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OPERATORS ON GENERALIZED BLOCH SPACE

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ABSTRACT. In [5], Zhu introduces a bounded operator T from $L^\infty(D)$ into Bloch space $\mathcal{B}$. In this paper, we will consider the generalized Bloch spaces $\mathcal{B}_q$ and find bounded operator from $L^\infty(D)$ into $\mathcal{B}_q$.

1. Introduction

Let $\mathbb{C}$ be the complex number plane and $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk in $\mathbb{C}$. Let $dA(z)$ be the area measure on D normalized so that the area is 1. For $1 \leq p < \infty$, $L^p(D, dA)$ will denote the Banach space of Lebesgue measurable functions f on D with

$$[\int_D |f(z)|^p dA(z)]^{\frac{1}{p}} \leq \infty.$$

$L^\infty(D, dA)$ will denote the Banach space of Lebesgue measurable functions f on D with

$$esssup\{|f(z)| : z \in D\} < \infty.$$

The Bloch space of D, denoted by $\mathcal{B}$, consists of analytic functions f on D such that $\sup\{(1 - |z|^2)|f'(z)| : z \in D\} < \infty$. The set $\mathcal{B}$ of Bloch functions (modulo constant functions) become a Banach space ([1],p.13). In [5], Zhu show that the integral operator T which is represented by

$$Tf(z) = \int_D \frac{f(w)}{(1 - z\overline{w})^2} dA(w)$$

is a bounded operator from $L^\infty(D)$ into $\mathcal{B}$.

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For each $q > 0$, the space $\mathcal{B}_q$ consist of analytic functions f on D with the property that

$$\sup\{(1-|z|^2)^q|f'(z)| : z \in D\} < \infty.$$

For each $q > 0$, let T_q denote the operator defined by

$$T_q f(z) = q\int_D \frac{f(w)}{(1-z\overline{w})^{1+q}}dA(w), \quad z \in D.$$

In this paper, we will show that generalized Bloch spaces $\mathcal{B}_q$ are Banach spaces. Also we will investigate some properties of T_q. In particular, we will show that T_q is a bounded operator from $L^\infty(D)$ into $\mathcal{B}_q$.

2. $\mathcal{B}_q$ is a Banach space

Let us define a norm on $\mathcal{B}_q$ as follows;

$$\| f \|_q = |f(0)| + \sup\{(1-|z|^2)^q|f'(z)| \mid z \in D\}.$$

Lemma 1. *If* $f \in \mathcal{B}_q, \quad q > 0$, *then*

$$|f(z)| \le |f(0)| + \| f \|_q (1-|z|^2)^{-q}.$$

Proof.

$$\begin{aligned}|f(z)-f(0)| &\le \int_0^1 |f'(tz)||z|dt \\ &\le \int_0^1 \frac{|f'(tz)|(1-|tz|^2)^q}{(1-|tz|^2)^q}dt \\ &\le \| f \|_q \int_0^1 \frac{1}{(1-t|z|^2)^q}dt \\ &\le \| f \|_q \frac{1}{(1-|z|^2)^q},\end{aligned}$$

since the first inequality follows from the followings

$$f(z)-f(0) = \int_0^1 f'(tz)zdt.$$

Thus the desired result follows. □

Theorem 1. *For each $q > 0$, $\mathcal{B}_q$ is a Banach space with norm $\|\|_q$.*

Proof. Let (f_n) be a Cauchy sequence in $\mathcal{B}_q$. By Lemma 1,

$$|(f_n - f_m)(z) - (f_n - f_m)(0)| \leq \| f_n - f_m \|_q (1 - |z|^2)^{-q}.$$

It follows that the sequence (f_n) is a Cauchy sequence in the topology of uniform convergence on compact sets. Thus there exists holomorphic function $f : D \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of D as $n \to \infty$.

Since $f_n \to f$ uniformly on compact subsets of D as $n \to \infty$, it follows that $f_n'(z) \to f'(z)$ uniformly on compact subsets of D as $n \to \infty$.

Thus, for each n

$$(1 - |z|^2)^q |(f_n - f_m)'(z)| \quad \to (1 - |z|^2)^q |(f_n - f)'(z)| \quad as \quad m \to \infty$$

for each $z \in D$. Therefore, for each sufficiently large n,

$$(1 - |z|^2)^q |(f_n - f)'(z)| \leq \epsilon.$$

Namely, $\| f_n - f \|_q \leq \epsilon$. □

3. Operator T_q on $L^\infty(D)$

In the sequal, $C_0(D)$ is the space of complex -valued continuous functions on D which vanish on the boundary.

Theorem 2. *If P is a polynomial, then there exists f in $C_0(D)$ such that $P = T_q f$.*

Proof. It suffices to show that $T_q g(z) = z^n$ for some $g \in C_0(D)$. In fact, if we consider the function $g(z) = (1 - |z|^2)^2 z^n$, then

$$\begin{aligned}
T_q g(z) &= q \int_D \frac{(1 - |w|^2) w^n}{(1 - z\overline{w})^{1+q}} dA(w) \\
&= q \int_D (1 - |w|^2) w^n \sum_{k=0}^{\infty} \frac{\Gamma(k + 1 + q)}{k! \Gamma(1 + q)} (z\overline{w})^k dA(w) \\
&= q \sum_{k=0}^{\infty} z^k \frac{\Gamma(k + 1 + q)}{k! \Gamma(1 + q)} \int_0^{2\pi} \int_0^1 (1 - r^2)^2 r^{n+k+1} e^{i(n-k)\theta} dr d\theta \\
&= q z^n \frac{\Gamma(n + 1 + q)}{n! \Gamma(1 + q)} \int_0^1 (1 - r^2)^2 r^{2n+1} dr \\
&= q \frac{\Gamma(n + 1 + q)}{n! \Gamma(1 + q)} \frac{1}{(n + 1)(n + 2)(n + 3)} z^n.
\end{aligned}$$

In third equality, if $n \neq k$, $\int_0^{2\pi}\int_0^1 (1-r^2)^2 r^{n+k+1} e^{i(n-k)\theta} dr d\theta = 0$. Thus the desired result follows. □

Theorem 3. *For each $q > 0$, the operator T_q maps each function of the form $z^n \overline{z}^m$ to a monomial where n and m are positive integers such that $n \geq m$.*

Proof.

$$\begin{aligned} T_q(z^n\overline{z}^m) &= q\int_D \frac{w^n\overline{w}^m}{(1-z\overline{w})^{1+q}} dA(w) \\ &= q\int_D w^n\overline{w}^m \sum_{k=0}^{\infty} \frac{\Gamma(k+1+q)}{k!\Gamma(1+q)} z^k \overline{w}^k dA(w) \\ &= q\sum_{k=0}^{\infty} \frac{\Gamma(k+1+q)}{k!\Gamma(1+q)} z^k \int_D w^n \overline{w}^{m+k} dA(w) \\ &= q\frac{\Gamma(n-m+1+q)}{(n-m)!\Gamma(1+q)} z^{n-m} \int_D w^n\overline{w}^n dA(w) \\ &= q\frac{\Gamma(n-m+1+q)}{(n-m)!\Gamma(1+q)} \frac{1}{(n+1)(n+2)(n+3)} z^{n-m}. \end{aligned}$$

Where, the fourth equality follows from the proof of Theorem 2. □

Lemma 2[13, p. 17]. *For $s > -1$ and $t \in R$, let*

$$I_{s,t}(z) = \int_D \frac{(1-|w|^2)^s}{|1-z\overline{w}|^{2+s+t}} dA(w), \quad z \in D$$

then we have

(1)$I_{s,t}(z)$ is bounded in z if $t < 0$;
(2) $I_{s,t}(z) \sim -log(1-|z|^2)$ as $|z| \to 1^-$ if $t = 0$;
(3) $I_{s,t}(z) \sim (1-|z|^2)^{-t}$ as $|z| \to 1^-$ if $t > 0$;

Theorem 4. *For each $q > 0$, the operator T_q maps $L^\infty(D)$ boundedly into $\mathcal{B}_q$.*

proof. For every g in $L^\infty(D)$,

$$T_q g(z) = q\int_D \frac{g(w)}{(1-z\overline{w})^{1+q}} dA(w).$$

$$\frac{d}{dz}(T_q g(z)) = q(q+1)\int_D \frac{\overline{w}g(w)}{(1-z\overline{w})^{2+q}} dA(w).$$

By Lemma 2,

$$\begin{aligned}|\frac{d}{dz}(T_q g(z))| &\leq q(q+1)||g||_\infty \int_D \frac{dA(w)}{|1-z\overline{w}|^{2+q}} \\ &\leq C||g||_\infty (1-|z|^2)^{-q}\end{aligned}$$

for some constant $C > 0$. Since

$$|T_q g(0)| \leq q| \int_D g(w) dA(w)| \leq q \parallel g \parallel_\infty,$$

we obtain the following desired result

$$||T_q g||_q \leq (C+q)||g||_\infty. \qquad \square$$

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