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EXTREMAL STRUCTURE OF $B(X^*)$

Joung Nam Lee

Abstract. In this note we consider some basic facts concerning abstract M spaces and investigate extremal structure of the unit ball of bounded linear functionals on σ-complete abstract M spaces.

1. Introduction

The original definition of an abstract L_1 or M space is given by S. Kakutani. The representation theorems of Kakutani are followed by several results which give joint characterizations of the abstract L_p spaces and M spaces among general Banach lattices.

A Banach lattice X(BL, for short) for which $\|x+y\| = \max(\|x\|, \|y\|)$, whenever $x, y \in X$ and $x \wedge y = 0$, is called an abstract M space. Let $1 \leq p < \infty$. A BL X for which $\|x+y\|^p = \|x\|^p + \|y\|^p$, whenever $x, y \in X$ and $x \wedge y = 0$, is called an abstract L_p space. It is obvious that every $L_p(\mu)$ space is an abstract L_p space if $p < \infty$ or an abstract M space if $p = \infty$. The converse is also true if $p < \infty$.

If $\{x_\alpha\}_{\alpha \in A}$ is a set in a BL, we denote by $\bigvee_{\alpha \in A} x_\alpha$ or by sup $\{x_\alpha\}_{\alpha \in A}$ the (unique) element $x \in X$ which has the following propeties: (1) $x \geq x_\alpha$ for all $\alpha \in A$ and (2) whenever $z \in X$ satisfies $z \geq x_\alpha$ for all $\alpha \in A$ then $z \geq x$. Unless the set A is finite, $\bigvee_{\alpha \in A} x_\alpha$ need not always exist in a BL [5].

For an element x in a BL X we put $x^+ = x \vee 0$ and $x^- = -(x \wedge 0) = (-x) \vee 0$. Obviously, $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Especially, if $x = u - v$, $u \geq 0$, $v \geq 0$ in X, then $u = x^+ + u \wedge v$ and $v = x^- + u \wedge v$. Also, if $u \wedge v = 0$, then $u = x^+$ and $v = x^-$ [6].

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The dual X^* of a BL X is also a BL provided that its positive cone is defined by $x^* \geq 0$ in X^* if and only if $x^*(x) \geq 0$, for every $x \geq 0$ in X. For any $x^*, y^* \in X^*$ and every $x \geq 0$ in X, we have

$$(x^* \vee y^*)(x) = \sup \{x^*(u) + y^*(x-u); 0 \leq u \leq x\}$$

and

$$(x^* \wedge y^*)(x) = \inf \{x^*(u) + y^*(x-u); 0 \leq u \leq x\}.$$

For a BL X, if X is an abstract M space, then X^* is an abstract L space and if X is an abstract L space, then X^* is an abstract M space, respectively. Also, if X is a BL, then X^* is a space of regular functionals [3]. Obviously, for a BL X and $x^* \in X^*$, $x^*(x) = \sup \{|x^*(y)| : |y| \leq x\}$ [4].

A BL X is said to be σ-complete if every order bounded set(sequence) in X has a sup, and a BL X is said to be bounded σ-complete, provided that any norm bounded and order monotone sequence in X is order convergent. Obviously, bounded σ-complete BL is σ-complete, but the inverse does not hold [5].

Since every $x^* \in X^*$ can be decomposed as a difference of two non-negative elements, it follows that every norm bounded monotone sequence $\{x_n\}_{n=1}^{\infty}$ in X is weak Cauchy. If, in addition, $x_n \xrightarrow{w} x$ for some $x \in X$ then $\|x_n - x\| \to 0$ as $n \to \infty$. This is a consequence of the fact that week convergecne to x implies the exitence of convex combinations of the x_n's which tend strongly to x.

For a Banach space X, we always denote by $B(X)$ and $S(X)$ the unit ball and the unit sphere of X respectively. $x \in S(X)$ is called an extreme point of $B(X)$ if for any given $y, z \in B(X)$ and $x = \lambda y + (1-\lambda)z$ for some $0 < \lambda < 1$, then $x = y = z$. The set of all extreme points of $B(X)$ is denoted by $\partial B(X)$. In this note we will investigate the extreme points of the unit ball of a dual space.

Now we show some propositions which will be needed in the sequel.

2. Main theorem

A BL X is an abstract L_p space if and only if for any $x, y \in X, x, y \geq 0$ implies $\|x+y\| = \|x\| + \|y\|$. Moreover, $\|u\| = \|x^+\| + \|u \wedge v\|$, $\|v\| = \|x^-\| + \|u \wedge v\|$, where $x, u, v \in X$, $x = u - v$ and $u \geq 0$, $v \geq 0$ [6]. Hence, we have the following result.

Lemma 1. *Let a BL X be an abstract L_p space and $x \in X$. Then $x = x^+ - x^-$ is unique in the sense that if $x = u - v, u \geq 0, v \geq 0$ and $\|u\| + \|v\| = \|x\|$, then $u = x^+$ and $v = x^-$.*

Proposition 2. *If a BL X is bounded σ-complete and $B(X)$ is order closed, then there exists $x \in S(X)$ such that $x^*(x) = \|x^*\|$ for every $x^*(\geq 0) \in X^*$, that is, x^* is norm attainable.*

Proof. Let x_n be a positive element in $S(X)$ such that $x^*(x_n) \to \|x^*\|$. Since X is bounded σ-complete and $B(X)$ is order closed, $y = \bigvee_n x_n$ exists in X and $\|y\| = 1$. Hence, $y \geq x_n \geq 0$ and $x^* \geq 0$ implies $\|x^*\| \geq x^*(y) \geq x^*(x_n) \to \|x^*\|$.

Note that the conclusion of Proposition 2 may not be true if an abstract M space X is not bounded σ-complete. For instance, let $X = c_0$ and $x^* = (c_n) \in l_1$ with infinitely $c_n \neq 0$. Then there does not exist $x \in S(X)$ such that $x^*(x) = \|x^*\|$.

If a BL X is bounded σ-complete and $B(X)$ is not order closed, then the conclusion of Proposition 2 is not true in general.

For a subset Y of a BL X, we define

$$Y^\perp = \{x \in X : |x| \wedge |y| = 0 \text{ whenever } y \in Y\}, \quad x^\perp = \{x\}^\perp.$$

If $x \in X = Y + Y^\perp$, then x can be uniquely decomposed into $x = y + z$, where $y \in Y$ and $z \in Y^\perp$. In this case, we write $x|_Y = y$ and $x^*|_Y(x) = x^*(y)$ for $x^* \in X^*$.

Proposition 3. *If an abstract M space X is σ-complete and $x^* \in X^*$, then for any $\varepsilon > 0$, there exists a subspace Y of X such that $X = Y + Y^\perp$ and $\|{x^*}^+|_{Y^\perp}\| < \varepsilon$, $\|{x^*}^-|_Y\| < \varepsilon$.*

Proof. Let x be in $S(X)$ such that $x^*(x) > \|x^*\| - \varepsilon$, and put $Y = (x^-)^\perp$. Then $x^+ \in Y$, $x^- \in Y^\perp$, and by [5] $X = Y + Y^\perp$. Moreover, by properties of an abstract M space X^*,

$$\begin{aligned}
&\|{x^*}^+|_Y\| + \|{x^*}^+|_{Y^\perp}\| + \|{x^*}^-|_Y\| + \|{x^*}^-|_{Y^\perp}\| \\
&\quad = \|{x^*}^+\| + \|{x^*}^-\| = \|x^*\| < x^*(x) + \varepsilon \\
&\quad = {x^*}^+|_Y(x) + {x^*}^+|_{Y^\perp}(x) - {x^*}^-|_Y(x) - {x^*}^-|_{Y^\perp}(x) + \varepsilon.
\end{aligned}$$

Since ${x^*}^+|_{Y^\perp}(x) \le 0$ and ${x^*}^-|_Y(x) \ge 0$ it follows that

$$\begin{aligned} &\|{x^*}^+|_{Y^\perp}\| + \|{x^*}^-|_Y\| \\ &\quad = \|{x^*}^+\| - \|{x^*}^+|_Y\| + \|{x^*}^-\| - \|{x^*}^-|_{Y^\perp}\| \\ &\quad \le \|{x^*}^+\| - {x^*}^+|_Y(x) + \|{x^*}^-\| - {x^*}^-|_{Y^\perp}(x) \\ &\quad < {x^*}^+|_{Y^\perp}(x) - {x^*}^-|_Y(x) + \varepsilon \le \varepsilon. \end{aligned}$$

Lemma 4. *Let an abstract M space X be bounded σ-complete and $B(X)$ order closed. Then $x^* \in X^*$ is norm attainable if and only if there exists a subspace Y of X satisfying ${x^*}^+ = x^*|_Y$, ${x^*}^- = -x^*|_{Y^\perp}$.*

Proof. Suppose that $x^* \in X^*$ is norm attainable. Then, by Proposition 2, there exist $x, y(\ge 0) \in S(X)$ such that ${x^*}^+(x) = \|{x^*}^+\|$ and ${x^*}^-(y) = \|{x^*}^-\|$. Since ${x^*}^+ = x^*|_Y$ and ${x^*}^- = -x^*|_{Y^\perp}$, we may assume $x \in Y$ and $y \in Y^\perp$ (otherwise we replace x, y by $x|_Y$, $y|_{Y^\perp}$ respectively). Now, we put $u = x - y$. Then $\|u\| = \|x - y\| = \max\{\|x\|, \|y\|\} = 1$ and thus, by properties of an abstract M space X^*, we get that

$$\begin{aligned} \|x^*\| &= \|{x^*}^+\| + \|{x^*}^-\| = {x^*}^+(x) + {x^*}^-(y) \\ &= x^*|_Y(x) + x^*|_{Y^\perp}(-y) = x^*(u). \end{aligned}$$

Conversely, choose $x \in S(X)$ such that $x^*(x) = \|x^*\|$, and define $Y = (x^-)^\perp$. Then $X = Y + Y^\perp$ and $x^+ \in Y$, $x^- \in Y^\perp$. Observe that $\|x^*\| = \|x^*|_Y\| + \|x^*|_{Y^\perp}\|$; to prove ${x^*}^+ = x^*|_Y$ and ${x^*}^- = -x^*|_{Y^\perp}$, it suffices to show $x^*|_Y \ge 0$ and $-x^*|_{Y^\perp} \ge 0$ thanks to Lemma 1. Indeed, if $x^*|_Y(y) < 0$ for some $y(\ge 0) \in S(X)$, then we may assume $y \in Y$. Therefore, $z = -x^- - y$ satisfies $\|z\| = \max\{\|x^-\|, \|y\|\} = 1$ and thus,

$$\begin{aligned} \|{x^*}^-\| &\ge {x^*}^-(-z) = x^*(z) - {x^*}^+(z) \ge x^*(z) \\ &= x^*|_{Y^\perp}(-x^-) - x^*|_Y(y) > x^*|_{Y^\perp}(-x^-) = -x^*|_{Y^\perp}(x). \end{aligned}$$

Since $\|{x^*}^+\| \ge x^*(x|_Y) = x^*|_Y(x)$, this clearly leads to a contradiction that

$$\|x^*\| = \|{x^*}^+\| + \|{x^*}^-\| > x^*|_Y(x) - x^*|_{Y^\perp}(x) = x^*(x) = \|x^*\|.$$

A similar argument would show that $-x^*|_{Y^\perp} \ge 0$.

Now we investigate the extreme points of the unit ball of a dual space. The sequence $\{x_n\}$ converges weakly to zero in a Banach space X if and only if $\{x_n\}$ is bounded, and $x^*(x_n) \to 0$ for every $x^* \in \partial B(X^*)$.

Theorem 5. *Let an abstract M space X be σ- complete and $x^* \in S(X^*)$. Then $x^* \in \partial B(X^*)$ if and only if $x^*(x)x^*(y) = 0$ for all $x, y \in X$ such that $x \wedge y = 0$.*

Proof. Sufficiency. First we show $\|{x^*}^+\|\|{x^*}^-\| = 0$. In fact, by Proposition 3, for any $\varepsilon > 0$, there exist orthogonal subspaces Y, Z, of X such that $X = Y + Z$ and $\|{x^*}^-|_Y\| < \varepsilon$, $\|{x^*}^+|_Z\| < \varepsilon$. Choose $x \in S(X)$ satisfying $x^*(x) > \|x^*\| - \varepsilon$, and let $x = u + v$, where $u \in Y$ and $v \in Z$. Then $x^*(u)x^*(v) = 0$ since $u \wedge v = 0$. If $x^*(v) = 0$, then

$$\begin{aligned}\|x^*\| - \varepsilon < x^*(x) &= {x^*}^+|_Y(u) - {x^*}^-|_Y(u)\\ &\le \|{x^*}^+|_Y\| + \|{x^*}^-|_Y\| < \|{x^*}^-\| + \varepsilon.\end{aligned}$$

Let $\varepsilon \to 0$. Then $\|{x^*}^-\| = \|x^*\| - \|{x^*}^+\| = 0$. Similarly, assume that $x^*(u) = 0$. Then $\|{x^*}^-\| = 0$. Hence, without loss of generality, we may assume $x^* = {x^*}^+$.

Let $y^*, z^* \in S(X^*)$ satisfy $2x^* = y^* + z^*$. Then $2x^* = ({y^*}^+ + {z^*}^+) - ({y^*}^- + {z^*}^-)$ and by properties of an abstract M space X^*,

$$\begin{aligned}\|2x^*\| &\le \|{y^*}^+\| + \|{z^*}^+\| + \|{y^*}^-\| + \|{z^*}^-\|\\ &= \|y^*\| + \|z^*\| = 2 = \|2x^*\|.\end{aligned}$$

Thus, by Lemma 1, we have ${y^*}^+ + {z^*}^+ = 2x^*$ and ${y^*}^- = {z^*}^- = 0$.

Now we show that $y^* = z^* = x^*$, i.e., $x^* \in \partial B(X^*)$. To this end we notice that $y^*(y) = z^*(y) = 0$ whenever $x^*(y) = 0$ (by [7], this means $x^* = ay^* = bz^*$, but $x^*, y^*, z^* \in S(X^*)$ and $2x^* = y^* + z^*$, so $a = b = 1$). First we assume $y \ge 0$; then from $y^*(y) \ge 0$, $z^*(y) \ge 0$, and $y^*(y) + z^*(y) = 2x^*(y) = 0$ we have $y^*(y) = z^*(y) = 0$. For the general case, since $x^*(y) = 0$ and by the condition given in the theorem, $x^*(y^+)x^*(y^-) = 0$, we have $x^*(y^+) = x^*(y^-) = 0$. Hence, $y^*(y) = z^*(y) = 0$ follows from the first case.

Necessity. Assume first that there exist $x, y \in X$ such that $x \wedge y = 0$ but $x^*(x) > 0$ and $x^*(y) > 0$. Then we put $Y = y^\perp$, and then by [5] $X = Y + Y^\perp$. Now, let $y^* = x^*|_Y$ and $z^* = x^*|_{Y^\perp}$. Then $\|y^*\| > 0$, $\|z^*\| > 0$ since $x \in Y$, $y \in Y^\perp$. Therefore, since

$$x^* = \|y^*\|\frac{y^*}{\|y^*\|} + \|z^*\|\frac{z^*}{\|z^*\|}$$

and $\|y^*\| + \|z^*\| = \|x^*\| = 1$ according to Lemma 1 and the intrinsic M space properties, we get that $x^* \in \partial B(X^*)$, which is desired result.

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DEPARMENT OF NATURAL SCIENCE, SEOUL NATIONAL UNIVERSITY OF TECHNOLOGY, SEOUL 139-743, KOREA.