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고체구조물의 비선형변형 수치해석에 대한 이론적 고찰(1)
-일반이론-

권 영 주*

A Study on the Numerical Technique for the Nonlinear Deformation Analysis of Solid Structures(1)
-General Theory Development-

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ABSTRACT

본 논문에서는 비선형 고체역학 이론에 대하여 특히 시간에 무관한 변형을 하는 초탄성 및 탄소성고체물질의 비선형 변형이론에 대하여 철저한 분석을 수행하였다. 특히 비선형 변형의 해석방법론에 대하여 특별한 관심을 가지고 분석하였다. 비선형 변형해석 방법론으로 널리 논의되고 있는 증분뉴튼랩슨 방법에 대하여 수정된 개념을 제시하여 비선형 변형 해석의 정확성을 향상시켰다.

Key Words : hyperelastic material(초탄성물질), hypoelastic material(탄소성물질), Lagrangian strain (라그란지안 변형도), Second Kirchhoff stress(두번째 키르히호프 응력), strain energy density (변형도에너지밀도), principle of virtual work(가상일원리), incremental Newton-Raphson method(증분뉴튼 랩슨방법)

1. Introduction

The nonlinear deformation theory of solids is very difficult to treat. Usually the nonlinear solids are classified as hyperelastic material, hypoelastic material, viscoelastic material and viscoplastic material. The treatment difficulties arise from the material nonlinearity and the geometric nonlinearity. Hence, the mathematical analysis is very difficult. Many engineers tried to devote to set up the nicely perfect mathematical nonlinear solid mechanics theory and to solve the resulted boundary value problems analytically. However, very few analytical solutions exist now. Modern mathematical nonlinear solid mechanics theories have been refined by some leading engi-

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neers like Rivlin[1], Truesdell[2], Pipkin[1] and recently Hill[3][4], Hutchinson[5], Budianski[6], Ogden[7], Rice[8], Needleman[9] et al. These engineers could have had the great successful achievements in the nonlinear solid mechanics analysis due to the numerical technique development[10] for the analysis. This numerical technique has been developed greatly by the appearance of computer. The topic of this paper is also about the numerical technique development for the nonlinear deformation analysis of solids. This paper proposes the modification of the usual incremental Newton-Raphson Scheme for the accuracy enhancement of the nonlinear finite element solution, especially for the hyperelastic and hypoelastic solids. The forthcoming paper (Part 2) will treat the application of the general theory developed here to a simple solid structure.

2. Kinematics of Deformation of Solids

2.1 Deformation Gradient, Polar Decomposition, Principal Stretches

In describing the deformation of solids, the Lagrangian material description is usually adopted. In this description, the independent variables are the position of material points in the reference configuration $\underline{P}$ and time t. And further, for this description two are the most widely used descriptions of the solid motion. The one is the convected coordinate description, and the other is the cartesian coordinate description. In convected coordinate description we follow the motion of the particle with coordinates $\{\theta^i\}$. If $\underline{r}(\theta^i, t)$ is the position vector of this point (say M) at time t, we introduce the following notation

$\underline{P} \equiv \underline{r}(\theta^i, 0) = x^j(\theta^i, 0)\ \underline{e}_j \equiv X^j(\theta^i)\underline{e}_j$ $\quad \underline{P}$: reference position vector

$\underline{p} \equiv \underline{r}(\theta^i, t) = x^j(\theta^i, t)\ \underline{e}_j$ $\quad \underline{p}$: current position vector

Since in any realistic motion the volume of a material element cannot be reduced to zero at any time the mapping from $\underline{P}$ to $\underline{p}$ will have to be always one-to-one. In view of the right handed coordinate system used here this condition is expressed as

$$J \equiv \det[\partial x_i / \partial X_j] = \det[\partial x_i / \partial \theta^k] / \det[\partial X_j / \partial \theta^\ell] > 0 \quad \forall t \geq 0$$

The covariant basis vectors in both configurations(reference undeformed configuration and current deformed configuration) are

$$\underline{g}_i = \partial \underline{p} / \partial \theta^i \text{ (deformed configuration)}$$
$$\underline{G}_i = \partial \underline{P} / \partial \theta^i \text{ (undeformed configuration)}$$

Consequently the metric tensor components are

$$g_{ij} = \underline{g}_i \cdot \underline{g}_j \text{ and } G_{ij} = \underline{G}_i \cdot \underline{G}_j$$

The deformation gradient tensor $\underline{F}$ is a second rank tensor relating the undeformed line element $\partial \underline{P}$ to the deformed one $\partial \underline{p}$

$$d\underline{p} = \underline{F} \cdot d\underline{P} = (\underline{p}\hat{\underline{\nabla}}) \cdot d\underline{P} \ \ i.e.,\ \underline{F} \equiv \underline{p}\ \hat{\underline{\nabla}}(with\ \hat{\underline{\nabla}} = \frac{\partial()}{\partial \theta^i}\ \underline{G}^i)$$

In convected coordinates $\underline{F}$ takes the following form

$$\underline{F} = \underline{p}(\frac{\partial}{\partial \theta^i})\ \underline{G}^i = (\frac{\partial \underline{p}}{\partial \theta^i})\ \underline{G}^i = \underline{g}_i\ \underline{G}^i$$

,where $\underline{g}^i$ and $\underline{G}^i$ are the contravariant basis vectors.

In cartesian coordinates, $\underline{G}_i = \underline{e}_i = \underline{G}^i$ and $\theta^i = X^i$, thus $\underline{g}_i = \partial x_j / \partial X_i \underline{e}_j$ and thus $\underline{F} = \frac{\partial x_i}{\partial X_j}\ \underline{e}_i\ \underline{e}_j$.

The tensor $\underline{F}$ characterises the deformation of the body around a small neighborhood of a given point. The linear transformation $\underline{F}$ from $d\underline{P}$ to

$d\underline{p}$ can be considered to be realised in two steps. First a transformation $\underline{U}$ stretches the material without rotation. Subsequently, a rigid body rotation $\underline{R}$ rotates the material without any further stretching. One could have considered the case where the rotation $\underline{R}$ takes place first and then the stretch $\underline{V}$ follows, i.e., $\underline{F}=\underline{U}\cdot\underline{R}$ or $\underline{F}=\underline{R}\cdot\underline{V}$. This is the so-called polar decomposition. The principal values of $\underline{F}$ are called principal stretches or stretch ratios of the material at the point in question.

2.2 Strain Measures

If we are interested in the change of length of an infinitesimal line element initially $d\underline{P}$ and currently $d\underline{p}$, then $\|d\underline{p}\|^2-\|d\underline{P}\|^2=d\underline{p}\cdot d\underline{p}-d\underline{P}\cdot d\underline{P}=d\underline{P}\cdot(\underline{F}^T\cdot\underline{F}-\underline{I})\cdot d\underline{P}\equiv 2d\underline{P}\cdot\underline{E}\cdot d\underline{P}$ with $\underline{E}=\frac{1}{2}(\underline{F}^T\cdot\underline{F}-\underline{I})$, $\underline{I}$: identity tensor. $\underline{E}$ is the so-called Lagrangian strain tensor. If we are interested in the elongation (or contraction) ratio Λ of the line element in equation initially at direction $\underline{N}$ and currently along direction $\underline{n}$, we have $\Lambda\equiv\|d\underline{p}\|/\|d\underline{P}\|$ and $\Lambda^2\equiv\|d\underline{p}\|^2/\|d\underline{P}\|^2\equiv\underline{N}\cdot\underline{C}\cdot\underline{N}$, where $\underline{C}=\underline{F}^T\cdot\underline{F}$ is called the right Cauchy-Green tensor. A useful expression of $\underline{E}$ in terms of the displacement $\underline{u}(\underline{u}=\underline{p}-\underline{P})$ is $\underline{E}=\frac{1}{2}[\underline{u}\ \underline{\nabla}+\underline{\nabla}\ \underline{u}+(\underline{\nabla}\ \underline{u})\cdot(\underline{u}\ \underline{\nabla})]$. The linearized version of $\underline{E}$ is called the small infinitesimal strain tensor $\underline{e}=\frac{1}{2}[\underline{u}\ \underline{\nabla}+\underline{\nabla}\ \underline{u}]$. We should emphasize here that the Lagrangian strain $\underline{E}$ is not the only possible strain measure that one could define. Other strain measures are, e.g., Eulerian or Almansi strain, Biot strain and Logarithmic strain etc.

2.3 Velocity Gradients and Strain Rates

The velocity gradients $\underline{L}$ is definied to be $d\underline{v}=\underline{L}\cdot d\underline{p}$ with $\underline{v}=\dot{\underline{p}}$ (velocity) or $\underline{L}=\underline{v}\ \underline{\nabla}$ $(\underline{\nabla}=\frac{\partial}{\partial\theta^i}\underline{g}^i)=\dot{\underline{g}}_i\ \underline{g}^i=\dot{\underline{F}}\cdot\underline{F}^{-1}$. In cartesian coordinates, one obtains

$$\underline{L}=\frac{\partial\dot{x}_i}{\partial X_k}\cdot\frac{\partial X_k}{\partial x_j}\ \underline{e}_i\ \underline{e}_j=\frac{\partial\dot{x}_i}{\partial x_j}\underline{e}_i\ \underline{e}_j=\frac{\partial v_i}{\partial x_j}\underline{e}_i\ \underline{e}_j$$

($v_i=\dot{x}_i$: velocity components).

The symmetric part of $\underline{L}$ is called the rate of deformation tensor $\underline{D}=\frac{1}{2}(\underline{L}+\underline{L}^T)=D_{ij}\ \underline{g}^i\ \underline{g}^j$ $(D_{ij}\equiv\frac{1}{2}\ \dot{g}_{ij})$. The antisymmetric part of $\underline{L}$ is called the spin tensor $\underline{W}=\frac{1}{2}(\underline{L}-\underline{L}^T)$. And then, the strain rate $\dot{\underline{E}}$ is given as $\dot{\underline{E}}=\underline{F}^T\cdot\underline{D}\cdot\underline{F}$

3. Equations Governing the Motion of Solid Structures

3.1 Conservation Laws in a Solid Structure

By D we will denote an arbitrary subregion of our solid structure of volume V and by ∂D we denote the boundary of the subregion. Always D contains the same material points. At a point with current position vector $\underline{p}$, the solid's density is $\rho=\rho(\underline{p},t)$, the body force is $\underline{b}(\underline{p},t)$ (the force per unit mass of the solid), and $\underline{t}$ denotes the traction vector acting on point $\underline{p}$ of ∂D, i.e., $\underline{t}$ is the contact force per unit area, acting on a small area $\Delta a(\ of\ \partial D)$ with unit outward normal $\underline{n}$. By $\underline{v}(\underline{p},t)$ we denote the velocity of a material point currently at $\underline{p}$, and ε is the internal energy per unit mass

3.1.1 Conservation of Mass

The mass enclosed in the material volume D is constant. This gives us the following continuity equation.

$$\rho J=\rho_o \text{ or } \partial\rho/\partial t+\underline{\nabla}\cdot(\rho\underline{v})=0(\rho_o:\text{ initial density})\quad(1)$$

3.1.2 Balance of Linear Momentum

The balance of linear momentum gives us the following equation of motion.

$$\rho \underline{a} = \rho \dot{\underline{v}} = \underline{\nabla} \cdot \underline{\sigma} + \rho \underline{b} \quad (\underline{a} = \dot{\underline{v}} \ : \ \text{acceleration}) \quad (2)$$

In convected curvilinear coordinates, the equation (2) becomes

$$\rho a^j = \sigma^{ij}\big|_i + \rho b^j \quad (3)$$

, where $\sigma^{ij}|_i$ is the covariant derivative with respect to θ^i of σ^{ij}. In cartesian coordiantes, the equation (2) becomes

$$\rho a_j = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j \quad (4)$$

3.1.3 Balance of Angular Momentum

The balance of angular momentum gives us the following symmetry condition

$$\underline{\sigma}^T = \underline{\sigma} \quad (5)$$

3.1.4. Energy Balance

The energy balance means that the change of energy in volume D is due to the power input by the body force and the contact force plus the heat input through the boudary surface plus the heat generated inside the volume. This energy balance gives us the following energy equation

$$\rho \dot{\varepsilon} = -\underline{\nabla} \cdot \underline{q} + \rho s + \underline{\sigma} : \underline{D} \quad (6)$$

, where $\underline{D}$ is the rate of deformation tensor, s is the heat generated per unit mass and $\underline{q} = -k\underline{\nabla}T$ (heat flux through the surface).

4. Alterantive Stress Measures

Especially in solid mechanics, we often work with the reference configuration and stress measures other than the Cauchy stress are more convenient, recalling the Cauchy stress tensor $\underline{\sigma}$ such that $\underline{t}(\underline{p}, t; \underline{n}) = \underline{n} \cdot \underline{\sigma}$ (the current traction vector). From the definition of the traction vector $\underline{t} = \frac{d\underline{f}_c}{da}$ ($\underline{f}_c$: contact force acting on the current surface), $d\underline{f}_c = \underline{t}\ da \equiv \underline{T}\ dA$ with $\underline{T} = d\underline{f}_c / dA$ (the pseudotraction vector), where A is the reference area with the relation $\underline{n}\ da = J(\underline{F}^T)^{-1} \cdot \underline{N} dA$ (Nanson's Relation) and $\underline{N}$ is the outward normal vector on the refrerence area. Using the above notations, we may define the First Piola Kirchhoff or nominal stress tensor $\underline{\Pi}$ to be

$$\underline{\Pi} = J\underline{F}^{-1} \cdot \underline{\sigma} \ \text{with} \ \underline{T} = \underline{N} \cdot \underline{\Pi} \quad (7)$$

Now we may define an element of force in the reference configuration $d\underline{F}_c$ related to $d\underline{f}_c$ by $\underline{F} \cdot d\underline{F}_c = d\underline{f}_c$. Then we may define the reference pseudotraction vector $\hat{\underline{T}}$ to be

$$\hat{\underline{T}} = \frac{d\underline{F}_c}{dA} = \underline{N} \cdot \underline{\Pi} \cdot (\underline{F}^{-1})^T$$

And then, the symmetric stress tensor, the so-called Second Piola Kirchhoff stress tensor, is defined as

$$\underline{S} = \underline{\Pi} \cdot (\underline{F}^{-1})^T = J\underline{F}^{-1} \cdot \underline{\sigma} \cdot (\underline{F}^{-1})^T \ \text{with} \ \hat{\underline{T}} = \underline{N} \cdot \underline{S} \quad (8)$$

Another convenient stress measure is introduced by the Kirchhoff stress $\underline{\tau}$ defined as

$$\underline{\tau} = J\underline{\sigma} \quad (9)$$

5. Principle of Virtual Work

We are now in a position to introduce the principle of virtual work. In spite of its resemblance to the energy equation we must emphasize that the principle of virtual work has nothing to do with energy. It is simply a different way of writing the linear momentum equations in mathematical terminology. The principle of virtual work is the weak formulation of the equation of motion of (2) (linear momentum balance). Assume that $\delta\underline{u} = \delta\underline{u}(\underline{p})$ is a continuous differentiable vector field such that $\delta\underline{u} = 0$ if $\underline{p} \in \partial V_u$ where ∂V_u is the part of the surface of the body on which the displacement $\underline{u}$ is prescribed. Also let $\underline{t}^*$ be the known traction vector in the part of the surface ∂V_t where the loads are specified (i.e., $\partial V = \partial V_u \cup \partial V_t$). Then the principle of virtual work in the reference configuration, using the symmetric Second Piola Kirchhoff stress $\underline{S}$, is

$$\int_{\partial V_t} (\underline{T}^* \cdot \delta\underline{u}) dA + \int_V \{\rho_o(\underline{b} - \underline{a}) \cdot \delta\underline{u}\} dV = \int_V (\underline{S} \cdot\cdot\ \delta\underline{E}) dV \tag{10}$$

,where $\underline{T}^*$ is the pseudotraction vector corresponding to $\underline{t}^*$ and "$\cdot\cdot$" means the double dot product which operates as, e.g., $\underline{T} \cdot\cdot \underline{S} = (T^{ijk}\ \underline{g}_i\ \underline{g}_j\ \underline{g}_k) \cdot\cdot\ (S_{\ell m} \underline{g}^\ell\ \underline{g}^m) = T^{ijk}\ S_{\ell m}(\underline{g}_k \cdot \underline{g}^\ell)(\underline{g}_j \cdot \underline{g}^m)\underline{g}_i = T^{ijk} S_{kj} \underline{g}_i$. It is also easily seen that the equilibrium equations in terms of $\underline{S}$ can be casted in the form

$$\hat{\underline{\nabla}} \cdot (\underline{S} \cdot \underline{F}^T) + \rho_o \underline{b} = \rho_o \underline{a} = \rho_o \frac{d^2 \underline{u}}{dt^2} \ \ in\ V,$$

$$\hat{\underline{T}}^* = \underline{N} \cdot \underline{S} \text{ on } \partial V_t \tag{11}$$

6. Constitutive Equations for the Solid Structure

6.1 General Considerations-Objectivity Criteria

Three are the fundamental postulates assumed to be valid for any constitutive theory of purely mechanical phenomena in a solid structure.

a) Principle of Determinism

The stress in a body at time t is determined by the history of deformation in that body.

b) Principle of Local Action

In determining the stress at a given particle with reference position $\underline{P}$ only the motion in a small neighborhood of $\underline{P}$ will affect the stress.

c) Principle of Material Frame Indifference

This principle states that the response of the material should be the same to any pair of equivalent observers. Since each observer is identified with a frame of reference, the principle of objectivity requires that the constitutive equations should be invariant under transformations that preserve the essential structure of space and time. Two observers are said to be equivalent if they agree on i) the distance between two arbitrary points ii) orientation iii) time elapsed between two events iv) the order in which the two events occur.

6.2 Change of Frame Transforms for Various Field Quantities-Rates

If $\underline{T}$ is a frame invariant tensor quantity, nothing guarantees that its time derivatives of any order will retain that property. Thus we will define certain appropriate objective rates for tensors only up to the second order (rates of higher order tensor quantities are very seldom used).

a) Convected Derivative : (°)

The convected derivative of a vector $\underline{V}$ is defined as

$$\overset{\circ}{\underline{V}} = \dot{\underline{V}} + \underline{L}^T \cdot \underline{V} \tag{12}$$

The convected derivative of a second order tensor $\underline{T}$ is defined as

$$\overset{\circ}{\underline{T}} = \dot{\underline{T}} + \underline{L}^T \cdot \underline{T} + \underline{T} \cdot \underline{L} \tag{13}$$

b) Jaumann (or Corotational) Derivative : (∇)

The Jaumann derivative of a vector $\underline{T}$ is defined as

$$\overset{\nabla}{\underline{V}} = \dot{\underline{V}} - \underline{W} \cdot \underline{V} \tag{14}$$

The Jaumann derivative of a second order tensor $\underline{T}$ is defined as

$$\overset{\nabla}{\underline{T}} = \dot{\underline{T}} - \underline{W}\cdot\underline{T} + \underline{T}\cdot\underline{W} \tag{15}$$

Note that in defining the above frame indifference rates the field quantities $\underline{V}$ and $\underline{T}$ had to be objective. In a similar fashion we could have defined different rate measures for $\underline{V}$, $\underline{T}$.

6.3 Constitutive Equations for Hyperelastic Solids.

The simplest possible solid that one can imagine is an elastic solid. In this case there exists a ground state for the material for which the stress $\underline{\sigma} = \underline{0}$. If we impose any deformation $\underline{F}$ to the material, the stress $\underline{\sigma}$ depends only on $\underline{F}$ and is independent of the history $\underline{F}(\tau)$ $0 < \tau < t$ of deformation. From the frame indifference requirements, we obtain

$$\underline{S} = \hat{\underline{S}}(\underline{E}) \quad \text{for elastic solid} \tag{16}$$

That is, we see that for a (path independent) elastic solid the second Piola Kirchhoff stress $\underline{S}$ depends only on the Lagrangian strain $\underline{E}$, An elastic material is called hyperelastic if there exists an internal energy of the material $\varepsilon = \hat{\varepsilon}(\underline{F})$ which depends only on $\underline{F}$ and a natural state of the material for which $\hat{\varepsilon}(\underline{F}_0) = 0$ and $\hat{\sigma}(\underline{F}_0) = 0$ (without loss of generality $\underline{F}_0 = \underline{I}$). For hyperelastic material, we may obtain

$$\underline{S} = \frac{\partial W}{\partial \underline{E}} \tag{17}$$

, where $W = \rho\varepsilon$ is called the strain energy density of the material (per unit reference volume). The equation (17) is the general form of the constitutive equation for the hyperelastic solid. Thus for a hyperelastic material if we know the strain energy density W in terms of $\underline{E}$(or $\underline{C}$), then we can find the stress-strain relationship from (17). Especially, for the isotropic hyperelastic material, we have

$$W = \hat{W}(\lambda_1, \lambda_2, \lambda_3) = W(\mathrm{I}_c, \mathrm{II}_c, \mathrm{III}_c) \tag{18}$$

, where λ_i are the principal stretches and I_c, II_c, III_c are the invariants of the Cauchy-Green tensor $\underline{C}$. Solid material may be either compressible or incompressible. For the compressible isotropic hyperelastic material, we have the following constitutive equation

$$\underline{S} = 2\left[\left(\frac{\partial W}{\partial \mathrm{I}_c} + \mathrm{I}_c \frac{\partial W}{\partial \mathrm{II}_c}\right)\underline{\mathrm{I}} - \frac{\partial W}{\partial \mathrm{II}_c}\underline{C}\right] + \frac{\partial W}{\partial \mathrm{III}_c}\mathrm{III}_c\,\underline{C}^{-1} \tag{19}$$

And for the incompressible isotropic hyperelastic material, we have the following constitutive equation

$$\underline{S} = 2\left[\left(\frac{\partial W}{\partial \mathrm{I}_c} + \mathrm{I}_c \frac{\partial W}{\partial \mathrm{II}_c}\right)\underline{\mathrm{I}} - \frac{\partial W}{\partial \mathrm{II}_c}\underline{C}\right] + p\underline{C}^{-1} \tag{20}$$

, where the unspecified constant p is the pressure.

6.4 Constitutive Equations for Hypoelastic Solids

For the case of a hypoelastic material, the path dependence of the deformation must be taken into account. Consequently, the rate form of the field equations should be used. We define the rate of a field quantity, denoted by($\dot{}$), to be its derivative with respect to some monotonically increasing parameter (i.e., time-like parameter). In the kind of incrementally linear hypoelastic material, the equilibrium field equation must take the following form

$$(\dot{S}^{ij} + S^{kj}\dot{u}^i|_k + \dot{S}^{kj}u^i|_k)|_j = 0 \tag{21}$$

For the incompressible solids, the incompressibility condition should be satisfied, additionally.

The constitutive equation for the hypoelastic material takes the following form which relates the stress rates to the strain rates

$$\begin{aligned} \dot{S}^{ij} &= L^{ijk\ell}\ \dot{E}_{k\ell} && \text{for the compressible material} \\ \dot{S}^{ij} &= L^{ijk\ell}\ \dot{E}_{k\ell} - \dot{p}(C^{ij})^{-1} && \text{for the incompressible material} \end{aligned} \tag{22}$$

, where $\underline{C}$ is the right Cauchy-Green deformation tensor while $\underline{\dot{E}}$ is the rate of the Lagrangian strain tensor $\underline{E}$ and has components

$$\dot{E}_{ij} = \frac{1}{2}(\dot{u}_{i|j} + \dot{u}_{j|i} + \dot{u}_{k|i}u_{k|j} + u_{k|i}\dot{u}_{k|j}) \tag{23}$$

For the Stören and Rice hypoelastic material the components of the incremental moduli tensor $\underline{L}$ are given by

$$\begin{aligned} L^{ijkl} = \frac{2}{3}\ E_s\Bigg[&\frac{1}{2}\{(C^{ik})^{-1}(C^{jl})^{-1} + (C^{il})^{-1}(C^{jk})^{-1}\} \\ &- \frac{3}{2}\left\{1 - \frac{E_t}{E_s}\right\}\frac{(S^{ij})'(S^{kl})'}{\sigma_2^{\,2}}\Bigg] \\ &- \frac{1}{2}[(C^{ik})^{-1}\ S^{jl} + (C^{jk})^{-1}S^{il} + (C^{il})^{-1}S^{jk} + (C^{jl})^{-1}S^{ik}] \end{aligned} \tag{24}$$

for the incompressible material

, where $(S^{ij})' = S^{ij} - \frac{1}{2}(C^{ij})^{-1}C^{k\ell}S_{k\ell}$, $\sigma_e^{\,2} = \frac{3}{2}C^{ik} C^{j\ell}S_{ij}S_{k\ell}$ and E_t and E_s are the tangent and secant moduli respectively of the uniaxial true stress-natural strain curve at a stress level equal to the equivalent stress σ_e. For the compressible material, $L^{ijk\ell}$ has the different form, i.e.,

$$\begin{aligned} L^{ijkl} = \frac{E_s}{1+\nu_s}\Bigg[&\frac{1}{2}\{(C^{ik})^{-1}(C^{jl})^{-1} + (C^{ik})^{-1}(C^{jl})^{-1}\} \\ &+ \frac{\nu_s}{1-2\nu_s}(C^{ij})^{-1}(C^{kl})^{-1} - \frac{3}{2}\cdot\frac{\frac{E_s}{E_t} - 1}{\frac{E_s}{E_t} - \frac{1-2\nu}{3}}\cdot\frac{(S^{ij})'\,(S^{kl})'}{\tau_e^{\,2}}\Bigg] \\ &- \frac{1}{2}\{(C^{ik})^{-1}\ S^{jl} + (C^{jk})^{-1}\ S^{il} + (C^{il})^{-1}\ S^{jk} + (C^{jl})^{-1}\ S^{ik}\} \end{aligned} \tag{25}$$

with
$$(S^{ij})' = S^{ij} - \frac{1}{3}(C^{ij})^{-1}C^{k\ell}S_{k\ell},$$
$$\tau_e^{\,2} = \frac{3}{2}\ C^{ik}C^{j\ell}(S_{ij})'(S_{k\ell})',$$
$$\frac{\nu_s}{E_s} = \frac{\nu}{E} + \frac{1}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right)$$

7. Numerical Methods for the Solution of a Boundary Value Problem in Nonlinear Solid Structural Deformation

For the more realistic boundary value problems in view of the corresponding high nonlinearities in the governing equations the only viable alternative is the numerical solution. The most widely used numerical method is the finite element method. The highly nonlinear equations resulted through the finite element discretization can be solved by the incremental Newton-Raphson method. In this paper the modified incremental Newton-Raphson Scheme is proposed for the accuracy increase of the solutions.

7.1 Modified Incremental Newton-Raphson Method

Assume that we want to solve the nonlinear equation

$$\underline{F}(\underline{u}) = \underline{f} \tag{26}$$

, which is resulted through the finite element discretization of the weak form of the equilibrium equation. $\underline{u} = (u_o,\ u_1,\ u_2, \cdots, u_M)$ is the discretized displacement vector and $\underline{f} = (f_0,\ f_1,\ f_2,\ \cdots,\ f_M)$ is the corresponding load vector. Introducing the load parameter λ such that $0 \le \lambda \le 1$ the equation (26) may be written as

$$\underline{F}(\underline{u}) = \lambda \underline{f} \tag{27}$$

Differentiating with respect to λ, $\frac{\partial \underline{F}}{\partial \underline{u}} \frac{\partial \underline{u}}{\partial \lambda} = \underline{f}$, where $\underline{F}(\underline{u}_m) = \lambda_m \underline{f}$

Hence, from the above equation, we may obtain the following approximate relation

$$\underline{u}_{m+1} = \underline{u}_m + \left[\frac{\partial \underline{F}}{\partial \underline{u}}(\underline{u}_m) \right]^{-1} (\lambda_{m+1} - \lambda_m)\underline{f} \qquad (28)$$

In other words, once we know the value $\underline{u}_m$ for the given load $\underline{f}_m$, we can evaluate the approximate value $\underline{u}_{m+1}$ for the given load $\underline{f}_{m+1}$ by (28). Here, we should notice that we start from $\underline{u}_0 = \underline{0}$ and $\lambda_0 = 0$ in order to guarantee that $\underline{F}(\underline{u}_m) = \lambda_m \underline{f}$. However, $\underline{u}_{m+1}$, here, is just the approximate value for the real $\underline{u}_{m+1}$. In order to obtain the exact $\underline{u}_{m+1}$, we apply the Newton-Raphson Scheme. That is, we take $\underline{u}_{m+1}$ as the first approximate value for the real $\underline{u}_{m+1}$ and denote $\underline{u}^{(1)}_{m+1}$. And then, the better estimated value is approximated by

$$\underline{u}^{(n+1)}_{(m+1)} = \underline{u}^{(n)}_{m+1} + \left[\frac{\partial \underline{F}}{\partial \underline{u}}(\underline{u}^{(n)}_{m+1}) \right]^{-1} [\lambda_{m+1}\underline{f} - \underline{F}(\underline{u}^{(n)}_{m+1})] \qquad (29)$$

, where $\underline{u}^{(1)}_{m+1} = \underline{u}_{m+1}$ in (28)

Since the nonlinear relation is not satisfied when $\underline{u}_0 = \underline{0}$, we cannot draw the tangent line at $\underline{u}_0 = \underline{0}$, thus we should draw the tangent line at the modified $\underline{u}_0$, in order to get the first approximate value for the real $\underline{u}_1$. Therefore, we obtain the following scheme for the first increment.

Modification of the initial value $\underline{u}_0$

$$\underline{u}_{01} = \underline{u}_0 + [\underline{K}]^{-1} \Delta\lambda_1 \underline{f} \qquad (30)$$

, where $\underline{K}$ is the linear global stiffness matrix corresponding to the nonlinear system. And the first iteration

First iteration

$$\underline{u}_1 = \underline{u}_{01} + \left[\frac{\partial \underline{F}}{\partial \underline{u}}(\underline{u}_{01}) \right]^{-1} \Delta\lambda_1 \underline{f} \qquad (31)$$

Higher iteration

$$\underline{u}_1^{(n+1)} = \underline{u}_1^{(n)} + \left[\frac{\partial \underline{F}}{\partial \underline{u}}(\underline{u}_1^{(n)}) \right]^{-1} [\lambda_1 \underline{f} - \underline{F}(\underline{u}_1^{(n)})]$$
$$(n = 1,\ 2,\ 3\cdots) \qquad (32)$$

, where $\underline{u}_1^{(1)} = \underline{u}_1$ in (31)

For higher increment,

First iteration

$$\underline{u}_{m+1} = \underline{u}_m + \left[\frac{\partial \underline{F}}{\partial \underline{u}}(\underline{u}_m) \right]^{-1} \Delta\lambda_m \underline{f} \qquad (33)$$

where $\underline{u}_1 = \underline{u}_1^{(n+1)}$ in (32)

Higher iteration

$$\underline{u}^{(n+1)}_{m+1} = \underline{u}^{(n)}_{m+1} + \left[\frac{\partial \underline{F}}{\partial \underline{u}}(\underline{u}^{(n)}_{m+1}) \right]^{-1} [\lambda_{m+1}\underline{f} - \underline{F}(\underline{u}^{(n)}_{m+1})]$$
$$(n = 1,\ 2,\ 3\cdots) \qquad (34)$$

where $\underline{u}^{(1)}_{m+1} = \underline{u}_{m+1}$ in (33)

This is the incremental Newton-Raphson Scheme. However, for the more accurate solutions, we modify the above incremental Newton-Raphson Scheme as follows. When the estimated value $\underline{u}^{(n+1)}_{m+1}$ is negative or less than $\underline{u}_m$, $\underline{u}^{(n+1)}_{m+1}$ is modified in such a way that it should locate near

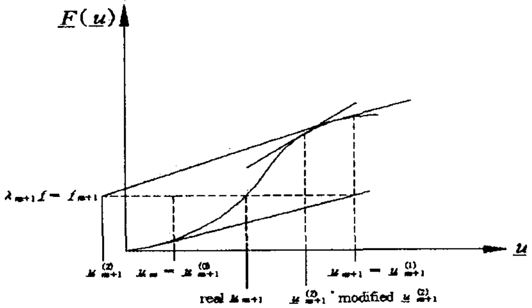

Fig. 1 Modification of $\underline{u}^{(n+1)}_{m+1}$

the real $\underline{u}_{m+1}$ (see the Fig.1). In this way, we may obtain the exact $\underline{u}_{m+1}$ corresponding to $\lambda_{m+1}f$.

However, we have another possibility that discretized strain may have the negative value. The case when such a situation occurs is that the discretized u_{i+1} is less than or equal to u_i, since in the finite element method we usually assume the linear variation for the discretized displacement $\underline{u}^{(e)}$. And so, in this case we should also modify the u_{i+1} such that u_{i+1} is greater than u_i in order to get the real $(u_{i+1})_{m+1}$ corresponding to the given load $\underline{f}_{m+1} = \lambda_{m+1}\underline{f}$. After imposing the condition $u_1 = 0$ or $u_0 = 0$ corresponding to $\lambda_0 = 0$, we may obtain the assembled tangential stiffness matrix $\underline{K}_T = \frac{\partial \underline{F}}{\partial \underline{u}}$. Using this $\underline{K}_T$, we can continue the incremental iteration procedure until $|\nabla \underline{u}^{(n)}_{m+1}| \leq$ error bound in order to get the desired accurate solutions.

7.2 Finite Element Discretization of a 3-Dimensional Solid

The starting point is the variational weak form of the equilibrium equation of (10) as discussed in the section 5. Or, the rate form of the principle of virtual work.

$$\int_{\partial V_t} \dot{T}^i \cdot \delta u_i dA + \int_V \rho_o \dot{b}^i \cdot \delta u_i dV = \int_V (\dot{S}^{ij} \delta E_{ij} + S^{ij} \dot{u}^k{}_{|i} \delta u_{k|j}) dV \qquad (35)$$

Usually, the equation (10) is appropriate for the hyperelastic material, while the equation (35) is proper for the hypoelastic material. A finite element discretization of the displacement field $\underline{u}$ is

$$\underline{u}(\underline{P}) = \sum_{I=1}^{M} u_I \ \underline{\Phi}^I(\underline{P}),\ \underline{\Phi}^I = \Phi^I_i \underline{G}^i \qquad (36)$$

, where u_I is the discretized nodal displacement to be solved and Φ^I_i is a proper shape function. Inserting (36) into (10) or (35), we may obtain the nonlinear equation (26) for $\underline{u} = (u_0, u_1, u_2, \cdots, u_M)$, whose solution was discussed just before.

8. Concluding Remarks

So far, we have discussed about the numerical technique for the nonlinear deformation analysis of solid structures. For this discussion, we developed the nonlinear solid mechanics theory especially for the time independent solid material, i.e., the hyperelastic solid material and the hypoelastic solid material. The theory development included the kinematics of solid deformations, strain and stress measures, governing equations, the variational formulation of governing equations, and the constitutive equations. In this paper, the incremental Newton-Raphson method as the methodology for the solution of the nonlinear equation resulted through the finite element discretization of 3-dimensional solid structures is discussed, and the modification of the incremental Newton-Raphson Scheme is proposed for the more accurate solutions. This accuracy improvement will be proved through the forthcoming paper(Part 2) which will discuss the application of the general theory developed in the current paper to a simple structure whose analytical solution is available.

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