논문
한국정밀공학회지 제15권 제12호 (1998년 12월)
Journal of the Korean Society of Precision Engineering, Vol. 15, No. 12, December 1998.

고체 평판의 비선형 순수굽힘변형에 대한 수학적 정해

권 영 주*

A Closed Form Nonlinear Solution for Large Pure Bending Deformation of Solid Plate

Youngjoo Kwon*

ABSTRACT

압축성 초탄성 평판의 순수굽힘에 대한 비선형 변형해석의 수학적 정해가 본 논문에 구해져 있다. 이차원 평면 변형도 상태가 해석을 위하여 가정되었으며, 비선형 순수굽힘 변형해석결과는 고전적인 선형 순수굽힘 변형해석결과와 비교되었다. 고전적인 선형굽힘 결과와는 다르게 비선형 순수굽힘 상태에서는 반경방향응력은 영(零)이 아니며 또한 각방향응력도 선형 상태가 아닌 것으로 규명되었다.

Key Words :pure bending(순수굽힘), hyperelastic solid plate(초탄성 고체평판), plane strain state(평면변형도 상태), principal stretch ratio(주신장률), principal stress(주응력), strain energy density(변형도 에너지 밀도)

1. Introduction

In this paper we treated the very classical problem, i.e., "pure bending of solid plate". Even though the pure bending problem is classical, and very important in its applications, e.g., in the application to the metal forming problem[1][5], its complete analytical nonlinear solution is not easy to obtain. Hence only the linear small deformation theory is available in the engineering textbook[1] and the numerical finite element method is currently the only tool for its analysis[2][3][4][5].

However, in this paper we obtained the closed form analytical solution for the nonlinear pure bending deformation of the isotropic hyperelastic compressible solid plate in the plane strain state. The closed form solution obtained in this paper is very important for the large bending deformation, since it is expected that even though the bending angle exceeds small degrees the linear solution is not correct. The contribution of this closed form nonlinear solution can be found in cold forming processes of metal sheets. More specifically, this solution is very useful in the investigation of the onset of bifurcation instabilities which occur usually in the highly nonlinear large pure bending

* 홍익대학교 기계설계학과

deformation of a ductile metal sheet[2].

Hence the solution process treated in this paper is very helpful for the accurate structural analysis of solids. The numerical values of analytical solutions are computed for the bending angle(ϕ) of 30 degrees, the Young's modulus(E) of 1.0 Pa and Poisson's ratio(ν) of 0.25, using the Simpson's rule[6].

2. Problem Description

A solid plate of initial thickness 2H is subject to plane strain pure bending. The solid plate is assumed to be made of a compressible isotropic hyperelastic material with the following strain energy density

$$W(I_c, II_c) = A\,I_c^{\,2} + B\,I_c - (4A+B)II_c - B \qquad (1)$$

where $I_c = \mathrm{tr}C$, $II_c = \mathrm{Det}\ \mathbf{C}$ ($\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$:the second order right Cauchy Green tensor, $\mathbf{F} = \mathbf{I} + \mathbf{u}\nabla$: the second order deformation gradient tensor, $\mathbf{u}$: the displacement vector), A and B are constants.

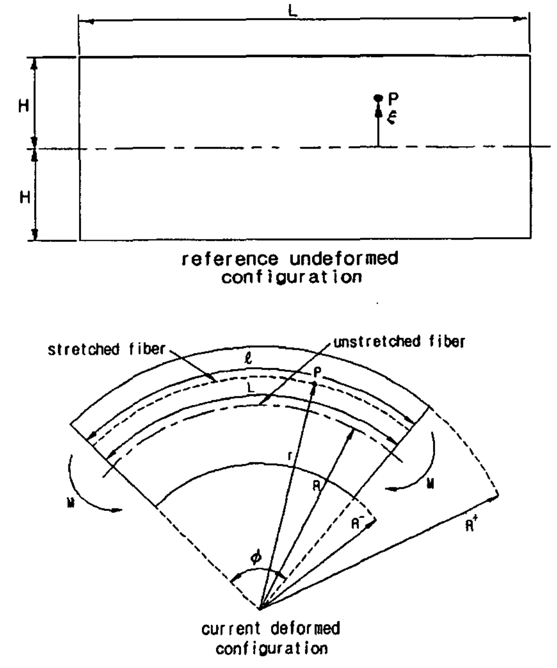


Fig. 1 Pure Bending of Solid Plate in Plane Strain State

2.1 Kinematics of Pure Bending

The pure bending deformation is assumed to occur in the plane strain state. Hence the deformation state is two dimensional. If λ_1 and λ_2 are principal stretch ratios in the radial and the angular direction respectively, then the following relations are satisfied

$$\lambda_1 = \frac{dr}{d\xi},\ \lambda_2 = \frac{\ell}{L} = \frac{r}{R} = \kappa r \qquad (2)$$

where, κ is the curvature of unstrecthed fiber, and $R = \frac{1}{\kappa}$ is the radius of curvature. ξ is the initial position of the material point P and r is the current position of the material point P. ℓ is the length of the current stretched fiber and L is the length of the unstretched fiber as shown in the Fig 1. ϕ is the bending angle and M is the bending monent.

2.2 Principal Stresses

If r and θ are the coordinates in the radial and the angular direction respectively, r and θ are the principal directions. Hence principal stresses are $\sigma_1 = \sigma_{rr}$, $\sigma_2 = \sigma_{\theta\theta}$, since $\sigma_{r\theta} = 0$. For the isotropic hyperelastic solid, the principal Cauchy stresses are

$$\sigma_i = \frac{\tau_i}{J} = \frac{1}{J}\lambda_i \frac{dW}{d\lambda_i}(\text{no sum}) \qquad (3)$$

, where $J = \lambda_1\ \lambda_2$ in two dimensional space.

Thus

$$\sigma_1 = \frac{1}{\lambda_2}\frac{dW}{d\lambda_1},\ \sigma_2 = \frac{1}{\lambda_1}\frac{dW}{d\lambda_2} \qquad (4)$$

2.3 The Equilibrium Equation

Since it is assumed that $\frac{\partial}{\partial\theta}(\ )=0\ \&\ \sigma_{r\theta}=0$, the only nontrivial equilibrium equation is along the radial direction,

$$i.e.,\ \frac{d\sigma_1}{dr}+\frac{\sigma_1-\sigma_2}{r}=0 \qquad (5)$$

2.4 Boundary Conditions

The radial stress is assumed to be zero on the upper and the lower surfaces, or the traction is free,

$$\text{i.e.,}\ \sigma_1=0\ \text{at}\ r=R^-\ \&\ r=R^+ \qquad (6)$$

And on the lateral surfaces,

$$\int_{R^-}^{R^+}\sigma_2\cdot r\ dr=M\ (M: bending\ moment) \qquad (7)$$

$$\&\ \int_{R^-}^{R^+}\sigma_2 dr=0(no:\ axial\ force) \qquad (8)$$

3. The Analytical Solution

3.1 The Solution in terms of Principal Stretch Ratios

Using the relation $I_c=\text{tr}\ \mathbf{C}=\lambda_1^2+\lambda_2^2$, II_c = Det $\mathbf{C}=\lambda_1^2\ \lambda_2^2$, $J=\text{Det}\ \mathbf{F}=\lambda_1\ \lambda_2$, the strain energy density $W(I_c,\ II_c)$ of (1) becomes

$$W(I_c,\ II_c)=A(\lambda_1^2+\lambda_2^2)^2+B(\lambda_1^2+\lambda_2^2)-(4A+B)\lambda_1^2\ \lambda_2^2-B \qquad (9)$$

Using (8), the stresses given by (4) become

$$\sigma_1=\sigma_{rr}=2\frac{\lambda_1}{\lambda_2}\{2A(\lambda_1^2-\lambda_2^2)+B(1-\lambda_2^2)\}$$

$$\sigma_2=\sigma_{\theta\theta}=2\frac{\lambda_2}{\lambda_1}\{2A(\lambda_2^2-\lambda_1^2)+B(1-\lambda_1^2)\} \qquad (10)$$

Thus, if we know the principal stretch ratios λ_1 and λ_2, we can obtain the solutions σ_1, σ_2 from (10).

3.2 The Analytical Closed Form Solution

Noting the relations (2) and (10), it is manifest that σ_1 and σ_2 are functions of r and $r'=\frac{dr}{d\xi}$.

And hence σ_1 and σ_2 are functions of only ξ. Inserting the relation (2) into (10), and then the resulted σ_1 and σ_2 into the equilibrium equation (5), we may obtain the equation of r in terms of ξ. Solving this resulted differential equation, we may obtain σ_1, σ_2 in terms of ξ and κ. The radius curvature κ is obtained by prescribing the bending moment M or the bending angle ϕ, i.e., by $L=R\phi=\frac{\phi}{\kappa}$

$$\text{or}\quad \kappa=\frac{\phi}{L} \qquad (11)$$

The curvature κ is also related to M by the equation (7), where $R^-\ \&\ R^+$ are evaluated by the condition $\sigma_1(R^-)=\sigma_1(R^+)=0$,

$$\text{or}\quad R^-=r(-H),\ R^+=r(H) \qquad (12)$$

Since σ_2 in (7) is a function of κ, (7) is the relation between κ & M. From the equation (7), we may obtain κ or ϕ for the given M. After some tedious algebra, the differential equation for $r(\xi)$, noting $r'(\xi)=\frac{dr}{d\xi}$ and $(\ )'=\frac{d}{d\xi}(\ )$, is obtained as

$$3A((r')^4)'+B((r')^2)'-(2A+B)\{(\kappa r)^2(r')^2\}'-A((\kappa r)^4)'-B((\kappa r)^2)'=0 \qquad (13)$$

The above equation is a highly nonlinear differential equation for $r(\xi)$. After quite some difficult integrating of the equation (13), we may obtain the following implicit closed form analytical solution for $r(\xi)$

$$\kappa(\xi+H)=\frac{1}{2}\int_{-\mu}^{y}[f(y,\ \mu)]^{\frac{1}{2}}dy \tag{14}$$

with $y=(\kappa r)^2-1,\ -\mu\le y\le\mu$. The μ is evaluated by the relation

$$4\kappa H=\int_{-\mu}^{\mu}[f(y,\ \mu)]^{\frac{1}{2}}dy \tag{15}$$

The $f(y,\ \mu)$ in (14) and (15) is given by the equation

$$f(y,\ \mu)=\frac{(y+1)\left\{2A(y+1)+By+\sqrt{4[2A(y+1)+By]^2-3B(4A+B)(y^2-\mu^2)}\right\}}{6A} \tag{16}$$

And then the principal stretch ratios λ_1 and λ_2 are given by

$$\lambda_1=\left[\frac{1}{y+1}f(y,\ \mu)\right]^{\frac{1}{2}},\quad \lambda_2=(y+1)^{\frac{1}{2}} \tag{17}$$

With above λ_1 and λ_2, we may compute the stresses σ_1 and σ_2 by the equation (10).

Using the above implicit closed form solution (14)~(17), we may compute σ_1 and σ_2 given by (10) by the following calculation procedure numerically.

Step 1. for given κ (or M) & H, we may evaluate μ by (15) numerically.
Step 2. for given κ (or M), ξ & H and μ obtained in Step 1, we may evaluate y by (14) numerically.
Step 3. with y & μ obtained in Step 1 & Step 2, we may evaluate λ_1, λ_2 by (17).
Step 4. with λ_1, λ_2 obtained in Step 3, we may evaluate σ_1 and σ_2 by (10).

That is, taking the above procedure, for given ξ, κ(or M) & H we may obtain the analytical solution σ_1 & σ_2. We can evaluate the integral in (14) and (15) by using the numerical method, e.g., Simpson's method & Romberg integration[6].

4. Computation Results and Discussions

For the compressible Neohooken material, we may compute the analytical solutions given in the previous section numerically. For the Neohookean material the material constants A and B are related to E(Young's Modulus) and ν(Poisson's Ratio) by

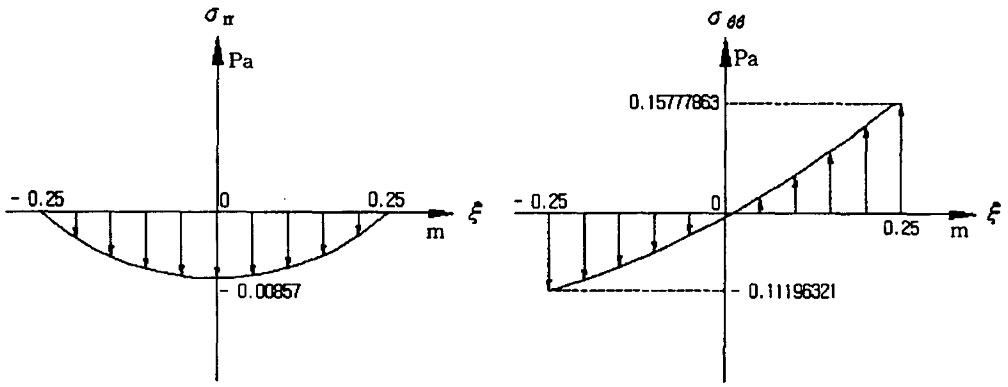
$$A=\frac{1}{8}\cdot\frac{E}{1+\nu}\cdot\frac{1-\nu}{1-2\nu},\quad B=-\frac{1}{4}\cdot\frac{E}{1+\nu}\cdot\frac{1}{1-2\nu} \tag{18}$$

Using thd above relation, the analytical results, computed numerically for $E=1.0$ Pa, and $\nu=0.25$, are given in the Table 1 and Fig. 2. For the guarantee of accuracy, the eight effective numbers are used in the numerical computation.

From the table and figure, we may note that the radial stress is not zero for the large pure bending deformation even though its absolute value is still smaller than the absolute value of the angular stress, while the radial stress is zero, i.e., $\sigma_{\theta\theta}=E\xi/R$, $\sigma_{rr}=0$ in the linear small deformation theory[1]. And the similar results are obtained in the analysis of bifurcation phenomena in pure bending of the incompressible plate[2]. The fact that the radial stress is not zero for the large pure bending deformation physically means that while the bending deformation becomes larger and larger the radial deformation towards the center of radius of curvature is required more and more and this is due to the negative nonzero radial stress, on the other hand the radial deformation in the linear bending theory is very small negligibly which does not need the radial stress. It is interesting that the unstretcched fiber ($\sigma_{\theta\theta}=0$) is higher than the center line fiber($\xi=0$) as shown in Fig. 2. This phenomenon means that

Table 1 Computed Analytical Results for Nonlinear Pure Bending

Input Data	Plate Length(L)	1.0 m	
	Bending Angle(ϕ)	30 °	
	Half Heigth(H)	0.25 m	
	Young's Modulus(E)	1.0 N/m^2	
	Poisson's Ratio(ν)	0.25	
Computed Curvature (κ)		0.52359867	
Bending Moment (M)		0.0052829826 N · m	
reference coordinate (ξ) (m)	current coordinate (r) (m)	radial stress(σ_1 or σ_{rr}) (N/m^2)	angular stress (σ_2 *or* $\sigma_{\theta\theta}$) (N/m^2)
-0.25	1.6507755	$-0.36781695\times10^{-15}$	-0.11196321
-0.22	1.6819258	-0.19854120×10^{-2}	-0.10229635
-0.19	1.7128982	-0.37059077×10^{-2}	-0.91830287×10^{-1}
-0.16	1.7436954	-0.51640001×10^{-2}	-0.80542658×10^{-1}
-0.13	1.7743166	-0.63615437×10^{-2}	-0.68412414×10^{-1}
-0.10	1.8047647	-0.73001148×10^{-2}	-0.55417403×10^{-1}
-0.70	1.8350419	-0.79808465×10^{-2}	-0.41535818×10^{-1}
-0.40	1.8651522	-0.84045067×10^{-2}	-0.26745521×10^{-1}
-0.10	1.8950961	-0.85714728×10^{-2}	-0.11026581×10^{-1}
0.0	1.9050417	-0.85701141×10^{-2}	-0.55765757×10^{-2}
0.10	1.9149688	-0.85402454×10^{-2}	-0.20680204×10^{-4}
0.40	1.9446468	-0.82794596×10^{-2}	0.17293534×10^{-1}
0.70	1.9741711	-0.77615394×10^{-2}	0.35591346×10^{-1}
0.10	2.0035448	-0.69858304×10^{-2}	0.54890428×10^{-1}
0.13	2.0327752	-0.59513382×10^{-2}	0.75209520×10^{-1}
0.16	2.0618687	-0.46568104×10^{-2}	0.96565466×10^{-1}
0.19	2.0908317	-0.31007793×10^{-2}	0.11897308
0.22	2.1196705	-0.12815764×10^{-2}	0.14244521
0.25	2.1377718	-0.3084676×10^{-15}	0.15777863



<radial stress> <angular stress>

Fig. 2 Stresses in Nonlinear Pure Bending

while the bending deformation becomes larger and larger the absolute value of angular stress in the upper part of the plate (above the unstretched fiber) becomes larger than the absolute value of angular stress in the lower part of the plate (below the unstretched fiber).

References

1. S. H. Crandall, N. C. Dahl and T. J. Lardner, "An Introduction to the Mechanics of Solids," McGraw-Hill Kogakusha, Ltd., pp 423~432, 1972.
2. N. Triantafyllidis, "Bifurcation Phenomena in Pure Bending," J. Mech. Phys. Solids Vol. 28, pp. 221~245, 1980.
3. R. Abeyaratne and N. Triantafyllidis, "On the Emergence of Shear Bands in Plane Strain," Materials Res. Lab. Rep. MRL E-125. Brown University, July, 1980.
4. J. W. Hutchinson and V. Tvergaard, "Shear Band Formation in Plane Strain," Int. J. Solids Structures, 17, p. 451, 1981.
5. N. Triantafyllidis, A. Needleman, "On the Development of Shear Bands in Pure Bending," Int. J. Solids Structures. Vol. 18. No. 2, pp. 121~138, 1982.
6. S. D. Conte, C. de Boor, "Elementary Numerial Analysis," Top Bublishing Co. Ltd, pp. 237~288, 1972.