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A NOTE ON THE EXISTENCE OF A LYAPUNOV FUNCTION

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ABSTRACT. We show that a real valued function ϕ defined by $\phi(x) = \sup\limits_{t \in \mathbb{R}^+} \frac{\ell(x,t)}{1+\ell(x,t)}$ is a Lyapunov function of compact asymptotically stable set M.

1. Introduction and Preliminaries

The subject of Lyapunov functions constitutes a control theme in the theory of differential equations. It provides powerful tools that can be used to study the behaviors of the solutions.

The classical theorem of Lyapunov on stability of the zero solutions for a given differential equation makes use of an auxiliary function which has to be positive definite. This auxiliary function is called a Lyapunov function in the theory of differential equations or more generally dynamical systems theory.

The aim of this paper is to show the existence of a Lyapunov function of compact asymptotically stable set. We recall some definitions from [2]. In this paper X will be locally compact metric space unless otherwise stated and $\mathbb{R}^+$ the set of nonnegative real numbers. A dynamical system on X is a continuous map $\pi : X \times \mathbb{R} \to X$ with the following properties

(i) $\pi(x, 0) = x$ for all $x \in X$,
(ii) $\pi(\pi(x, s), t) = \pi(x, s+t)$ for all $x \in X$ and $s, t \in \mathbb{R}$.

We call a family of dynamical systems $\{\pi_i | i \in I\}$ a dynamical polysystem, where I is an arbitrary of indices. Let $\{\pi_i | i \in I\}$ be a dynamical polysystem on X. The

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reachable map of the polysystem $\{\pi_i | i \in I\}$ is the multivalued map $R : X \times \mathbb{R}^+ \to 2^X$ defined by

$$\begin{aligned} R(x,t) = \{y \in X|\ & \text{there exist an integer} \quad n, t_1, \ldots, t_n \in \mathbb{R}^+ \quad \text{and} \\ & i_1, \ldots, i_n \in I \quad \text{such that} \quad \sum_{i=1}^{n} t_i = t \quad \text{and} \\ & y = \pi_{i_n}(\pi_{i_{n-1}}(\ldots \pi_{i_2}(\pi_{i_1}(x,t_1),t_2),\ldots,t_{n-1}),t_n)\} \end{aligned}$$

of X, where 2^X denotes the set of all subsets of X. Also, we define $R(A,t) = \bigcup_{x \in A} R(x,t)$ for $A \subset X$ and $t \in \mathbb{R}^+$ and we define $R(x,\mathbb{R}^+)$ by $R(x)$. For $A \subset X$, we let $R(A) = \bigcup_{x \in A} R(x)$. For a point $x \in X$, a subset M in X and $\delta > 0$, we denote

$$\begin{aligned} d(x,M) &= \inf\{d(x,y) | y \in M\} \\ B(M,\delta) &= \{x \in X | d(x,M) < \delta\}. \end{aligned}$$

We say that a compact subset M of X is

stable if for any neighborhood U of X, there exists a neighborhood V of M such that $R(V) \subset U$,

an attractor if $A(M)$ contains a neighborhood of M, where

$$\begin{aligned} A(M) = \{x \in X|\ & \text{there exists} \ \ t \in \mathbb{R}^+ \ \text{such that} \ R(x,[t,\infty)) \subset U \\ & \text{for any neighborhood } U \text{ of } M\ \}, \end{aligned}$$

asymptotically stable if M is stable and an attractor.

Proposition 1.1 [5]. *If R is uniformly bounded on $A \subset X$ and cluster, then R is upper semicontinuous on A.*

Proposition 1.2[5]. *Let M be a compact subset of X. Then M is stable if and only if, for any neighborhood U of M, there is a compact positively invariant neighborhood V of M such that $V \subset U$.*

Proposition 1.3[5]. *Let M be a compact subset of X and suppose M is stable. Then there is a neighborhood W of M such that a cluster map R is uniformly bounded on W.*

Proposition 1.4 [5]. *Let M be a compact subset M of X be a asymptotically stable. Then M is a uniform attractor.*

2. Lyapunov function

We define a function $\ell : X \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\ell(x,t) = \sup_{y \in R(x,t)} d(y, M).$$

The following theorem gives the existence of a neighborhood W of M which ℓ is continuous in $W \times \mathbb{R}^+$ for a compact stable subset M of X.

Proposition 2.1. *Let a compact subset M of X be stable. Then there is a neighborhood W of M such that ℓ is continuous in $W \times \mathbb{R}^+$.*

Proof. By Proposition 1.1 and 1.3, there is a $\delta > 0$ such that R is upper semicontinuous on $B(M,\delta)$. By stability of M, there is a neighborhood W of M such that $R(W,\mathbb{R}^+) \subset B(M,\delta)$. Let $(x,t) \in W \times \mathbb{R}^+$. For all $y \in R(x,t)$, since $R(x,t) \subset R(W,\mathbb{R}^+) \subset B(M,\delta)$, we have $d(y,M) < \delta$ and so $\ell(x,t) \leq \delta$. Let $(x_n,t_n) \in W \times \mathbb{R}^+$ with $(x_n,t_n) \to (x,t)$. For each $\varepsilon > 0$, there is $y \in R(x,t)$ such that $\ell(x,t) < d(y,M) + \varepsilon$. Thus, for each integer n, there is $z_n \in R(x_n,t_n)$ such that $\ell(x_n,t_n) < d(z_n,M) = \frac{\varepsilon}{2}$. Since R is lower semicontinuous at (x,t), there is a sequence $y_n \in R(x_n,t_n)$ such that $y_n \to y$. Therefore we have $d(y,M) \leq d(y,y_n) + d(y_n,M) \leq d(y,y_n) + \ell(x_n,t_n)$ and so $\ell(x,t) < \ell(x_n,t_n) + d(y,y_n) + \varepsilon$. Since R is upper semicontinuous at (x,t), there is a sequence $w_n \in R(x,t)$ such that $d(z_n,w_n) \to 0$. Thus, it follows that $d(z_n,M) \leq d(z_n,w_n) + d(w_n,M) \leq d(z_n,w_n) + \ell(x,t)$ and so $\ell(x_n,t_n) < \ell(x,t) + d(z_n,w_n) + \varepsilon$. Since $y_n \to y$ and $d(z_n,w_n) \to 0$, there is an integer m such that for all $n \geq m, d(y,y_n) < \varepsilon$ and $d(z_n,w_n) < \varepsilon$. Clearly, we have $\ell(x,t) < \ell(x_n,t_n) + 2\varepsilon$ and $\ell(x_n,t_n) < \ell(x,t) + 2\varepsilon$. It follows that $|\ell(x_n,t_n) - \ell(x,t)| < 2\varepsilon$. Hence ℓ is continuous at (x,t). The theorem is proved

Definition 2.2. Let M be a compact subset X and W positively invariant neighborhood of M. A Lyapunov function of M is a continuous function $\phi : W \to \mathbb{R}^+$ such that

(i) $\phi(x) = 0$ if and only if $x \in M$,

(ii) for each $y \in R(x,t), \phi(y) \le \phi(x)$.

We need the following lemma.

Lemma 2.3. *Let the subsets A, B of $\mathbb{R}$ be bounded above. Then*

$$|\sup_{a\in A} a - \sup_{b\in B} b| \le \sup_{a\in A, b\in B} |a-b|$$

Proof. Let $p = \sup_{a\in A} a$, $q = \sup_{b\in B} b$ and $r = \sup_{a\in A, b\in B} |a-b|$. For each $\varepsilon > 0$, there is $a \in A$ and $b \in B$ such that $p-\varepsilon < a \le p, q-\varepsilon < b \le q$ and $a-b-\varepsilon < p-q < a-b+\varepsilon$. Since $|a-b| \le r$, we have $-r-\varepsilon < p-q < r+\varepsilon$. Clearly, $|p-q| < r+\varepsilon$. We have $|p-q| \le r$. Hence the lemma is completed.

The main theorem of this paper is the following.

Theorem 2.4. *Suppose that a compact subset M of X is asymptotically stable. Let a real valued function ϕ define by*

$$\phi(x) = \sup_{t\in\mathbb{R}^+} \frac{\ell(x,t)}{1+\ell(x,t)}.$$

Then a function ϕ is a Lyapunov function of M.

Proof. It is obvious that $\phi(x) = 0$ if and only if $x \in M$ and for each $y \in R(x,t), \phi(y) \le \phi(x)$. Next we claim that ϕ is continuous on W. By Proposition 1.1, 1.2, 1.3, 1.4 and 2.1, there is a positively invariant neighborhood W of M such that $W \subset A_u(M)$, R is upper semicontinuous on W and ℓ is continuous $W \times \mathbb{R}^+$. Let $x \in M, x_n \in W$ and $x_n \to x$. We claim that $\phi(x_n) \to 0$. Assume that $\phi(x_n) \not\to 0$. Since $0 \le \phi(x_n) \le 1$, we have $\phi(x_n) \to r, 0 < r \le 1$. We may assume that $\phi(x_n) > \frac{r}{2}$. Then there is a sequence $t_n \in \mathbb{R}^+$ such that $\frac{\ell(x_n,t_n)}{1+\ell(x_n,t_n)} > \frac{r}{2}$. We have $\ell(x_n,t_n) > \frac{r}{2-r}$. Thus there is a sequence $y_n \in R(x_n,t_n)$ such that $d(y_n,M) > \frac{r}{2-r}$. Since M is stable, there is a neighborhood U of M such that $R(U,\mathbb{R}^+) \subset B(M,\frac{r}{2-r})$. We can choose an integer m so that $x_m \in U$. Clearly, we have $y_m \in R(x_m,t_m) \subset R(U,\mathbb{R}^+) \subset B(M,\frac{r}{2-r})$. Thus $d(y_n,M) < \frac{r}{2-r}$. This contradiction shows that $\phi(x_n) \to 0$. Let $x \in W-M, x_n \in W$ and $x_n \to x$. Set $\alpha = \frac{d(x,M)}{2} > 0$. Since $x \in A_u(M)$, there are $\beta, s \in \mathbb{R}^+$ with $0 < \beta \le \alpha$ such that $R(B(x,\beta),[s,\infty)) \subset B(M,\alpha)$. We may assume that $x_n \in B(x,\beta)$. For all $t \ge s$, since $R(x,t) \subset R(B(x,\beta),[s,\infty)) \subset B(M,\alpha)$ and $R(x_n,t) \subset R(B(x,\beta),[s,\infty)) \subset B(M,\alpha)$, we have $\ell(x,t) \le \alpha$ and $\ell(x_n,t) \le \alpha$.

Since $x \in R(x,0)$, we have $\ell(x,0) \geq d(x,M) > \alpha$. Since $x_n \in R(x_n,0), \ell(x_n,0) \geq d(x_n,M)$. From the fact that $2\alpha = d(x,M) \leq d(x_n,M) + d(x,x_n) < d(x_n,M) + \beta$, we have $d(x_n,M) > 2\alpha - \beta \geq \alpha$. Clearly, $\ell(x_n,0) > \alpha$. Thus we have

$$\phi(x) = \sup_{0 \leq t \leq s} \frac{\ell(x,t)}{1+\ell(x,t)} \text{ and } \phi(x_n) = \sup_{0 \leq t \leq s} \frac{\ell(x_n,t)}{1+\ell(x_n,t)}.$$

For any $\varepsilon > 0$ and $t \in [0,s]$, since ℓ is continuous at (x,t), there are neighborhood V_t of x and I_t of t such that $|\ell(y,r) - \ell(x,t)| < \varepsilon$ for each $y \in V_t$ and $r \in I_t$. Since (I_t) is an open cover of $[0,s]$ and $[0,s]$ is compact, there are $t_1, \ldots, t_k \in [0,s]$ such that $[0,s] \subset \bigcup_{i=1}^{k} I_{t_i}$. Let $V = \bigcap_{i=1}^{k} V_{t_i}$. Clearly, V is a neighborhood of x. Let $y \in V$ and $t \in [0,s]$. Then there is an integer i such that $t \in I_{t_i}$ and $y \in V \subset V_{t_i}$. We have $|\ell(y,t) - \ell(x,t)| \leq |\ell(y,t) - \ell(x,t_i)| + |\ell(x,t) - \ell(x,t_i)| < 2\varepsilon$. We choose an integer m such that for all $n \geq m, x_n \in V$. By Lemma 2.3, we have

$$\begin{aligned} |\phi(x_n) - \phi(x)| &= |\sup_{0 \leq t \leq s} \frac{\ell(x_n,t)}{1+\ell(x_n,t)} - \sup_{0 \leq t \leq s} \frac{\ell(x,t)}{1+\ell(x,t)}| \\ &\leq \sup_{0 \leq t \leq s} |\frac{\ell(x_n,t)}{1+\ell(x_n,t)} - \frac{\ell(x,t)}{1+\ell(x,t)}| \\ &\leq \sup_{0 \leq t \leq s} |\ell(x_n,t) - \ell(x,t)| \leq 2\varepsilon. \end{aligned}$$

It follows that $\phi(x_n) \to \phi(x)$. Therefore, ϕ is continuous at (x,t). Hence the function ϕ is a Lyapunov function of M. The theorem is completed.

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