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고체구조물의 비선형변형 수치해석에 대한 이론적고찰(2)
-단순구조물에의 적용-

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A Study on the Numerical Technique for the Nonlinear Deformation Analysis of Solid Structures(2)
-Application to a Simple Solid Structure-

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ABSTRACT

본 논문에서는 고체구조물의 비선형변형해석에 대하여 일반이론으로 개발된 수치해법을 단순 고체 구조물인 일차원 봉 문제에 적용하여 그 변형해석을 수행 하였다. 정확한 해를 구하기 위하여 증분 뉴톤-랩슨방법이 수정 보완 사용되었다. 또한 개발된 비선형유한요소법의 검증을 위하여 수학적인 정해가 존재하는 균일한 체력이 작용하는 단순봉의 변형을 해석하여 그 결과를 수학적인 정해와 비교하였다. 비교 결과 본 논문을 통하여 개발된 비선형 유한요소법의 정확성이 입증되었다.

Key Words : One Dimensional Bar(일차원봉), Strain Energy Density(변형도에너지 밀도), Body Force(체력), Load Parameter(하중매개변수), Finite Element Discretization(유한요소분할), Material Nonlinearity (물질비선형성), Incremental Tangential Stiffness Matrix(증분접선강도행렬).

1. Introduction

Recent advances in nonlinear solid deformation theories[(1)(2)], the numerical technique[(3)] and the computer capability make the numerical computations of solid deformations possible. These advances allow mechanical engineers to develop the mechanical computer aided engineering system (so-called MCAE System). These developed MCAE Systems are now commercially available to be used in the industry for the enhancement of the design productivity. And so, the development of powerful numerical computation technique becomes very important to mechanical engineers nowadays. For this kind of purpose, in this paper, numerical technique developed in the previous paper for the nonlinear solid deformation analysis is applied to a simple solid structure, i.e, one dimensional bar. The incremental Newton-Raphson Scheme which is modified in this paper has

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been used for other applications[4].

2. Problem Formulation

Consider the following problem, i.e., the one dimensional bar of length L exerted by a distributed external load $P(x)$ as in Fig. 1. The equilibrium equation will reduce to the following form for the case in Fig. 1

$$\frac{\partial\sigma}{\partial x}+P(x)=0 \tag{1}$$

,where we denote $\sigma_{xx}=\sigma$:normal stress, $f_x=P(x)$ the given body force. And the boundary condition is $\sigma(L)=0$, i.e., the traction is free at the end of the bar. If $P(x)$ is prescribed, the solution of (1) is very simple, that is,

$$\sigma(x)=-\int P(x)dx+\text{const} \tag{2}$$

For the linear elasticity, the linear stress-strain relation is satisfied until the limit point : this limit point is usually characterized by the yield strain $\varepsilon_y=0.2\%(=0.002)$. Beyond this limit the linear relation cannot be satisfied. Instead, $\sigma=\sigma(\varepsilon)$ has the nonlinear form, e.g., $\sigma=k\varepsilon^n$ more specifically, e.g., $\sigma=k\sqrt{\varepsilon}=k\varepsilon^{\frac{1}{2}}$ for $n=\frac{1}{2}$ Usually, $k=E\ \varepsilon_y^{\ 1-n}$ or $E=k\ \varepsilon_y^{\ n-1}$. And hence, we obtain the following two categories. For small strain region,

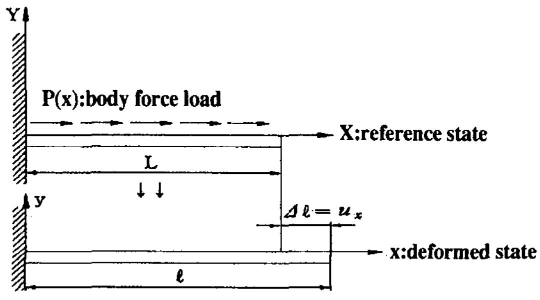

Fig. 1 Simple bar exerted by the body force

$$\sigma=E\varepsilon \tag{3}$$

For large strain region,

$$\sigma=k\varepsilon^n=E\ \ \varepsilon_y^{\ 1-n}\ \ \varepsilon^n \tag{3)*}$$

For infinitesimal displacement, we get the following strain-displacement relation

$$\varepsilon=\frac{du}{dx} \tag{4}$$

, since $v=w=0$ & $\partial u/\partial y=\partial u/\partial x=0$($u$,v,w are displacements in x, y, z respectively) for this simple bar case. Hence, we consider only the material nonlinearity. Thus we are ready to obtain the exact solution of (1) if the external body force $P(x)$ is prescribed. Here, let us assume

$$P(x)=a+bx^2+dx^4(N/m^3) \tag{5}$$

, where a, b, d are positive values.
Then (2) becomes

$$\sigma(x)=a(L-x)+\frac{b}{3}(L^3-x^3)+\frac{d}{5}(L^5-x^5) \tag{6}$$

We have two solutions according to the constitutive relations, i.e., linear and nonlinear solutions. The linear solution is

$$u(x)=\frac{1}{E}[(aL+\frac{b}{3}\ L^3+\frac{d}{5}L^5)x-\frac{a}{2}x^2-\frac{b}{12}x^4-\frac{d}{30}x^6] \tag{7}$$

The nonlinear solution is

$$\begin{aligned}u(x)=&\frac{1}{k^2}[(a^2L^2+\frac{b^2}{9}L^6+\frac{d^2}{25}L^{10}+\frac{2}{3}abL^4+\frac{2}{15}bdL^8+\frac{2}{5}adL^6)x\\&-(a^2L+\frac{1}{3}abL^3+\frac{1}{5}adL^5)x^2+\frac{a^3}{3}x^3-\frac{1}{2}(\frac{1}{9}b^2L^3+\frac{1}{15}bdL^5\\&+\frac{1}{3}abL)x^4+\frac{2}{15}abx^5-\frac{1}{3}(\frac{1}{25}d^2L^5+\frac{1}{15}bdL^3+\frac{1}{5}adL)x^6\\&+\frac{1}{7}(\frac{b^2}{9}+\frac{2}{5}ad)x^7+\frac{2}{135}bdx^9+\frac{d^2}{275}x^{11}]\quad for\quad n=\frac{1}{2}\end{aligned} \tag{8}$$

3. Weak Form of the Problem (Variational Formulation of the Problem)

To solve the problem using the finite element method, we need the integral form of (1). For this, we should know the potential energy of the system, because the system will be in the equilibrium state when the potential energy is minimum. The potential energy of the bar is given by

$$I(u) = \int_0^L W(\varepsilon, x)dx - \int_0^L p(x)u(x)dx \tag{9}$$

, where $W(x, \varepsilon)$ is the strain energy density. $W(x, \varepsilon)$ is given by $\sigma(x) = \frac{dW}{d\varepsilon}$.

Thus, for linear solid deformation

$$\sigma(x) = \frac{dW}{d\varepsilon} = E\varepsilon \quad \text{or} \quad W(x,\varepsilon) = \frac{E}{2}\varepsilon^2 \tag{10}$$

and for nonlinear solid deformation

$$\sigma(x) = \frac{dW}{d\varepsilon} = k\varepsilon^n \quad \text{or} \quad W(x,\varepsilon) = \frac{k}{n+1}\varepsilon^{n+1} \tag{10*}$$

Therefore, we obtain the following integral form

$$I(u) = \int_0^L \frac{1}{2}E\varepsilon^2 dx - \int_0^L P(x)u(x)dx \tag{11}$$

for linear deformation

$$I(u) = \int_0^L \frac{k}{n+1}\varepsilon^{n+1} dx - \int_0^L P(x)u(x)dx \tag{11*}$$

for nonlinear deformation

4. Finite Element Discretization

To evaluate the integral in Eq (9), we will break it up into E subintegrals over each of the E elements. That is, we will consider

$$I = I^{(1)} + I^{(2)} + \cdots + I^{(e)} + \cdots I^{(E)} = \sum_{e=1}^{E} I^{(e)} \tag{12}$$

, where

$$I^{(e)} = \int_{x_i}^{x_j} W(x, \frac{du}{dx})dx - \int_{x_i}^{x_j} P(x)u(x)dx \tag{12*}$$

with $x_i = x_e$ & $x_j = x_{e+1}$

To evaluate this elemental integral, we will need to know or assume something about the displacement distribution within the element. The easiest assumption to make is that it varies linearly over each individual element. And hence, the complete integral of (12) will be a function of the complete set of unknown nodal displacements. That is,

$$I = I(u_1, u_2, \cdots u_i, u_j, \cdots, u_M) \tag{13}$$

This relation can be written in a more condensed form by using matrix notation. We will want to differentiate I with respect to each of the nodal-point displacements to find a minimum of I. To minimize I, we will then set each of these M partial derivatives equal to zero, i.e.,

$$\frac{dI}{d\underline{u}} = \underline{0} \tag{14}$$

Where $\underline{0}$ is a column matrix of M zeros in this case. Instead of working with I over the entire interval of integration, we will break it up into elements by writing it as given by (12). Then (14) becomes

$$\frac{dI}{d\underline{u}} = \frac{d}{d\underline{u}}\sum_{e=1}^{E} I^{(e)} = \sum_{e=1}^{E} \frac{dI^{(e)}}{d\underline{u}} = \underline{0} \tag{15}$$

And we denote

$$\frac{dI^{(e)}}{d\underline{u}} = \underline{D}^{(e)} \frac{dI^{(e)}}{d\underline{u}^{(e)}} \tag{16}$$

, where the displacement matrix $\underline{D}^{(e)}$ is given by

$$\underline{D}^{(e)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dot{1} & \dot{0} \\ 0 & 1 \\ \dot{0} & \dot{0} \end{bmatrix} \tag{16*}$$

Using (16), we get

$$\frac{dI}{d\underline{u}}=\sum_{e=1}^{E}\underline{D}^{(e)}\frac{dI^{(e)}}{d\underline{u}^{(e)}}=\underline{0} \tag{17}$$

Within a typical element (e) we will assume a linear variation of $u^{(e)}$. This equation may be written in matrix notation as the product of a row matrix and a column matrix. That is, we may get, defining the following matrices

$$\underline{P}^{T}=[1\ \ x]$$

$$\underline{P}^{(e)}=\begin{bmatrix}1 & x_i\\ 1 & x_j\end{bmatrix},\ \underline{R}^{(e)}=\underline{P}^{(e)-1}=\frac{1}{x_{ij}}\begin{bmatrix}x_j & -x_i\\ -1 & 1\end{bmatrix}$$

(where we have adopted the shorthand notation, that is, $x_{ij}=x_j-x_i$)

$$u^{(e)}(x)=\underline{P}^{T}\ \underline{R}^{(e)}\ \underline{u}^{(e)} \tag{18}$$

Using the following notation,

$$\frac{d\underline{P}^{T}}{dx}=\underline{P}_x{}^{T}=[0\ \ 1]$$

, we may write

$$\varepsilon^{(e)}(x)=\frac{du^{(e)}}{dx}=\underline{P}_x{}^{T}\ \underline{R}^{(e)}\ \underline{u}^{(e)} \tag{19}$$

We are now ready to consider the integral $I^{(e)}$ in Eq (12). It will be convenient to consider the integral in two parts by

$$I^{(e)}=I_N^{(e)}-I_L^{(e)} \tag{20}$$

, where

$$I_N^{(e)}=\int_{x_i}^{x_j}W(x,\frac{du^{(e)}}{dx})dx \tag{21}$$

$$I_L^{(e)}=\int_{x_i}^{x_j}P(x)u^{(e)}(x)\,dx \tag{22}$$

Equation (20) may be substituted into Eq (15) to give

$$\frac{dI}{d\underline{u}}=\sum_{e=1}^{E}\underline{D}^{(e)}\frac{dI_N^{(e)}}{d\underline{u}^{(e)}}-\sum_{e=1}^{E}\underline{D}^{(e)}\frac{dI_L^{(e)}}{d\underline{u}^{(e)}}=\underline{0} \tag{23}$$

We are now ready to evaluate each of these derivatives. This will be done for a typical element (e), and then the results will be summed up over the entire set of E elements. The results will be as follows. Defining,

$$\underline{k}^{(e)}=\frac{E}{x_{ij}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

: elemental stiffness matrix for linear elasticity

$$\underline{a}^{(e)}\doteq\begin{bmatrix}a_1\\ a_2\end{bmatrix}^{(e)}$$

with

$$a_1^{(e)}=\frac{1}{x_{ij}}\left[\frac{(x_j^2-x_i^2)^2}{2}a+\frac{x_j^4-4x_jx_i^3+3x_i^4}{12}b+\frac{x_j^6-6x_jx_i^5+5x_i^6}{30}d\right]$$

$$a_2^{(e)}=\frac{1}{x_{ij}}\left[\frac{(x_j^2-x_i^2)^2}{2}a+\frac{3x_j^4-4x_ix_j^3+x_i^4}{12}b+\frac{5x_j^6-6x_ix_j^5+x_i^6}{30}d\right]$$

, we get

$$\sum_{e=1}^{E}\underline{D}^{(e)}\underline{d}^{(e)}\varepsilon^{(e)} \tag{24}$$

$$=\sum_{e=1}^{E}\underline{D}^{(e)}\underline{a}^{(e)}(\text{for linear deformation})$$

$$\sum_{e=1}^{E}\underline{D}^{(e)}\underline{d}^{(e)}(\varepsilon^{(e)})^n \tag{24*}$$

$$=\sum_{e=1}^{E}\underline{D}^{(e)}\underline{a}^{(e)}(\text{for nonlinear deformation})$$

,where

$$\underline{K}^{(e)}=\frac{E}{x_{ij}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\ \ \&\ \ \underline{d}^{(e)}=\begin{bmatrix}-k\\ k\end{bmatrix}^{(e)}$$

And hence, for linear deformation,

$$\underline{K}\underline{u}=\underline{f} \tag{25}$$

, where

$\underline{K}=\sum_{e=1}^{E}\underline{D}^{(e)}\underline{K}^{(e)}\ \underline{D}^{(e)T}$: global (assembled) stiffness matrix, and

$\underline{f}=\sum_{e=1}^{E}\underline{D}^{(e)}\underline{a}^{(e)}$: load vector.

For nonlinear deformation, defining,

$$\underline{F}(\underline{u}) = \sum_{e=1}^{E} \underline{D}^{(e)} \underline{d}^{(e)} (\underline{P}_x^T \underline{R}^{(e)} \ \underline{D}^{(e)T} \underline{u})^n$$

$$= \sum_{e=1}^{E} \underline{D}^{(e)} \underline{d}^{(e)} (\underline{\varepsilon}^{(e)})^n$$

, where

$$\varepsilon^{(e)} = \underline{P}_x^T \underline{R}^{(e)} \ \underline{D}^{(e)T} \underline{u} = \frac{1}{x_{ij}} (u_j - u_i)$$

(24)* becomes

$$\underline{F}(\underline{u}) = \underline{f} \qquad (26)$$

Introducing the load parameter λ such that $0 \le \lambda \le 1$, (26) may be written as

$$\underline{F}(\underline{u}) = \lambda \underline{f} \qquad (26)^*$$

This is the equation discussed in the general theory developed in the previous paper(Part 1). And the numerical general solution for this equation was treated in detail also in the previous paper(Part 1).

Now the incremental tangential stiffness matrix $\underline{K}_T$ defined by $\underline{K}_T = \frac{\partial \underline{F}}{\partial \underline{u}}$ becomes for this one dimensional bar, defining

$$\underline{K}_T^{(e)} = \begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} \end{bmatrix} \qquad (27)$$

with $K_{11}^{(e)} = K_{22}^{(e)} = -K_{12}^{(e)} = -K_{21}^{(e)} = \frac{nk}{x_{ij}} (\varepsilon^{(e)})^{n-1}$,

$$\underline{K}_T = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & \cdots & \cdots & 0 \\ K_{21}^{(1)} & K_{22}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} & 0 & \cdots & \cdots & \cdots \\ 0 & K_{21}^{(2)} & K_{22}^{(2)}+K_{11}^{(3)} & K_{12}^{(3)} & 0 & \cdots & \cdots \\ 0 \quad 0 & & K_{21}^{(e-1)} & K_{22}^{(e-1)}+K_{11}^{(e)} & K_{12}^{(e)} & 0 & \cdots \\ & & & K_{21}^{(e)} & K_{22}^{(e)}+K_{11}^{(e+1)} & K_{12}^{(e+1)} & \\ 0 \quad 0 & & & & & & \cdots \\ 0 \quad 0 & \cdots & & & K_{21}^{(E-1)} & K_{22}^{(E-1)}+K_{11}^{(E-1)} & K_{12}^{(E)} \\ 0 \quad 0 & \cdots & & & & K_{21}^{(E)} & K_{22}^{(E)} \end{bmatrix}$$

: (banded symmetric matrix)

Thus we are ready to compute the displacement field using the above scheme, together with the boundary condition, e.g., $u_1 = 0$ or $u(0) = 0$ for this simple bar case.

5. Results and Discussions

Using the modified incremental Newton-Raphson Scheme developed in the previous paper(Part 1), we may compute the displacements of the one dimensional simple bar exerted by the uniform body force. These computed results may be compared with the analytical results. These computed results and the analytical results are shown in Table 1 & 2.

Table 1. Nonlinear deformation analysis results for mild material

Input Data	Node No.	Displacements(m)			
		linear solution		nonlinear solution	
		computed results	analytical results	computed results	analytical results
Length of bar =1.0m	1	0.0	0.0	0.0	0.0
Yield strain =0.2%	2	0.28019803	0.28019803	0.31404375 $\times 10^{3}$	0.31437004 $\times 10^{3}$
	3	0.52899577	0.52899577	0.56164520 $\times 10^{3}$	0.56230315 $\times 10^{3}$
Young's Modulus =1.0 Pa	4	0.74595833	0.74595833	0.74993601 $\times 10^{3}$	0.75093924 $\times 10^{3}$
	5	0.93020833	0.93020833	0.88572827 $\times 10^{3}$	0.88710197 $\times 10^{3}$
Element length =0.125m	6	1.0801970	1.0801970	0.97571469 $\times 10^{3}$	0.97750415 $\times 10^{3}$
	7	1.1933838	1.1933838	0.10269597 $\times 10^{4}$	0.10292426 $\times 10^{4}$
Body force : P(x)= 2.0+ $0.5x^2$+x^4 N/m³	8	1.2658243	1.2658243	0.10479502 $\times 10^{4}$	0.10508574 $\times 10^{4}$
n = 0.5	9	1.2916667	1.2916667	0.10506215 $\times 10^{4}$	0.10543710 $\times 10^{4}$

Table 2. Nonlinear deformation analysis results for very stiff material

Input Data	Node No.	Displacements(m)			
		linear solution		nonlinear solution	
		computed results	analytical results	computed results	analytical results
	1	0.0	0.0	0.0	0.0
	2	0.20354885 $\times 10^{-4}$	0.20354886 $\times 10^{-4}$	0.31074100 $\times 10^{-5}$	0.31090751 $\times 10^{-5}$
	3	0.39675837 $\times 10^{-4}$	0.39675837 $\times 10^{-4}$	0.5907154 $\times 10^{-5}$	0.59084910 $\times 10^{-5}$
Length of bar =1.0m	4	0.57954920 $\times 10^{-4}$	0.57954920 $\times 10^{-4}$	0.84130904 $\times 10^{-5}$	0.84151134 $\times 10^{-5}$
	5	0.75178232 $\times 10^{-4}$	0.75178232 $\times 10^{-4}$	0.10637909 $\times 10^{-4}$	0.10640641 $\times 10^{-4}$
Yield strain =0.2%	6	0.91324910 $\times 10^{-4}$	0.91324910 $\times 10^{-4}$	0.12593273 $\times 10^{-4}$	0.12596746 $\times 10^{-4}$
Young's Modulus	7	0.10636571 $\times 10^{-3}$	0.10636571 $\times 10^{-3}$	0.14289966 $\times 10^{-4}$	0.14294229 $\times 10^{-4}$
=0.194×10[illegible] Pa	8	0.12026122 $\times 10^{-3}$	0.12026122 $\times 10^{-3}$	0.15738105 $\times 10^{-4}$	0.15743221 $\times 10^{-4}$
Element length	9	0.13295963 $\times 10^{-3}$	0.13295963 $\times 10^{-3}$	0.16947476 $\times 10^{-4}$	0.16753534 $\times 10^{-4}$
=0.0667m	10	0.14439411 $\times 10^{-3}$	0.14439411 $\times 10^{-3}$	0.17928082 $\times 10^{-4}$	0.17935200 $\times 10^{-4}$
Body force : P(x)=(0.45+	11	0.15447985 $\times 10^{-3}$	0.15447985 $\times 10^{-3}$	0.18690997 $\times 10^{-4}$	0.18699336 $\times 10^{-4}$
$0.25x^2$+$0.37x^4$) $\times 10^4$ N/m³	12	0.16311057 $\times 10^{-3}$	0.16311057 $\times 10^{-3}$	0.19249667 $\times 10^{-4}$	0.19259442 $\times 10^{-4}$
n = 0.5	13	0.17015475 $\times 10^{-3}$	0.17015475 $\times 10^{-3}$	0.19621822 $\times 10^{-4}$	0.19633322 $\times 10^{-4}$
	14	0.17545143 $\times 10^{-3}$	0.17545143 $\times 10^{-3}$	0.19832232 $\times 10^{-4}$	0.19845846 $\times 10^{-4}$
	15	0.17880551 $\times 10^{-3}$	0.17880551 $\times 10^{-3}$	0.19916607 $\times 10^{-4}$	0.19932851 $\times 10^{-4}$
	16	0.17996282 $\times 10^{-3}$	0.17996282 $\times 10^{-3}$	0.19927002 $\times 10^{-4}$	0.19946575 $\times 10^{-4}$

The solid material in Table 2 is very stiff (Young's modulus = 0.194×10^8 Pa). Hence, the deformation is expected to occur in the linear small deformation region. Displacement in Table 2 is exactly such results as expected. However, material in Table 1 is not stiff (Young's modulus = 1.0 Pa). Therefore, deformation occurs in the nonlinear large deformation region as shown in Table 1. Hence, the linear solutions are meaningless in this case. As shown, in Tables, computed results exactly agree with analytical results. And so, the modified incremental Newton-Raphson Scheme applied here is very successful.

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