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THE HYPERBOLIC METRIC ON K-CONVEX REGIONS

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ABSTRACT. Mejia and Minda proved that if a hyperbolic simply connected region Ω is k-convex, then $|\nabla \log \lambda_\Omega(z)| \leq \lambda_\Omega(z)\sqrt{1-\frac{2k}{\lambda_\Omega(z)}}, z \in \Omega$. We show that this inequality actually characterizes k-convex regions.

1. Introduction

We begin with a short introduction to hyperbolic regions in the complex plane $\mathbf{C}$. A general discussion of this subject can be found in [1] and [4]. A region Ω in $\mathbf{C}$ is called hyperbolic if the complement of Ω with respect to $\mathbf{C}$ contains at least two points. A hyperbolic simply connected region Ω is said to be k-convex $(k>0)$ if $|a-b|<2/k$ for any pair of distinct points $a, b \in \Omega$ and the intersection of two closed disks of radii $1/k$ that have both a and b on their boundaries lies in Ω. Mejia and Minda [6] proved that if Ω is a hyperbolic simply connected region bounded by a simple closed curve $\partial\Omega$ of class C^2 and if $K_e(z, \partial\Omega) \geq k$ for all $z \in \partial\Omega$, then Ω is k-convex. Here $K_e(z, \partial\Omega)$ denotes the euclidean curvature of $\partial\Omega$ at the point z. If a region Ω is hyperbolic, then, by the uniformization theorem [2, p.39], there exists a holomorphic universal covering projection f of the open unit disk $D = \{z : |z| < 1\}$ onto Ω. Note that f is a conformal mapping of D onto Ω when Ω is simply connected. The hyperbolic metric on D, normalized to have Gaussian curvature -1, is $\lambda_D(z)|dz| = 2|dz|/\left(1-|z|^2\right)$. If $f : D \to \Omega$ is any holomorphic universal covering projection, then the hyperbolic metric $\lambda_\Omega(w)|dw|$ on Ω is determined from

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$$\lambda_\Omega\left(f(z)\right)|f'(z)| = \frac{2}{1-|z|^2}.$$

The hyperbolic metric is invariant under holomorphic covering projections. In particular, the hyperbolic metric is a conformal invariant.

The gradient $\nabla g(z)$ is the complex vector $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$ and its squared length is $|\nabla g|^2 = \left|\frac{\partial g}{\partial x}\right|^2 + \left|\frac{\partial g}{\partial y}\right|^2$. If g is a real-valued differentiable function, then we have $|\nabla g| = 2\left|\frac{\partial g}{\partial z}\right|$, where $\frac{\partial}{\partial z}$ is the differential operator

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\ z = x + iy.$$

Mejia and Minda [6] proved that if a hyperbolic region Ω in $\mathbf{C}$ is k-convex, then for $z \in \Omega$

$$|\nabla \log \lambda_\Omega(z)| \le \lambda_\Omega(z)\sqrt{1 - \frac{2k}{\lambda_\Omega(z)}}.$$

In this paper we show that this inequality actually characterizes k-convex regions.

2. Euclidean and hyperbolic curvature

Let $\gamma : z = z(t)$, $t \in [a,b]$, be a C^2 curve in the complex plane with $z'(t) \neq 0$ for $t \in [a,b]$. The euclidean curvature $K_e(z,\gamma)$ of the curve γ at the point $z = z(t)$ is the rate of change of the angle θ that the tangent vector makes with the positive real axis respect to arc length :

$$K_e(z,\gamma) = \frac{d\theta}{ds} = \frac{d\theta}{dt}\frac{dt}{ds} = \frac{1}{|z'(t)|}\mathrm{Im}\left\{\frac{z''(t)}{z'(t)}\right\}.$$

The value of the euclidean curvature is independent of the parametrization of γ. If f is holomorphic and locally univalent in a neighborhood of γ, then $f \circ \gamma$ is also a C^2 curve with nonvanishing tangent. Let $w = f(z)$ and $\sigma = f \circ \gamma$. Then $w = w(t) = f\left(z(t)\right)$, $t \in [a,b]$ is a parametrization of σ. We have

$$K_e(w,\sigma) = \frac{1}{|w'(t)|}\mathrm{Im}\left\{\frac{w''(t)}{w'(t)}\right\}$$
$$= \frac{1}{|f'(z)|\,|z'(t)|}\mathrm{Im}\left\{\frac{f''(z)z'(t)^2 + f'(z)z''(t)}{f'(z)z'(t)}\right\}.$$

This yields the formula for the change of the euclidean curvature under f :

$$K_e(f(z), f\circ\gamma)\,|f'(z)| = K_e(z,\gamma) + \mathrm{Im}\left\{\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right\}. \tag{1}$$

Next, we define the hyperbolic curvature. More details, see [3] and [7]. If γ is a C^2 curve in a hyperbolic region Ω with nonvanishing tangent, then the hyperbolic curvature of γ at the point $z = z(t)$ is given by

$$K_\Omega(z,\gamma) = \frac{1}{\lambda_\Omega(z)}\left[K_e(z,\gamma) + 2\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}\right].$$

For the unit open disk D we have

$$K_D(z,\gamma) = \frac{1-|z|^2}{2}K_e(z,\gamma) + \mathrm{Im}\left\{\frac{\overline{z(t)}z'(t)}{|z'(t)|}\right\}.$$

Let us determine the hyperbolic curvature of the positively oriented circle γ in D with center 0 and radius $r \in (0,1)$. A parametrization of γ is $z = z(t) = re^{it}$, $0 \le t \le 2\pi$. Then

$$K_e(z,\gamma) = \frac{1}{|z'(t)|}\mathrm{Im}\left\{\frac{z''(t)}{z'(t)}\right\} = \frac{1}{r}.$$

As $\overline{z(t)}z'(t)/|z'(t)| = ir$ so that

$$K_D(z,\gamma) = \frac{1-r^2}{2}\frac{1}{r} + r = \frac{1}{2}\left(r + \frac{1}{r}\right).$$

Thus, any circle in D with center origin has the hyperbolic curvature strictly larger than 1. The following result is well known. We include a proof for the convenience.

Lemma 1. *Suppose Ω and Δ are hyperbolic regions and f is a conformal mapping of Ω onto Δ. Then $K_\Omega(z,\gamma) = K_\Delta(f(z), f\circ\gamma)$ for any C^2 curve γ in Ω with nonvanishing tangent.*

Proof. Let $w = f(z)$ and $\sigma = f\circ\gamma$. From the conformal invariance of the hyperbolic metric, we obtain $\lambda_\Omega(z) = \lambda_\Delta(f(z))\,|f'(z)|$. Then

$$\log \lambda_\Omega(z) = \log \lambda_\Delta(f(z)) + \frac{1}{2}\log f'(z) + \frac{1}{2}\log \overline{f'(z)}$$

so that

$$\frac{\partial \log \lambda_\Omega(z)}{\partial z} = \frac{\log \lambda_\Delta(w)}{\partial w} f'(z) + \frac{1}{2}\frac{f''(z)}{f'(z)}.$$

From $w'(t) = f'(z)z'(t)$, we obtain $|w'(t)| = |f'(z)|\,|z'(t)|$ and

$$f'(z)\frac{z'(t)}{|z'(t)|} = |f'(z)|\frac{w'(t)}{|w'(t)|}.$$

Hence

$$2\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\} = 2\,|f'(z)|\,\mathrm{Im}\left\{\frac{\log \lambda_\Delta(w)}{\partial w}\frac{w'(t)}{|w'(t)|}\right\} + \mathrm{Im}\left\{\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right\}. \tag{2}$$

From (1) and (2), we obtain

$$\lambda_\Omega(z)\,K_\Omega(z,\gamma) - K_e(z,\gamma) = |f'(z)|\left[\lambda_\Delta(w)\,K_\Delta(w,\sigma) - K_e(w,\sigma)\right] + |f'(z)|\,K_e(w,\sigma) - K_e(z,\gamma).$$

This identity yields $K_\Omega(z,\gamma) = K_\Delta(w,\sigma) = K_\Delta(f(z), f\circ\gamma)$ since $\lambda_\Omega(z) = \lambda_\Delta(w)$ $|f'(z)|$.

3. A characterization of k-convex regions

A holomorphic univalent function f in D is called k-convex provided $f(D)$ is a k-convex region. If a region Ω is k-convex, then, by the uniformization theorem, there exists a k-convex function f such that $f(D) = \Omega$. Ma, Minda and Mejia [5] proved that if f is a k-convex function, normalized by $f(0) = 0$ and $f'(0) = \alpha > 0$, then

$$|f''(0)| \leq 2\alpha\sqrt{1-\alpha k} \tag{3}$$

with equality if and only if

$$f(z) = \frac{\alpha z}{1 - \sqrt{1 - \alpha \overline{k} e^{i\theta} z}}$$

for some real number θ. We now establish a characterization of k-convex regions.

Theorem 2. *A hyperbolic simply connected region Ω is k-convex if and only if*

$$|\nabla \log \lambda_\Omega(z)| \le \lambda_\Omega(z) \sqrt{1 - \frac{2k}{\lambda_\Omega(z)}} \tag{4}$$

for all z in Ω.

Proof. Mejia and Minda [6] showed the necessity. We establish a new proof. Suppose Ω is k-convex. Fix $a \in \Omega$ and let $z = f(w)$ be a k-convex function of $(D, 0)$ onto (Ω, a) such that

$$\lambda_\Omega(f(w))\,|f'(w)| = \frac{2}{1 - |w|^2},\ w \in D.$$

In particular, $\lambda_\Omega(a) = 2/|f'(0)|$. Also

$$\log \lambda_\Omega(f(w)) + \frac{1}{2}\log f'(w) + \frac{1}{2}\log \overline{f'(w)} = \log 2 - \log(1 - w\overline{w}).$$

Applying the operator $\frac{\partial}{\partial w}$ to both sides of this identity, we obtain

$$\frac{\partial \log \lambda_\Omega(f(w))}{\partial z} f'(w) + \frac{1}{2}\frac{f''(w)}{f'(w)} = \frac{\overline{w}}{1 - w\overline{w}}.$$

For $w = 0$, this gives

$$\frac{\partial \log \lambda_\Omega(a)}{\partial z} f'(0) = -\frac{1}{2}\frac{f''(0)}{f'(0)},$$

so that

$$|\nabla \log \lambda_\Omega(a)| = 2\left|\frac{\partial \log \lambda_\Omega(a)}{\partial z}\right| = \frac{1}{2}\frac{|f''(0)|}{|f'(0)|}\lambda_\Omega(a). \tag{5}$$

The function

$$g(z) = (f(z) - f(0)) \exp(-\arg f'(0))$$

is a normalized ($g(0) = 0$ and $g'(0) = |f'(0)| > 0$) k-convex function. Therefore, by (3), we obtain

$$\frac{|f''(0)|}{|f'(0)|} = \frac{|g''(0)|}{|g'(0)|} \leq 2\sqrt{1-|f'(0)|\,k} = 2\sqrt{1-\frac{2k}{\lambda_\Omega(a)}}. \tag{6}$$

From (5) and (6), we obtain the inequality (4).

Conversely, suppose that the inequality (4) holds. Let $z = f(w)$ be a conformal mapping of D onto Ω and let δ be the circle $w = w(t) = re^{it}$, $0 \leq t \leq 2\pi$, $0 < r < 1$. Then $K_D(w,\delta) > 1$. Let $\sigma = f \circ \delta$. Then $z = z(t) = f(\delta(t))$, $0 \leq t \leq 2\pi$ is a parametrization of the curve σ. Lemma 1 yields

$$K_\Omega(z,\sigma) = K_\Omega(f(w), f\circ\delta) = K_D(w,\delta) > 1.$$

From the definition of the hyperbolic curvature and the inequality (4), we obtain

$$\begin{aligned} K_e(z,\sigma) &= K_\Omega(z,\sigma)\,\lambda_\Omega(z) - 2\mathrm{Im}\left\{\frac{\partial \log\lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\} \\ &> \lambda_\Omega(z) - |\nabla \log\lambda_\Omega(z)| \\ &\geq \lambda_\Omega(z)\left(1-\sqrt{1-\frac{2k}{\lambda_\Omega(z)}}\right) \\ &= \frac{2k}{1+\sqrt{1-\frac{2k}{\lambda_\Omega(z)}}} > k. \end{aligned}$$

As $z(t) \in \sigma$ is arbitrary, we have $K_e(z,\sigma) > k$ for all $z \in \sigma$ and so $f(\{w : |w| < r\})$ is k-convex. Then $\Omega = f(D)$ is k-convex since it is an increasing union of k-convex regions.

Let f be a conformal mapping of D onto a k-convex region Ω. We note that the inequality (4) holds since Ω is k-convex. Then, by the proof of the sufficiency of Theorem 2, $K_D(z,\gamma) \geq 1$ implies $K_e(f(z), f\circ\gamma) \geq k$. Thus, we obtain the following results.

Corollary 3. *Let γ be a C^2 curve in D with nonvanishing tangent and $z \in \gamma$. If $K_D(z,\gamma) \geq 1$, then $K_e(f(z), f\circ\gamma) \geq k$ for any conformal mapping of D onto a k-convex region Ω.*

Corollary 4. *Let Δ be a disk in D. If f is a conformal mapping of D onto a k-convex region Ω, then $f(\Delta)$ is also k-convex.*

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