J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
5(1998), no. 2, 115–122

PARTIAL DIFFERENTIAL EQUATIONS AND SCALAR CURVATURE ON SEMIRIEMANNIAN MANIFOLDS(I)

YOON-TAE JUNG, YUN-JEONG KIM,
SOO-YOUNG LEE, AND CHEOL-GUEN SHIN*

ABSTRACT. In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct timelike or null future(or past) complete Lorentzian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures.

1. Introduction

By the results of Kazdan and Warner ([6, 7, 8]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three catagories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [6, 7, 8], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question of how to determine which smooth functions are scalar curvatures

Received by the editors Aug. 31, 1998.
1991 *Mathematics Subject Classifications.* 53C21, 53C50, 58D30, 58G03.
Key words and phrases. Warped product, Scalar curvature, Upper and lower solution method.
The first author was supported in part by Research Funds from Chosun University, 1997.

of complete Riemannian metrics on an open manifold. Results of Gromov and Lawson ([5]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds. Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded ([5], [11] p. 322).

In [9, 10], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [4], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature.

Ironically, even though there exists some obstruction of positive or zero scalar curvature on a Riemannian manifold, results of [4] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may exist some obstruction of negative or zero scalar curvature.

In this paper, when N is a compact Riemannian manifold without boundary, we discuss the method of using warped products to construct timelike or null future(or past) complete Lorentzian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. It is shown that if the fiber manifold N belongs to class (A) or (B), then M admits a Lorentzian metric with negative scalar curvature approaching zero near the end outside a compact set.

2. Main results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive number. The Lorentzian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)f'(t)^2\} \tag{2.2}$$

for $t \in [a, \infty)$ and $x \in N$. (For details, cf. [3] or [4])

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u''(t) - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [4], we have the following proposition.

Proposition 2.1. *If the scalar curvature of the fiber manifold N is arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $[a, \infty)$ such that the resulting Lorentzian warped product metric on $[a, \infty) \times_f N$ produces positive constant scalar curvature.*

However, the results of [4] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature when the fiber manifold has constant scalar curvature.

Remark 2.2. *By Remark 2.58 in [1] and Corollary 5.6 in [12], if (a, b) is a finite interval and $n = 3$, then all nonspacelike geodesics are incomplete. But on $(-\infty, +\infty)$ there exists a warping function so that all non-spacelike geodesics are complete. For Theorem 5.5 in [12] implies that all timelike geodesics are future (resp. past) complete on $(-\infty, +\infty) \times_{v(t)} N$ if and only if $\int_{t_0}^{+\infty} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$) and Remark 2.58 in [1] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty} v^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} v^{\frac{1}{2}} dt = +\infty$) (cf. Theorem 4.1 and Remark 4.2 in [2]).*

If N admits a Riemannian metric of negative or zero scalar curvature, then we let $u(t) = t^{\alpha}$ in (2.3), where $\alpha \in (0,1)$ is a constant, and we have

$$R(t,x) \leq -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > a.$$

Therefore, from the above fact, Remark 2.2 implies the following:

Theorem 2.3. *For $n \geq 3$, let $M = [a, \infty) \times_f N$ be the Lorentzian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (A) or (B), then on M there is a future geodesically complete Lorentzian metric of negative scalar curvature outside a compact set.*

We note that the term $\alpha(1-\alpha)$ achieves its maximum when $\alpha = \frac{1}{2}$. And when $u = t^{\frac{1}{2}}$ and N admits a Riemannian metric of zero scalar curvature, we have

$$R = -\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}, \quad t > a.$$

If $R(t,x)$ is the function of only t-variable, then we have the following lemma whose proof is similar to that of Lemma 1.8 in [10].

Theorem 2.4. *If $R(g) = 0$, then there is no positive solution to equation (2.3) with*

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. Assume that

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

with $c > 1$. Equation (2.3) gives

$$t^2 u'' + \frac{c}{4} u \leq 0.$$

Let

$$u(t) = t^{\alpha} v(t), \quad t \geq t_0,$$

where $\alpha > 0$ is a constant and $v(t) > 0$ is a smooth function. Then we have

$$u'' = \alpha(\alpha-1)t^{\alpha-2}v(t) + 2\alpha t^{\alpha-1}v'(t) + t^{\alpha}v''(t).$$

And we obtain

$$t^{\alpha}v(t)[\alpha(\alpha-1) + \frac{c}{4}] + 2\alpha t^{\alpha+1}v'(t) + t^{\alpha+2}v''(t) \leq 0. \tag{2.4}$$

Let δ be a positive constant such that $\delta^2 = \frac{c-1}{4}$. Then we have

$$\alpha(\alpha-1) + \frac{c}{4} = (\alpha - \frac{1}{2})^2 + \frac{c-1}{4} \geq \delta^2.$$

Then δ is a constant independent on α. Equation (2.4) gives

$$2\alpha t v'(t) + t^2 v''(t) \leq -\delta^2 v(t). \tag{2.5}$$

Let $\beta = 2\alpha$ and we choose $\alpha > 0$ such that $\beta < 1$, that is, $\alpha < \frac{1}{2}$. Then (2.5) becomes

$$(t^\beta v'(t))' \leq -\frac{\delta^2 v(t)}{t^{2-\beta}}.$$

Upon integration we have

$$t^\beta v'(t) - \tau^\beta v'(\tau) \leq -\int_\tau^t \frac{\delta^2 v(s)}{s^{2-\beta}} ds, \quad t > \tau > t_0. \tag{2.6}$$

Here we have two following cases:
i) If $v'(\tau) \leq 0$ for some $\tau > t_0$, then (2.6) implies that

$$t^\beta v'(t) \leq -C$$

for some positive constant C and for large t. We have

$$v(t) \leq v(\tau) - \int_\tau^t \frac{C}{s^\beta} ds = v(\tau) - \frac{s^{1-\beta}}{1-\beta}|_\tau^t \to -\infty,$$

as $\beta < 1$. Hence $v(t) < 0$ for some t, contradicting that $v(t) > 0$ for all $t \geq t_0$.
ii) We have $v'(t) > 0$ for all $t > t_0$. Equation (2.6) implies that

$$\tau^\beta v'(\tau) - \int_\tau^t \frac{\delta^2 v(s)}{s^{2-\beta}} ds \geq 0$$

for all $t > \tau > t_0$. As $v'(t) > 0$ for all $t > t_0$, we have

$$\tau^\beta v'(\tau) \geq v(\tau) \int_\tau^t \frac{\delta^2}{s^{2-\beta}} ds = v(\tau)[\frac{1}{s^{1-\beta}}[-\frac{\delta^2}{1-\beta}]|_\tau^t$$

Let $t \to \infty$ we have

$$\tau^\beta v'(\tau) \geq \frac{v(\tau)}{\tau^{1-\beta}} \frac{\delta^2}{1-\beta}.$$

Or after changing the parameter we have

$$\frac{v'(t)}{v(t)} \geq \frac{1}{t} \frac{\delta^2}{1-\beta}, \quad t > t_0.$$

Choosing $\alpha < \frac{1}{2}$ close to $\frac{1}{2}$ so that $\beta < 1$ is close to 1 and using the fact that δ is independent on α or β, we have

$$\frac{v'(t)}{v(t)} \geq \frac{N}{t}$$

for a big integer $N > 2$. This gives

$$v(t) \geq Ct^N, \quad t > t_0,$$

where C is a positive constant. (2.6) implies that

$$t^\beta v'(t) \leq \tau^\beta v'(\tau) - \int_\tau^t \frac{C\delta^2 s^N}{s^{2-\beta}} ds \to -\infty \quad \text{as} \quad t \to \infty.$$

Thus $v'(t) < 0$ for t large, which is also a contradiction. Hence there is no solution to equation (2.3). □

In particular, if $R(g) = 0$, then using Lorentzian warped product it is impossible to obtain a Lorentzian metric of uniformly negative scalar curvature outside a compact subset. The best we can do is when $u(t) = t^{\frac{1}{2}}$, or $f(t) = t^{\frac{1}{n+1}}$, where the scalar curvature is negative but goes to zero at infinity.

Theorem 2.5. *Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a negative function such that*

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \leq 0 \quad \textit{for } t > t_0,$$

where $t_0 > a$ and $0 < c < 1$ are constants. Then equation (2.3) has a positive solution on $[a,\infty)$.

Proof. Since $R(g) = 0$ and $R(t,x) \leq 0$, the lower solution $u_-(t)$ is a small positive constant.

Put $u_+(t) = (c_+ + t^{-1})^m$, where $-1 < m < 0$ is determined later. Then

$$u_+''(t) = m(m-1)(c_+ + t^{-1})^{m-2}\frac{1}{t^4} + m(c_+ + t^{-1})^{m-1}\frac{2}{t^3}.$$

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) - R(t)u_+(t) \\
&= \frac{4n}{n+1}[m(m-1)(c_+ + t^{-1})^{m-2}\frac{1}{t^4} + m(c_+ + t^{-1})^{m-1}\frac{2}{t^3}] - R(t)(c_+ + t^{-1})^m \\
&= \frac{4n}{n+1}(c_+ + t^{-1})^m[m(m-1)(c_+ + t^{-1})^{-2}\frac{1}{t^4} + m(c_+ + t^{-1})^{-1}\frac{2}{t^3} - \frac{n+1}{4n}R(t)] \\
&\leq \frac{4n}{n+1}(c_+ + t^{-1})^m[m(m-1)(c_+ + t^{-1})^{-2}\frac{1}{t^4} + m(c_+ + t^{-1})^{-1}\frac{2}{t^3} + \frac{c}{4}\frac{1}{t^2}]
\end{aligned}$$

For $0 < c < c_1 < 1$ there exists $t_0 > a$ such that

$$m(m-1)(c_+ + t^{-1})^{-2}\frac{1}{t^4} + m(c_+ + t^{-1})^{-1}\frac{2}{t^3} + \frac{c}{4}\frac{1}{t^2} \le \frac{m(m-1)}{t^2} + \frac{2m}{t^2} + \frac{\frac{c_1}{4}}{t^2}$$

for $t > t_0$. Since $0 < c < c_1 < 1$, there exists $m < 0$ such that

$$m^2 + m + \frac{c_1}{4} \le 0.$$

Hence there exists $t_0 > a$ such that if $t > t_0$, then

$$m(m-1)(c_+ + t^{-1})^{-2}\frac{1}{t^4} + m(c_+ + t^{-1})^{-1}\frac{2}{t^3} + \frac{c}{4}\frac{1}{t^2} \le 0.$$

Therefore $u_+(t)$ is our upper solution. Since $t > t_0 > a$, we can take the lower solution $u_-(t)$ so that $0 < u_-(t) < u_+(t)$. □

Remark 2.6. *When $R(g) = 0$, the results in Theorem 2.4 and Theorem 2.5 are almost sharp. For if $f(t) = t^{\frac{1}{n+1}}$ for $t > a$, then we have*

$$R(t,x) = R(t) = -\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}.$$

References

1. J. K. Beem and P. E. Ehrlich, *Global Lorentzian Geometry*, Pure Appl. Math. **67**, Dekker, New York, 1981.
2. J. K. Beem, P. E. Ehrlich and Th. G. Powell, *Warped product manifolds in relativity*, Selected Studies (Th.M. Rassias, G.M. Rassias, eds.), North-Holland, 1982, 41-56.
3. F. Dobarro and E. Lami Dozo, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 295-408.
4. P. E. Ehrlich, Yoon-Tae Jung and Seon-Bu Kim, *Constant Scalar Curvatures on Warped Product Manifolds*, Tsukuba J. Math. **20(1)** (1996), 239-256.
5. M. Gromov and H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 295-408.
6. J. L. Kazdan and F. W. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Differential Geom. **10** (1975), 113-134.
7. J. L. Kazdan and F. W. Warner, *Existence and conformal deformation of metrics with prescribed Guassian and scalar curvature*, Ann. of Math. **101** (1975), 317-331.
8. J. L. Kazdan and F. W. Warner, *Curvature functions for compact 2 - manifolds*, Ann. of Math. **99** (1974), 14-74.
9. M. C. Leung, *Conformal scalar curvature equations on complete manifolds*, Comm. Partial Differential Equations **20** (1995), 367-417.
10. M. C. Leung, *Conformal deformation of warped products and scalar curvature functions on open manifolds*, preprint.
11. H. B. Lawson and M. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, 1989.

12. T. G. Powell, *Lorentzian manifolds with non-smooth metrics and warped products*, Ph.D thesis, Univ. of Missouri-Columbia, 1982.

DEPARTMENT OF MATHEMATICS, CHOSUN UNIVERSITY, KWANGJU, KOREA.
E-mail address: ytajung@chosun.ac.kr

DEPARTMENT OF MATHEMATICS, CHOSUN UNIVERSITY, KWANGJU, KOREA.

DEPARTMENT OF MATHEMATICS, CHOSUN UNIVERSITY, KWANGJU, KOREA.

*DEPARTMENT OF COMPUTER SCIENCE INFORMATION PROCESS, SUNCHON CHONGAM JUNIOR COLLEGE, SUNCHON, KOREA.