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A NOTE ON MINIMAL SETS OF THE CIRCLE MAPS

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ABSTRACT. For continuous maps f of the circle to itself, we show that (1) every ω-limit point is recurrent (or almost periodic) if and only if every ω-limit set is minimal, (2) every ω-limit set is almost periodic, then every ω-limit set contains only one minimal set.

1. Introduction

Let I be the unit interval, S^1 the circle and X a topological space. And let $C^0(X,X)$ denote the set of continuous maps from X into itself.

Let $f \in C^0(X,X)$. For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^{n+1} = f \circ f^n$. Let f^0 denote the identity map of X.

For any $f \in C^0(X,X)$, let $P(f), AP(f), R(f), \Lambda(f)$ and $\Omega(f)$ denote the collection of the periodic points, almost periodic points, recurrent points, ω-limit points and nonwandering points of f, respectively.

$Y \subset X$ is called an invariant subset of f if $f(Y) \subset Y$; and strongly invariant if $f(Y) = Y$. Suppose $Y \subset X$ is non-void, closed, and invariant relative to f. If Y has no peoper subset which is non-void and invariant relative to f then Y is said to be a minimal set of f.

In 1986, J.C.Xiong [5] proved that for any interval map f, every ω-limit point is recurrent (or almost periodic) if and only if every ω-limit set is minimal. We have the same result for map of the circle.

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Theorem 5. *Let $f \in C^0(S^1, S^1)$. Then the followings are equivalent.*
(1) $\Lambda(f) = AP(f)$.
(2) $\Lambda(f) = R(f)$.
(3) For every $x \in S^1$, the ω-limit set $\omega(x, f)$ of x is minimal.

In 1966, A.N.Sarkovskii [3] showed that for any interval map f, the following conditions (i) and (ii) are equivalent.

(i) The periods of all periodic points of f are powers of 2.
(ii) For every $x \in I$ either the ω-limit set $\omega(x)$ of x is a periodic orbit of f or the set $\omega(x)$ contains no periodic orbit of f.

In [5], J.C.Xiong showed that the condition (i) is equivalent to the following condition (iii).

(iii) For every point $x \in I$, the ω-limit set $\omega(x, f)$ of x contains only one minimal set.

In this paper, we will prove the following theorem.

Theorem 8. *Let $f \in C^0(S^1, S^1)$. Suppose that $\Lambda(f) = AP(f)$. Then we have that for any $x \in S^1$, the ω-limit set $\omega(x, f)$ of x contains only one minimal set.*

2. Preliminaries and definitions

Let (X, d) be a metric space and $f \in C^0(X, X)$. The *forward orbit* $O(x)$ of $x \in X$ is the set $\{f^k(x) \mid k = 1, 2, \cdots\}$.

A point $x \in X$ is called a *periodic point* of f if for some positive integer n, $f^n(x) = x$. The period of x is the least such integer n. We denote the set of periodic points of f by $P(f)$.

A point $x \in X$ is called a *recurrent point* of f if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent points of f by $R(f)$.

A point $x \in X$ is called a *nonwandering point* of f if for every neighborhood U of x, there exists a positive integer m such that $f^m(U) \cap U \neq \phi$. We denote the set of nonwandering points of f by $\Omega(f)$.

A point $x \in X$ is called a *almost periodic point* of f if for any $\epsilon > 0$ one can find an integer $N > 0$ with the following property that for any integer $q > 0$ there exists an integer r, $q \leq r < q + N$, such that $d(f^r(x), x) < \epsilon$, where d is the metric of X. We denote the set of almost periodic points of f by $AP(f)$.

A point $y \in X$ is called an *ω-limit point* of x if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of ω-limit points of x by $\omega(x, f)$. Define $\Lambda(f) = \bigcup_{x \in X} \omega(x, f)$.

3. Main Results

Lemma 1[4]. *Let $f \in C^0(S^1, S^1)$. Then we have that $x \in AP(f)$ if and only if $x \in \omega(x, f)$ and $\omega(x, f)$ is minimal.*

The following lemma follows from [1].

Lemma 2. *Let $f \in C^0(S^1, S^1)$. Then we have*

$$P(f) \subset AP(f) \subset R(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f).$$

The following lemma found in [2]

Lemma 3. *Let $f \in C^0(S^1, S^1)$ and let $P(f)$ be empty. Then we have $\Omega(f) = R(f)$.*

Corollary 4. *Let $f \in C^0(S^1, S^1)$ with $P(f) \neq \phi$. Then the followings are equivalent.*
(1) $\overline{R(f)} = P(f)$.
(2) $\Omega(f) = P(f)$.
(3) $\Lambda(f) = P(f)$.

Theorem 5. *Let $f \in C^0(S^1, S^1)$. Then the following conditions are equivalent.*
(1) $\Lambda(f) = AP(f)$.
(2) $\Lambda(f) = R(f)$.
(3) For every $x \in S^1$, the ω-limit set $\omega(x, f)$ of x is minimal.

Proof. (1) $\Rightarrow$ (2) : Obvious by Lemma 2.

(2) $\Rightarrow$ (3) : Let x be arbitrary point in S^1, and let y be any point in $\omega(x, f)$. Then there exists a sequence $n_i \to \infty$ such that $f^{n_i}(x) \to y$. Suppose that $z \in \omega(y, f)$. Then there exists a sequence $m_i \to \infty$ such that $f^{m_i}(y) \to z$. Therefore $f^{m_i+n_i}(x) \to z$, and hence $z \in \omega(x, f)$. Thus $\omega(y, f) \subset \omega(x, f)$. Now we show that $\omega(x, f) \subset \omega(y, f)$. Since y is arbitrary point in $\omega(x, f)$, it suffices to show that $y \in \omega(y, f)$. We know that $y \in \Lambda(f)$ by definition. By assumption, $y \in R(f)$, and hence $y \in \omega(y, f)$. Therefore $\omega(x, f)$ is minimal for any $x \in S^1$.

$(3) \Rightarrow (1)$: Suppose that $\omega(x, f)$ is minimal for any $x \in S^1$. Let $y \in \Lambda(f) \setminus AP(f)$. Then there exists $z \in S^1$ such that $y \in \omega(z, f)$. Since $\omega(z, f)$ is minimal, $\omega(y, f) = \omega(z, f)$. Hence $y \in \omega(y, f)$ and $\omega(y, f)$ is minimal, and hence $y \in AP(f)$ by Lemma 1. This is a contradiction.

Corollary 6. *Let $f \in C^0(S^1, S^1)$. Suppose that $P(f)$ is closed. Then the followings are equivalent.*

(1) $R(f) = AP(f)$.

(2) $\Omega(f) = AP(f)$.

(3) $\Lambda(f) = AP(f)$.

(4) $\Lambda(f) = R(f)$.

(5) For every $x \in S^1$, the ω-limit set $\omega(x, f)$ of x is minimal.

Corollary 7. *Let $f \in C^0(S^1, S^1)$. Suppose that for any $x \in S^1$, the ω-limit set $\omega(x, f)$ of x contains a minimal set containing x. Then we have $\Lambda(f) = AP(f)$.*

Theorem 8. *Let $f \in C^0(S^1, S^1)$. Suppose that $\Lambda(f) = AP(f)$. Then we have that for any $x \in S^1$, the ω-limit set $\omega(x, f)$ of x contains only one minimal set.*

Proof. Suppose that $\Lambda(f) = AP(f)$. Let $x \in S^1$. Assume that there exist two minimal sets M, N with $M \subset \omega(x, f)$ and $N \subset \omega(x, f)$. Then for every $a \in M$ and $b \in N$, $M = \omega(a, f)$ and $N = \omega(b, f)$. We know that the ω-limit set $\omega(x, f)$ of x is minimal by Theorem 5. Since $a, b \in \omega(x, f)$,

$$M = \omega(a, f) = \omega(x, f) = \omega(b, f) = N.$$

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