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FIXED POINT THEOREMS ON GENERALIZED CONVEX SPACES

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ABSTRACT. We obtain new fixed point theorems on maps defined on "locally G-convex" subsets of a generalized convex spaces. Our first theorem is a Schauder-Tychonoff type generalization of the Brouwer fixed point theorem for a G-convex space, and the second main result is a fixed point theorem for the Kakutani maps. Our results extend many known generalizations of the Brouwer theorem, and are based on the Knaster-Kuratowski-Mazurkiewicz theorem. From these results, we deduce new results on collectively fixed points, intersection theorems for sets with convex sections and quasi-equilibrium theorems.

0. Introduction

There have appeared many generalizations of the concept of convex subsets of a topological vector space (t.v.s.). Especially, the convex spaces due to Lassonde [18] and the H-spaces [1-3] due to Horvath [11-13] were shown to be very useful in many fields in mathematics such as the KKM theory, fixed point and coincidence theory, variational inequalities, best approximations, and minimax theory.

Motivated by recent works of Park [20-23] on convex spaces and H-spaces, Park and the author [25] introduced the notion of *generalized convex spaces* or *G-convex spaces* which extend many of topological spaces having generalized convexity structures. In the present paper, we obtain fixed point theorems for "locally G-convex" subsets of G-convex spaces.

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Our first main result, Theorem 1, is a Schauder-Tychonoff type generalization of the Brouwer fixed point theorem for a G-convex space with certain "local convexity." Our second main result, Theorem 2, is a fixed point theorem for Kakutani maps. From Theorem 2, we deduce new results on collectively fixed points, intersection theorems for sets with convex sections and quasi-equilibrium theorems.

1. Preliminaries

Throughout this paper, we assume that any topological space is Hausdorff.

A *multifunction* (or *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^{-}y = \{x \in X : y \in Fx\}$ for $y \in Y$. For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) : x \in X, y \in Tx\}$ is closed in $X \times Y$.

For $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$. For any $B \subset Y$, the *lower inverse* of B under F is defined by $F^{-}(B) = \{x \in X : Fx \cap B \neq \emptyset\}$. The (*lower*) *inverse* of $F : X \multimap Y$ is the multifunction $F^{-} : Y \multimap X$ defined by $x \in F^{-}y$ if and only if $y \in Fx$.

For topological spaces X and Y, a map $F : X \multimap Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^{-}(B)$ is closed in X; *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, $F^{-}(B)$ is open in X; and *compact* provided $F(X)$ is contained in a compact subset of Y.

If Y is a compact topological space and $F : X \multimap Y$ a closed multifunction with nonempty values, then F is u.s.c.

Let $\overline{}$ denote the closure.

Let X be a topological space, $\mathcal{R}$ a cover of X, and $\mathrm{St}(B, \mathcal{R}) = \bigcup\{V \in \mathcal{R} : B \cap V \neq \emptyset\}$ for each $B \subset X$. A cover $\mathcal{R}$ of X is called a *star refinement* [*barycentric refinement,* resp.] of a cover $\mathcal{U}$ whenever the cover $\{\mathrm{St}(V, \mathcal{R}) : V \in \mathcal{R}\}$ [$\{\mathrm{St}(x, \mathcal{R}) : x \in X\}$, resp.] refines $\mathcal{U}$. For any $A \subset X$ or $A \in X$, let $\mathcal{V}_A(X)$ be the set of all open neighborhoods of A.

A topological space X is said to be *contractible* if the identity function 1_X of X is homotopic to a constant function.

Let X be a topological space and $\langle X \rangle$ the set of all nonempty finite subsets of X. A pair (X, F) is called an *H-space* [1] or a *c-space* [13] if $F = \{F(A)\}$ is a family of contractible subsets of X indexed by $A \in \langle X \rangle$ such that $F(A) \subset F(B)$ whenever $A \subset B \in \langle X \rangle$.

For an H-space (X, F), a subset C of X is said to be *H-convex* (or *F-set*) if for each $A \in \langle X \rangle$, $A \subset C$ implies $F(A) \subset C$. Given a nonempty subset A of an H-space (X, F), the H-convex hull of A, denoted by H-co A, is defined by Tarafdar [33] as follows:

$$H\text{-co}\, A = \bigcap \{Y : A \subset Y \subset X \text{ and } Y \text{ is } H\text{-convex}\}.$$

For a set A, let $|A|$ denote the cardinality of A. Let Δ_n denote the standard n-simplex; that is,

$$\Delta_n = \{u \in \mathbb{R}^{n+1} : u = \sum_{i=1}^{n+1} \lambda_i(u) e_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{n+1} \lambda_i(u) = 1\},$$

where e_i is the i-th unit vector in $\mathbb{R}^{n+1}$.

A *generalized convex space* or a *G-convex space* $(X; \Gamma)$ consists of a topological space X and a function $\Gamma : \langle X \rangle \multimap X$ such that

(1) for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A = \Gamma(A) \subset \Gamma_B$; and
(2) for each $A \in \langle X \rangle$ with $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J$.

Here Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$.

Note that Γ_A does not need to contain A for $A \in \langle X \rangle$. For details, see [25].

For an $(X; \Gamma)$, a subset C of X is said to be *G-convex* if $A \in \langle C \rangle$ implies $\Gamma_A \subset C$. For a nonempty subset S of X, the *G-convex hull* of S, G-co S, is defined by

$$G\text{-co}\, S = \bigcap \{Y : S \subset Y \subset X \text{ and } Y \text{ is } G\text{-convex}\}.$$

A subset C of X is said to be *of type I* if for any $x \in C$ and $V \in \mathcal{V}_x(X)$, there exists a $U \in \mathcal{V}_x(X)$ such that G-co$\,(U \cap C) \subset V$. And a subset C of X is said to be *of type II* if $\{x\}$ is G-convex for $x \in C$ and for any compact G-convex subset A of C and $V \in \mathcal{V}_A(X)$, there exists a $U \in \mathcal{V}_A(X)$ such that G-co$\,(U \cap C) \subset V$.

A convex subset X in a t.v.s. is a G-convex space $(X;\Gamma)$ by putting $\Gamma_A = \operatorname{co} A$, where co denotes the convex hull in the usual sense, and every subset of a locally convex t.v.s. is type II. A subset C of a t.v.s. E is said to be *of Z-type* if for every $V \in \mathcal{V}_0(E)$ there exists a $U \in \mathcal{V}_0(E)$ such that $\operatorname{co}(U \cap (C - C)) \subset V$ [10]. Note that every subset C of Z-type is of type II and any type II subset of X is of type I.

Any convex space X becomes a G-convex space $(X;\Gamma)$ by putting $\Gamma_A = \operatorname{co} A$. An H-space (X, F) is a G-convex space $(X;\Gamma)$. In fact, by putting $\Gamma_A = F(A)$ for each $A \in \langle X \rangle$ with $|A| = n+1$, there exists a continuous map $\phi_A : \Delta_n \to X$ such that for all $J \subset A$, $\phi_A(\Delta_J) \subset F(J)$ [13, Theorem 1].

The other major examples of G-convex spaces are metric spaces with Michael's convex structure [19], S-contractible spaces [26-28], Horvath's pseudo-convex spaces [11], Komiya's convex spaces [17], Bielawski's simplicial convexities [4], and Joó's pseudoconvex spaces [14].

From now on, we may write $(X;\Gamma)$ for H-spaces instead of (X, F) as in [1-3], where $\Gamma_A = F(A)$ for each $A \in \langle X \rangle$.

2. The Schauder-Tychonoff type fixed point theorems

The classical KKM theorem [16] can be stated as follows:

LEMMA. *Let $R_0, \cdots, R_n$ be closed subsets of Δ_n. If for any $\{i_0, \cdots, i_k\} \subset \{i : i = 0, \cdots, n\}$, the face Δ_k of Δ_n corresponding to $\{i_0, \cdots, i_k\}$, is contained in $\bigcup\{R_j : j = i_0, \cdots, i_k\}$, then $\bigcap\{R_i : i = 0, \cdots, n\} \neq \emptyset$.*

From Lemma we obtain the following Schauder-Tychonoff type [31, 34] fixed point theorem for G-convex spaces:

THEOREM 1. *Let $(X;\Gamma)$ be a regular G-convex space. Then any compact continuous function $g : X \to X$ such that $\overline{g(X)}$ is of type* I, *has a fixed point.*

Proof. Let us assume that g has no fixed point. Then for each $x \in X$, there exists a $U_x \in \mathcal{V}_x(X)$ such that $x \notin g^-(U_x)$. Otherwise $g(x) \in V$ for all $V \in \mathcal{V}_x(X)$, which means that $x = g(x)$. Since X is regular, there is a $V_0 \in \mathcal{V}_x(X)$ such that

$$x \in V_0 \subset \overline{V_0} \subset U_x.$$

And there exist $V_1 \in \mathcal{V}_{g^-(\overline{V_0})}(X)$ and $V_2 \in \mathcal{V}_x(X)$ such that $V_1 \cap V_2 = \emptyset$.

Let $V_x := V_2 \cap V_0$. Then $V_x \cap g^-(V_x) \subset V_2 \cap g^-(V_0) \subset V_2 \cap V_1 = \emptyset$. Since $\overline{g(X)}$ is of type I compact subset of X, there exist a family of open neighborhoods $\mathcal{W} = \{W_x\}_{x \in \overline{g(X)}}$ such that

(1.1) for every $x \in \overline{g(X)}$, $G\text{-co}\,(W_x \cap \overline{g(X)}) \bigcap g^-(W_x) = \emptyset$

and an open star refinement $\mathcal{U}$ of $\{W_x \cap \overline{g(X)} : W_x \in \mathcal{W}\}$. Choose a finite cover $\mathcal{R} = \{U_i \in \mathcal{U}\}_{i=0,1,\cdots,n}$ of $\overline{g(X)}$, $\xi_i \in U_i$, and $W_i := W_{x_i} \in \mathcal{W}$ such that $\mathrm{St}(U_i, \mathcal{R}) \subset W_i \cap \overline{g(X)}$ for $i = 0, 1, \cdots, n$. Let $A = \{\xi_0, \xi_1, \cdots, \xi_n\}$ and $X_i = X \backslash g^-(U_i)$. Since $U_i = \tilde{U}_i \cap \overline{g(X)}$ for some open subset $\tilde{U}_i$ of X and $g^-(U_i) = g^-(\tilde{U}_i)$, X_i is closed in X. By the definition of G-convex spaces, there exists a continuous function $\phi_A : \Delta_n \to X$ such that $\phi_A(\Delta_J) \subset \Gamma_{\{\xi_j : j \in J\}}$ for each $J \subset \{0, 1, \cdots, n\}$.

If $\bigcap\{U_j : j \in J\} \neq \emptyset$ for some $J \subset \{0, 1, \cdots, n\}$, we have

$$\phi_A(\Delta_J) \subset \Gamma_{\{\xi_j : j \in J\}} \subset G\text{-co}\,(W_{j_0} \cap \overline{g(X)})$$

for each $j_0 \in J$, since $\{\xi_j : j \in J\} \subset \bigcup\{U_j : j \in J\} \subset W_{j_0} \cap \overline{g(X)}$. And by (1.1), we have

$$\begin{aligned} G\text{-co}\,(W_{j_0} \cap \overline{g(X)}) &\subset X \backslash g^-(W_{j_0}) \subset X \backslash g^-(W_{j_0} \cap \overline{g(X)}) \\ &\subset X \backslash \bigcap\{g^-(U_j) : j \in J\} = \bigcup_{j \in J} X_j. \end{aligned}$$

On the other hand, if $\bigcap\{U_j : j \in J\} = \emptyset$ for some $J \subset \{0, 1, \cdots, n\}$, then $\bigcap\{g^-(U_j) : j \in J\} = \emptyset$. Hence $\phi_A(\Delta_J) \subset X = \bigcup_{j \in J} X_j$; that is,

$$\Delta_J \subset {\phi_A}^-(\bigcup_{j \in J} X_j) \subset \bigcup_{j \in J} {\phi_A}^-(X_j).$$

Note that ${\phi_A}^-(X_j)$ is closed in Δ_n for each $j = 0, 1, \cdots, n$. By Lemma, $\bigcap_{i=0}^n {\phi_A}^-(X_i) \neq \emptyset$, and hence $\bigcap_{i=0}^n X_i \neq \emptyset$. This implies

$$X \neq X \backslash \bigcap_{i=0}^n X_i = \bigcup_{i=0}^n g^-(U_i) = X.$$

This contradiction shows that g has a fixed point. $\square$

Particular Forms. A particular form of Theorem 1 for a paranormed space (which is not locally convex) is due to Zima [35]. This is extended by Rzepecki [29, Theorem 1] for a Hausdorff t.v.s. Note that a Hausdorff t.v.s. is regular.

Since an H-space is a G-convex space, the following holds by Theorem 1.

Corollary 1.1. *Let $(X;\Gamma)$ be a regular H-space and $g : X \to X$ be a compact continuous function such that*

(1.2) *for any $x \in \overline{g(X)}$ and $V \in \mathcal{V}_x(X)$, there exists a $U \in \mathcal{V}_x(X)$ such that H-co$(U \cap \overline{g(X)}) \subset V$.*

Then g has a fixed point.

Particular Forms. For a locally convex t.v.s., Theorem 1 and Corollary 1.1 include earlier works of Brouwer [6], Schauder [30,31], and Tychonoff [34]. They also contain Pasicki [26, Theorem 1] for regular S-contractible spaces, Park [23, Theorem 3] and Horvath [13, Theorem 4.4 and Corollary 4.4] for H-spaces since a uniform space is regular.

Corollary 1.2. *Let $(X;\Gamma)$ be a complete metric H-space such that any open ball of X is H-convex and for any $\epsilon > 0$, $\{x \in X : d(x, C) < \epsilon\}$ is H-convex whenever C is H-convex. Then any compact l.s.c. multifuction $T : X \multimap X$ with nonempty closed H-convex values has a fixed point.*

Proof. By [13, Theorem 3.3], T has a continuous selection $g : X \to X$. By Corollary 1.1, g has a fixed point $x = gx \in Tx$. □

Particular Form. Horvath [13, Corollary 4.5].

3. Fixed points for Kakutani factorizable maps

We define the following map from a topological space X into a G-convex space Y:

$F \in \mathbb{K}(X,Y) \iff F$ is a *Kakutani* map; that is, F is u.s.c. with nonempty compact G-convex values.

From Theorem 1, we obtain the following result for Kakutani maps on G-convex spaces:

THEOREM 2. *Let $(X;\Gamma)$ be a compact G-convex space of type* II. *Then any $T \in \mathbb{K}(X,X)$ has a fixed point.*

Proof. Suppose $x \notin Tx$ for $x \in X$. Since X is regular, there are $U \in \mathcal{V}_x(X)$ and $V \in \mathcal{V}_{Tx}(X)$ such that $U \cap V = \emptyset$. For V, there is a $V_1 \in \mathcal{V}_{Tx}(X)$ such that G-co$V_1 \subset V$ by assumption. The upper semicontinuity of T implies the existence of a $P \in \mathcal{V}_x(X)$ for which $T(P) \subset V_1$. Put $W_x = U \cap P$, then

(1) $W_x \cap G\text{-co}T(W_x) \subset U \cap G\text{-co}V_1 \subset U \cap V = \emptyset$.

For the cover $\mathcal{W} = \{W_x\}_{x\in X}$ of X, there exists an open star refinement $\mathcal{U}$ subordinated to $\mathcal{W}$. Choose a finite cover $\mathcal{R} = \{U_i \in \mathcal{U}\}_{i=0,1,\cdots,n}$ of X, $x_i \in U_i$ and a $y_i \in T(U_i)$ such that $\mathrm{St}(U_i,\mathcal{R}) \subset W_{x_i} \in \mathcal{W}$ for each $i = 0,1,\cdots,n$. Let $A = \{y_0, y_1, \cdots, y_n\}$. For a partition of unity $\{p_i : i = 0,1,\cdots,n\}$ subordinated to $\mathcal{R}$, define $p : X \to \Delta_n$ by

$$p(x) = (p_0(x), p_1(x), \cdots, p_n(x))$$

for each $x \in X$. Since X is a G-convex space, there is a continuous function $\phi_A : \Delta_n \to \Gamma_A \subset X$ such that $\phi_A(\Delta_J) \subset \Gamma_J$ for each $J \subset A$, where Δ_J is the face of Δ_n corresponding to J.

Define $h : X \to X$ by

$$h(x) = (\phi_A p)x$$

for each $x \in X$. Since X is of type II and h is continuous, h has a fixed point $x_0 = h(x_0)$. And for some $i \in N_{x_0} = \{y_j \in A : p_j(x_0) \neq 0\}$,

$$x_o = h(x_0) \in \phi_A(\Delta_{N_{x_0}}) \subset \Gamma_{N_{x_0}} \subset G\text{-co}T(\mathrm{St}(U_i,\mathcal{R})) \subset G\text{-co}T(W_{x_i}).$$

Since $x_0 \in U_i$, it contradicts (1). This completes our proof. □

REMARK. By Theorem 2, for any simplex P, each finite composites of maps in $\mathbb{K}(P,P)$ has a fixed point.

PARTICULAR FORMS. 1. For a locally convex t.v.s., Theorem 2 includes particular forms of Corollary 1.1.

2. If X is a compact convex subset of a locally convex t.v.s., Theorem 2 contains earlier works of Kakutani [15], Bohnenblust and Karlin [5], Fan [7], and Glicksberg [8].

3. If X is of Z-type subset of a t.v.s., Theorem 2 includes Hadžić [9].

4. If X is a compact S-contractible space and $T \in \mathbb{K}(X,X)$, Theorem 2 includes Pasicki [28, Theorem 2. 13].

4. Collectively fixed point theorem and its applications

Let I be a finite index set.

Let $\{X_i\}_{i\in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j\in I} X_j \quad \text{and} \quad X^i = \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I\setminus\{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $(x^i, x_i) \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$, the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = (x^i, x_i)$ for any $i \in I$, where x^i denotes the projection of x in X^i.

For $i \in I$, let $(X_i; \Gamma^i)$ be a G-convex space and $X = \prod_{i\in I} X_i$. Define $\Gamma : \langle X\rangle \multimap X$ by $\Gamma_A = \prod_{i\in I} \Gamma^i_{A_i}$, where $A_i = \pi_i(A)$ and $\pi_i : X \to X_i$ is a projection. Then $(X;\Gamma)$ becomes a G-convex space [32, Theorem 4.1].

Note that if $(X_i;\Gamma^i)$ is a G-convex space of type II for each $i \in I$, then $(X;\Gamma)$ be a G-convex space of type II with the product topology.

From Theorem 2 we obtain the following collectively fixed point theorem:

THEOREM 3. *Let $\{(X_i;\Gamma^i)\}_{i\in I}$ be a family of compact G-convex spaces of type* II, *and $T_i \in \mathbb{K}(X, X_i)$ for each $i \in I$. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i\hat{x}$ for each $i \in I$.*

Proof. Define $T : X \multimap X$ by $Tx = \prod_{i\in I} T_i x$ for each $x \in X$. Then T is u.s.c. by virtue of Ky Fan [7, Lemma 1] and $T \in \mathbb{K}(X, X)$. By Theorem 2, T has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T\hat{x}$ and hence $\hat{x}_i \in T_i\hat{x}$ for each $i \in I$. $\square$

The collectively fixed point theorem can be reformulated to generalization of von Neumann type intersection theorem for sets with G-convex sections as follows:

THEOREM 4. *Let $\{(X_i;\Gamma^i)\}_{i\in I}$ be a familly of compact G-convex spaces of type* II, *and A_i a closed subset of X such that $A_i(x^i)$ is a nonempty G-convex subset of X_i for each $x^i \in X^i$ and $i \in I$. Then $\bigcap_{i\in I} A_i \neq \emptyset$.*

Proof. We use Theorem 3 with $T_i : X \multimap X_i$ defined by $T_i x = A_i(x^i)$ for $x \in X$. Then

$$\begin{aligned} X_i \times A_i &= \{(y_i, x) \in X_i \times X^i \times X_i : x \in A_i\} \\ &= \{(y_i, x^i, x_i) \in X_i \times X : x_i \in A_i(x^i)\} \\ &= \{(y_i, x^i, x_i) \in X_i \times X : x_i \in T_i(y_i, x^i)\}, \end{aligned}$$

which implies that T_i is a closed map with nonempty G-convex values. Since X_i is compact, T_i is u.s.c. Therefore, by Theorem 3, there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i\hat{x}$ for all $i \in I$. So we have $\hat{x} = (\hat{x}^i, \hat{x}_i) \in A_i$ for all $i \in I$. This completes our proof. □

PARTICULAR FORM. Ky Fan [7, Theorem 2]: X_i are convex subsets of locally convex t.v.s. for all $i \in I$. This result was applied in [7] to obtain a manimax theorem generalizing von Neumann's and Ville's.

Theorem 3 can be refomulated to the form of a quasi-equilibrium theorem as follows:

THEOREM 5. *Let $\{(X_i;\Gamma^i)\}_{i\in I}$ be a family of compact G-convex spaces of type* II, *$S_i : X \multimap X_i$ a closed map, and $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ u.s.c. functions for each $i \in I$. Suppose that for each $x \in I$,*

(1) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*
(2) *the function M_i defined on X by*

$$M_i(x) = \max_{y \in S_i x} g_i(x^i, y)$$

is l.s.c.; and

(3) *for each $x \in X$, the set*

$$\{y \in S_i x : f_i(x^i, y) \geq M_i(x)\}$$

is G-convex.

Then there exists an $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in S_i\hat{x} \quad \text{and} \quad f_i(\hat{x}^i, \hat{x}_i) \geq M_i(\hat{x}).$$

Proof. For each $i \in I$, define a map $T_i : X \multimap X_i$ by

$$T_i x = \{y \in S_i x : f_i(x^i, y) \geq M_i(x)\}$$

for $x \in X$. Note that each $T_i x$ is nonempty by (1) since $S_i x$ is compact and $g_i(x^i, \cdot)$ is u.s.c. on $S_i x$. We show that $\mathrm{Gr}(T_i)$ is closed in $X \times X_i$. In fact, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(T_i)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$\begin{aligned} f_i(x^i, y) &\geq \limsup_\alpha f_i(x^i_\alpha, y_\alpha) \geq \limsup_\alpha M_i(x_\alpha) \\ &\geq \liminf_\alpha M_i(x_\alpha) \geq M_i(x) \end{aligned}$$

and, since $\mathrm{Gr}(S_i)$ is closed in $X \times X_i$, $y_\alpha \in S_i x_\alpha$ implies $y \in S_i x$. Hence $(x, y) \in \mathrm{Gr}(T_i)$. Since X_i is compact, T_i is u.s.c. Now we apply Theorem 3. Then there exists an $\hat{x} \in X$ such that $\hat{x} \in T_i \hat{x}$ for each $i \in I$; that is, $\hat{x}_i \in S_i \hat{x}$ and $f_i(\hat{x}^i, \hat{x}_i) \geq M_i(\hat{x})$. This completes our proof. □

From Theorem 5, we have the following quasi-equilibrium theorem:

THEOREM 5′. *Let $(X; \Gamma)$ be a compact G-convex space of type* II, *$f, g : X \times X \to \mathbf{R}$ u.s.c. functions, and $S : X \multimap X$ a closed map. Suppose that*

(1) *$g(x) \leq f(x)$ for each $x \in X$;*
(2) *the function M defined on X by*

$$M(x) = \max_{y \in Sx} g(x, y) \quad \text{for} \quad x \in X$$

is l.s.c.; and

(3) *for each $x \in X$, the set*

$$\{y \in Sx : f(x, y) \geq M(x)\}$$

is G-convex.

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in S\hat{x} \quad \text{and} \quad f(\hat{x}, \hat{x}) \geq M(\hat{x}).$$

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