we use the existing procedure[6]. Intuitively, a variation of linear programming is used to determine whether given a reference point z^*. And a $w \in C(H)$ can be found so that one non-basic variable is attractive for entry into the basis, whereas none of the other non-basic variables is attractive for entry. If such a vector can be found, its associated facet is efficient, and otherwise not. To find adjacent extreme points along that facet, we need only determine the level at which a variable corresponding to an efficient facet enters the basis.

Step 3 *Ask the DM to choose the preferred one between z^* and a distinctly different adjacent efficient extreme point.*

The purpose of this step is to identify the DM's partial preference structure through pairwise comparisons.

Step 4 *If there exists at least one extreme point preferred to the reference point, then add an essential generators to NC(H) corresponding to the DM's responses, and return to Step 1. Otherwise, go to next step.*

If the DM prefers z^* to an adjacent z^i, we generate a ray $d = z^i - z^*$ and add it to $NC(H)$ as an essential generator. Similarly, preferring z^i to z^*, a ray $d = z^* - z^i$

is added to *NC(H)* as an essential generator. When all of adjacent efficient points are less preferred to z^*, we find out the MPEP according to the DM's partial preference structure. Otherwise, we should regenerate a reference point and repeat the previous steps, because there exist points on a certain facet superior to the reference point.

Step 5 *Stop with the current reference point as the MPEP*

For pseudo-concave utility function., the MPEP is the reference point itself as proven on theorem 3.

4. An Example

As an illustration, consider the following three-objective MOLPP in the objective space.

$$\text{Max} \ \{z_1, z_2, z_3\}$$

$$\text{s.t.} \quad z \in Z$$

The feasible region Z in objective space has been made for the graphical representation as shown in Figure 3. The feasible objective space Z is a convex set composed of 8 constraints.

The set of Pareto-efficient facets of which outer normal vectors are positive is determined as shown in Table 1 with the set of efficient extreme points. The essential generators of initial MCDC are composed of $\{g^A, g^C, g^D, g^E\}$. With this MCDC, a reference point is selected as z^6(i.e. $z^* = z^6$). The adjacent extreme point of z^6 is z^4, z^5, z^9. Assume the preference relations between the reference point ($z^* = z^6$) and its adjacent extreme points are as follows :

$$z^9 < z^6, \ z^5 < z^6, \ z^4 > z^6.$$

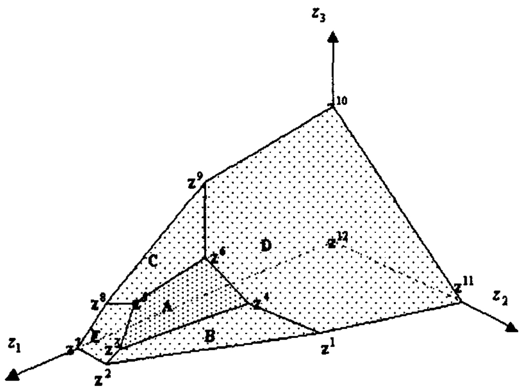
Then the MCDC under such partial preference has the following efficient facets as essential generators :

$$\{g^A, \ g^B, \ g^E\}.$$

Another new reference point is selected as z^4 based on this MCDC. Assume that the preference relations between the reference point and its adjacent extreme points are as follows :

$$z^1 < z^4, \ z^3 < z^4.$$

Hence the most preferred extreme point becomes z^4.



[Figure 3] The feasible objective space

〈Table 1〉 The maximally efficient extreme points

z^1 : (2.73, 2.73, 0.00)	z^2 : (5.56, 0.37, 0.00)
z^3 : (5.33, 0.44, 0.22)	z^4 : (3.75, 1.50, 0.75)
z^5 : (5.00, 0.33, 0.67)	z^6 : (3.88, 0.71, 1.41)
z^7 : (5.71, 0.00, 0.00)	z^8 : (5.00, 0.00, 0.83)
z^9 : (3.33, 0.00, 2.22)	z^{10} : (0.00, 0.00, 3.33)
z^{11} : (0.00, 3.75, 0.00)	z^{12} : (0.00, 0.00, 0.00)

〈Table 2〉 The maximally efficient facets

2-dim. facets	1-dim. facets
g^A, g^B, g^C, g^D, g^E	$g^{1,2}$, $g^{1,4,}$, $g^{1,11}$, $g^{2,3}$, $g^{2,7}$, $g^{3,4}$, $g^{3,5}$, $g^{4,6}$, $g^{5,6}$, $g^{5,8}$, $g^{6,9}$, $g^{7,8}$, $g^{8,9}$, $g^{9,10}$, $g^{10,11}$

5. Conclusion

This study focuses on the practical utilization of the concept of the maximally changeable dominance cone and improvement of extracting preference information with easier way than earlier work. This algorithm shows how to reflect the DM's responses to cone structure. Comparing with the existing method [6], this algorithm starts with a set of Pareto-efficient extreme points near at the MPEP, so finds out the MPEP with less effort. It is worthwhile to expand the notion of the maximally changeable dominance cone into the multi-objective non-linear programming problem. We are developing the concerned theory and a DSS for the problem.

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