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THE SZEGŐ KERNEL AND A SPECIAL SELF-CORRESPONDENCE

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ABSTRACT. For a smoothly bounded n-connected domain Ω in $\mathbb{C}$, we get a formula representing the relation between the Szegő kernel associated with Ω and holomorphic mappings obtained from harmonic measure functions. By using it, we show that the coefficient of the above holomorphic map is zero in doubly connected domains.

1. Introduction

The Szegő kernel associated to a bounded domain in the plane carries plenty of information about the domain as the Bergman kernel does. Conformal mappings onto canonical domains can be expressed simply in terms of the Bergman kernel and the Szegő kernel. Hence it is possible to know the property of the conformal mappings by inspecting the Bergman kernel and the Szegő kernel. So our concern is to find the transformation formulas of the Bergman kernel and the Szegő kernel and to know the relation between them and other classical functions.

Now we suppose that Ω_1 and Ω_2 are two bounded domains in $\mathbb{C}$ and that f is a proper holomorphic mapping of Ω_1 onto Ω_2. There are a positive integer m and holomorphic mappings $F_1, F_2, \cdots, F_m$ which are m local inverses to f defined locally on $\Omega_2 - V$ where $V = \{f(z) | f'(z) = 0\}$. Bell [1] proved that the Bergman kernel functions transform under proper holomorphic mappings exactly as under biholomorphic mappings as follows:

$$\sum_{j=1}^{m} K_{\Omega_1}(z, F_j(w))\, \overline{F_j'(w)} = K_{\Omega_2}(f(z), w) f'(z) \tag{1}$$

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for $z \in \Omega_1$ and $w \in \Omega_2$ where K_{Ω_i} denotes the Bergman kernel function associated to Ω_i for $i = 1, 2$. From the above formula, we can get many important applications (see [1, 2, 3, 9]).

But, we got only a few results for the Szegő kernel. The author [8] proved the transformation formula for the Szegő kernel under proper holomorphic map of a multiply connected planar domain onto a simply connected planar domain and it was generalized under proper holomorphic correspondence between multiply connected planar domains (see [7]). Since the zeroes of the Szegő kernel are parts of the zeroes of the Ahlfors map and give rise to a particular basis for the Hardy space $H^2(b\Omega)$ (see [5]), they can be the powerful tools for getting the properties of the mapping for planar domains.

In this note, we prove an important formula representing the relation between the Szegő kernel function and holomorphic mappings obtained from harmonic measure functions by using the behavior of the zeroes of the Szegő kernel. We can use it to show that the coefficient of the holomorphic map is zero in doubly connected domains.

2. Preliminaries

Suppose that Ω is a smoothly bounded, n-connected domain in $\mathbb{C}$ and $b\Omega$ denotes the boundary of Ω.

We shall let $L^2(b\Omega)$ denote the space of square integrable complex valued functions on $b\Omega$ with the inner poduct given by $< u, v >= \int_{b\Omega} u\bar{v}ds$ where ds denotes the arc length measure.

The Hardy space of functions in $L^2(b\Omega)$ that are the L^2 boundary values of holomorphic functions on Ω shall be written $H^2(b\Omega)$.

The orthogonal projection $S : L^2(b\Omega) \to H^2(b\Omega)$ called the Szegő projection is well-defined and represented by the Szegő kernel $S_\Omega(z, w)$ on $\Omega \times \overline{\Omega}$ via

$$S\varphi(z) = \int_{b\Omega} S_\Omega(z, w)\varphi(w)\, ds_w$$

for φ in $L^2(b\Omega)$ and z in Ω. Here we have identified $S\varphi \in H^2(b\Omega)$ with its unique holomorphic extension to Ω. The Szegő kernel is holomorphic in z, anti-holomorphic in w, and $S_\Omega(z, w) = \overline{S_\Omega(w, z)}$.

Let $\{\gamma_j\}_{j=1}^n$ denote the n non-intersecting boundary curves of Ω. Without loss of generality, assume that γ_n is the outer boundary curve which bounds the unbounded component of the complement of Ω in $\mathbb{C}$. Let $\{\omega_j\}_{j=1}^n$ denote the harmonic measure functions associated to Ω. They are harmonic functions on Ω which extend C^∞ smoothly to $\overline{\Omega}$ and $\omega_j(\gamma_i) = \delta_{ij}$ (see [10; p.38]). We can get a multi-valued holomorphic function W_j by analytically continuing around Ω a germ of $\omega_j + i\omega_j^*$ where ω_j^* is a local harmonic conjugate for ω_j. Then $W_j' = 2\partial\omega_j/\partial z$ is also a holomorphic function. The Szegő kernel and the Bergman kernel are related via

$$K_\Omega(z,w) = 4\pi S_\Omega(z,w)^2 + \sum_{j=1}^{n-1} \lambda_j W_j'(z) \tag{2}$$

where λ_j's are constants in z which depend on w (see [6; p.119]).

Let $a \in \Omega$ be given. The function $g_a(z) = S_\Omega(z,a)/L_\Omega(z,a)$ maps Ω onto the unit disc and is n-to-one map (counting multiplicities) where $S_\Omega(z,a)$ denotes the Szegő kernel and $L_\Omega(z,a)$ denotes the Garabedian kernel (see [10; p.390]). Among all holomorphic functions h that map Ω into the unit disc, the functions that maximize the quantity $|h'(a)|$ are given by $e^{i\theta}g_a(z)$ for some real constant θ. Furthermore, g_a is uniquely characterized as the solution to this extremal problem such that $g_a'(a) > 0$. Also, g_a extends to be in $C^\infty(\overline{\Omega})$, g_a' is nonvanishing on the boundary, and g_a maps each boundary curve one-to-one onto the boundary of the unit disc .

The n zeroes of g_a are given by the simple pole of $L_\Omega(z,a)$ at a and $n-1$ zeroes of $S_\Omega(z,a)$ at $a_1, a_2, \ldots, a_{n-1}$ in $\Omega - \{a\}$ (counted with multiplicities).

of Ω onto the unit disc. It is an n-to-$C^\infty(\overline{\Omega})$, Garabedian kernel $L_\Omega(z,a)$ via

Bell [4; p.105] proved that if a is close to one of the boundary curves, the zeroes $a_1, a_2, \ldots, a_{n-1}$ become distinct simple zeroes. If a is a point in the boundary of Ω, $S_\Omega(z,a)$ is nonvanishing on Ω as a function of z and has exactly $n-1$ zeroes on the boundary of Ω, one on each boundary component not containing a.

3. Results

First we mention the following transformation formula (3) in [9] which is the result similar to the formula (1).

For a proper anti-holomorphic correspondence f between two bounded domains

Ω_1 and Ω_2 in $\mathbb{C}$, there are subvarieties V_1 and V_2 of Ω_1 and Ω_2^* where $\Omega_2^* = \{w|\bar{w} \in \Omega_2\}$ and positive integers p and q satisfying the following conditions:

(i) Near a point $z \in \Omega_1 - V_1$, there are p anti-holomorphic mappings $f_1, f_2, \cdots, f_p$ which are defined locally near z and represent f.

(ii) Near a point $w \in \Omega_2 - V_2^*$, there are q local inverses $F_1, F_2, \cdots, F_q$ to f which are defined locally near w.

Then the Bergman kernels transform via

$$\sum_{j=1}^{p} K_{\Omega_2}(f_j(z), w)\frac{\partial f_j}{\partial \bar{z}}(z) = \sum_{i=1}^{q} K_{\Omega_1}(F_i(w), z)\frac{\partial F_i}{\partial \bar{w}}(w) \tag{3}$$

for $z \in \Omega_1$ and $w \in \Omega_2$ where K_{Ω_i} denotes the Bergman kernel function associated to Ω_i for $i = 1, 2$.

Let Ω be a smoothly bounded, n-connected domain in $\mathbb{C}$. The multi-valued map $a \mapsto a_1, a_2, \ldots, a_{n-1}$ is a proper anti-holomorphic self-correspondence of Ω where $a_1, a_2, \ldots, a_{n-1}$ are the $n-1$ zeroes of $S_\Omega(z, a)$. From now on, let f denote this multi-valued map. There exists a subvariety V_1 of Ω such that $\{f_i\}_{i=1}^{n-1}$ denote the mappings that locally define f and $f_i(a) = a_i$ for $a \in \Omega - V_1$. The inverse correspondence f^{-1} is equal to f.

Let $\{\omega_j\}_{j=1}^{n}$ denote the harmonic measure functions associated with Ω such that $\omega_j(\gamma_i) = \delta_{ij}$ where $b\Omega = \cup_{i=1}^{n}\gamma_i$. Then we have $0 < \omega_j < 1$ in Ω for each $j = 1, \ldots, n$ and $\sum_{j=1}^{n} \omega_j \equiv 1$ on $\overline{\Omega}$ (see [10; p.38]). the Szegő kernel. By the properties of harmonic measures, $\sum_{i=1, i\neq j}^{n} \omega_i = 1 - \delta_{jk}$ on γ_k for each $k = 1, 2, \ldots, n$.

On the other hand, Bell [4; p.105] proved that if a is a point in the boundary of Ω, $S_\Omega(z, a)$ is nonvanishing on Ω as a function of z and has exactly $n-1$ zeroes on the boundary of Ω, one on each boundary component not containing a. Therefore $\sum_{i=1}^{n-1} \omega_j \circ f_i$ is a harmonic function which equals to $1 - \delta_{jk}$ on γ_k for each $k = 1, 2, \ldots, n$. Hence,

$$\sum_{i=1}^{n-1} \omega_j \circ f_i \equiv \sum_{i=1, i\neq j}^{n} \omega_i$$

on $\overline{\Omega}$ for each $j = 1, \ldots, n$ by the maximum principle (see [12; p.271]).

We summarize this result in the following proposition.

Proposition 1. *For a smoothly bounded, n-connected domain Ω in $\mathbb{C}$, the harmonic* *measures* $\{\omega_j\}_{j=1}^n$ *satisfy that*

$$\sum_{i=1}^{n-1} \omega_j \circ f_i \equiv \sum_{i=1, i\neq j}^{n} \omega_i$$

for each $j = 1, \ldots, n$.

We express the coefficients λ_j's in Bergman's formula (2) explicitly in Proposition 2 by using the zeroes of the Szegő kernel and it helps to understand c_j's in Theorem 3.

Proposition 2. *For a smoothly bounded, n-connected domain* Ω *in* $\mathbb{C}$, *the Bergman kernel and the Szegő kernel are related via*

$$K_\Omega(z,a) = 4\pi S_\Omega(z,a)^2 + \sum_{j=1}^{n-1}\{\sum_{k=1}^{n-1} K_\Omega(a_k, a) H_{kj}\} W_j'(z)$$

for $z, a \in \Omega$ *where* $[H_{kj}]$ *denotes the inverse matrix of* $[W_j'(a_k)]$.

Proof. Let $L_\Omega(z,a)$ denote the Garabedian kernel. Without loss of generality, we assume that for $a \in \Omega$, the zeroes a_i of the Szegő kernel $S_\Omega(z,a)$ are distinct. Since $span\{W_j' : j = 1, \ldots, n-1\} = span\{L_\Omega(\cdot, a_i)S_\Omega(\cdot, a) : i = 1, \ldots, n-1\}$, there exists a non-singular matrix $[m_{ji}]$ such that $W_j'(z) = \sum_{i=1}^{n-1} m_{ji} L_\Omega(z, a_i) S_\Omega(z,a)$ (see [11]).

On the other hand, let $G_i(z) = L_\Omega(z, a_i)S_\Omega(z,a)$ for $i = 1, \ldots, n-1$. Since $L_\Omega(z, a_i)$ has a simple pole at $z = a_i$ with residue $1/2\pi$ and $S_\Omega(a_i, a) = 0$ for each i, $G_i(a_k) = \delta_{ik} S_\Omega'(a_i, a)/2\pi$. Hence $[W_j'(a_k)] = [m_{ji}][G_i(a_k)]$ is a non-singular matrix. Therefore the equality $K_\Omega(a_k, a) = \sum_{j=1}^{n-1} \lambda_j(a) W_j'(a_k)$ got from (2) implies that λ_j is represented by $\lambda_j(a) = \sum_{k=1}^{n-1} K_\Omega(a_k, a) H_{kj}$ where $[H_{kj}]$ denotes the inverse matrix of $[W_j'(a_k)]$. □

The following theorem is the one which relates the Szegő kernel and holomorphic maps $W_j'(z)$ obtained from the harmonic measures on Ω.

Theorem 3. *For a smoothly bounded, n-connected domain* Ω *in* $\mathbb{C}$, *the Szegő kernel satisfies the following identity*

$$\sum_{i=1}^{n-1} S_\Omega(f_i(z), w)^2 \frac{\partial f_i}{\partial \bar{z}}(z) = \sum_{i=1}^{n-1} S_\Omega(f_i(w), z)^2 \frac{\partial f_i}{\partial \bar{w}}(w) + \sum_{j=1}^{n-1} c_j(w) \overline{W}_j'(z)$$

for $z, w \in \Omega$ *with coefficients* c_j*'s depending on* w.

Proof. By (2), the Szegő kernel and the Bergman kernel are related via

$$K_\Omega(z,w) = 4\pi S_\Omega(z,w)^2 + \sum_{j=1}^{n-1} \lambda_j(w) W_j'(z).$$

By the transformation formula (3) for the Bergman kernel,

$$\sum_{i=1}^{n-1} K_\Omega(f_i(z), w)\frac{\partial f_i}{\partial \bar z}(z) = \sum_{i=1}^{n-1} K_\Omega(f_i(w), z)\frac{\partial f_i}{\partial \bar w}(w)$$

for $z, w \in \Omega$.

The above two equations yield that

$$\begin{aligned}4\pi \sum_{i=1}^{n-1} S_\Omega(f_i(z), w)^2 \frac{\partial f_i}{\partial \bar z}(z) + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \lambda_j(w) W_j'(f_i(z))\frac{\partial f_i}{\partial \bar z}(z) \\ = 4\pi \sum_{i=1}^{n-1} S_\Omega(f_i(w), z)^2 \frac{\partial f_i}{\partial \bar w}(w) + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \overline{\lambda}_j(f_i(w))\overline{W_j'}(z)\frac{\partial f_i}{\partial \bar w}(w)\end{aligned}$$

by the conjugate symmetric properties of the Bergman kernel and the Szegő kernel.

On the other hand, $\sum_{i=1}^{n-1} \partial(\omega_j \circ f_i)/\partial \bar z = -\partial \omega_j/\partial \bar z$ by Proposition 1. It implies that

$$\begin{aligned}\sum_{i=1}^{n-1} W_j'(f_i(z))\frac{\partial f_i}{\partial \bar z}(z) &= \sum_{i=1}^{n-1} 2\omega_j'(f_i(z))\frac{\partial f_i}{\partial \bar z}(z) \\ &= \sum_{i=1}^{n-1} 2\partial(\omega_j \circ f_i)(z)/\partial \bar z \\ &= -2\partial\omega_j(z)/\partial \bar z \\ &= -\overline{W_j'}(z)\end{aligned}$$

since ω_j is real-valued. Hence,

$$\sum_{i=1}^{n-1} S_\Omega(f_i(z), w)^2 \frac{\partial f_i}{\partial \bar z}(z) = \sum_{i=1}^{n-1} S_\Omega(f_i(w), z)^2 \frac{\partial f_i}{\partial \bar w}(w) + \sum_{j=1}^{n-1} c_j(w)\overline{W_j'}(z)$$

where $c_j(w) = \frac{1}{4\pi}\{\lambda_j(w) + \sum_{i=1}^{n-1} \overline{\lambda}_j(f_i(w))\frac{\partial f_i}{\partial \bar w}(w)\}$. □

As an application, we show that for a doubly connected domain, the coefficient $c_1(w)$ in Theorem 3 is zero.

Corollary 4. *For a smoothly bounded, doubly connected domain* Ω *in* $\mathbb{C}$, $c_1(w)$ *in Theorem 3 is zero for each* $w \in \Omega$.

Proof. Since Ω is doubly connected, for fixed $w \in \Omega$ $S(z,w)$ has only one zero w_1. Hence $f = f_1$ is bi-anti-holomorphic map of Ω onto Ω. By the method similar to the transformation formula for the Szegő kernel under biholomorphic map (see [4; p.46]), we get

$$S_\Omega(f_1(z),w)\sqrt{\frac{\partial f_1}{\partial \bar z}(z)} = S_\Omega(f_1(w),z)\sqrt{\frac{\partial f_1}{\partial \bar w}(w)}$$

for $z, w \in \Omega$.

On the other hand, by Theorem 3

$$S_\Omega(f_1(z),w)^2\frac{\partial f_1}{\partial \bar z}(z) = S_\Omega(f_1(w),z)^2\frac{\partial f_1}{\partial \bar w}(w) + c_1(w)\overline{W_1'}(z)$$

for $z, w \in \Omega$. It also holds for $z \in \overline{\Omega}$. Hence $c_1(w)\overline{W_1'}(z) = 0$. By taking its conjugate and integrating it with respect to z, $\bar c_1(w)\int_{\gamma_1}\frac{\partial \omega_1}{\partial n}ds = 0$. Since $\int_{\gamma_1}\frac{\partial \omega_1}{\partial n}ds \neq 0$ (see [10;p.40]), $c_1(w) = 0$ for each $w \in \Omega$. □

Remark. For a doubly connected domain, the formula in Theorem 3 is reduced to the transformation formula for the Szegő kernel under a bi-anti-holomorphic map.

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