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THE CURVATURE TENSORS IN THE EINSTEIN'S $^*g$-UNIFIED FIELD THEORY I. THE SE-CURVATURE TENSOR OF $^*g$-SEX$_n$

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ABSTRACT. Recently, Chung and et al. ([11], 1991c) introduced a new concept of a manifold, denoted by $^*g$-SEX$_n$, in Einstein's n-dimensional $^*g$-unified field theory. The manifold $^*g$-SEX$_n$ is a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda\nu}$ through the SE-connection which is both Einstein and semi-symmetric. In this paper, they proved a necessary and sufficient condition for the unique existence of SE-connection and presented a beautiful and surveyable tensorial representation of the SE-connection in terms of the tensor $^*g^{\lambda\nu}$.

This paper is the first part of the following series of two papers:

I. The SE-curvature tensor of $^*g$-SEX$_n$

II. The contracted SE-curvature tensors of $^*g$-SEX$_n$

In the present paper we investigate the properties of SE-curvature tensor of $^*g$-SEX$_n$, with main emphasis on the derivation of several useful generalized identities involving it. In our subsequent paper, we are concerned with contracted curvature tensors of $^*g$-SEX$_n$ and several generalized identities involving them. In particular, we prove the first variation of the generalized Bianchi's identity in $^*g$-SEX$_n$, which has a great deal of useful physical applications.

1. Introduction

In Appendix II to his last book Einstein ([12], 1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its ex-

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position is mainly geometrical. It may be characterized as a set of geometrical postulates for the space-time X_4. Characterizing Einstein's unified field theory as a set of geometrical postulates in X_4, Hlavatý ([13], 1957) gave its mathematical foundation for the first time. Since then Hlavatý and number of mathematicians contributed for the development of this theory and obtained many geometrical consequences of these postulates.

Generalizing X_4 to n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory, so called *Einstein's n-dimensional unified field theory* (denoted by n-g-UFT hereafter), had been attempted by Wrede ([15], 1958) and Mishra ([14], 1959). On the other hand, corresponding to n-g-UFT, Chung ([1], 1963) introduced a new unified field theory, called *the Einstein's n-dimensional *g-unified field theory* (denoted by n-$^*g$-UFT hereafter). This theory is more useful than n-g-UFT in some physical aspects. Chung and et al obtained many results concerning this theory ([2], 1969; [3], 1981; [6], 1988), particularly proving that n-$^*g$-UFT is equivalent to n-g-UFT so far as the classes and indices of inertia are concerned ([4], 1985). However, in both n-dimensional generalizations it has been unable yet to represent the general n-dimensional Einstein's connection in a surveyable tensorial form. This is probably due to the complexity of the higher dimensions.

Recently, Chung and et al ([5], 1987) introduced a new concept of n-dimensional SE-manifold (denoted by SEX_n hereafter), imposing the semi-symmetric condition to the Einstein's connection of X_n, and displayed a unique representation of the n-dimensional Einstein's connection in a beautiful and surveyable form in terms of $g_{\lambda\mu}$. Many results concerning SEX_n have been obtained since then ([7]-[10], 1989a-1991b).

Corresponding to SEX_n , Chung and et al ([11], 1991c) also introduced a manifold $^*g$-SEX_n in n-$^*g$-UFT. The manifold $^*g$-SEX_n is a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by the unified field tensor $*g^{\lambda\nu}$ through the SE-connection which is both Einstein and semi-symmetric. In this paper, they proved a necessary and sufficient condition for the unique existence of SE-connection in n-$^*g$-UFT and presented a surveyable tensorial representation of the SE-connection in terms of the tensor $*g^{\lambda\nu}$.

In this paper, Part I of a series of two papers, we investigate the properties of the SE-curvature tensor of $^*g$-SEX$_n$. Part II deals with its contracted curvature tensors, with main emphasis on the derivation of several useful generalized identities involving them.

2. Preliminaries

This section is a brief collection of basic concepts, notations, and results, which are needed in our subsequent considerations. They are due to Chung ([1], 1963; [4], 1985; [11], 1991c) and Mishra ([14], 1959), mostly due to [11].

(a) n-dimensional $^*g$-unified field theory

Corresponding to the Einstein's n-g-UFT[1], our n-$^*g$-UFT, initiated by Chung ([1], 1963), is based on the following three principles.

Principle A. Let X_n be an n-dimensional generalized Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformation $x^\nu \to x^{\nu'}$ for which

$$\det\left(\frac{\partial x'}{\partial x}\right) \neq 0. \tag{2.1}$$

In n-g-UFT the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[2]:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2a}$$

where

$$\mathfrak{g} = det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = det(h_{\lambda\mu}) \neq 0. \tag{2.2b}$$

[1] Hlavatý characterized Einstein's 4-dimensional unified field theory (4-g-UFT) as a set of geometrical postulates in X_4 for the first time [13] and gave its mathematical foundation.

[2] Throughout the present paper, Greek indices are used for the holonomic components of tensors in X_n. They take the values $1, 2, \cdots, n$, and follow the summation convention. *We also assume that* $n > 1$ in this paper.

In n-$^*g$-UFT the algebraic structure on X_n is imposed by the basic real tensor $^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}\,{}^*g^{\lambda\nu} \stackrel{\text{def}}{=} g_{\mu\lambda}\,{}^*g^{\nu\lambda} = \delta^\nu_\mu. \tag{2.3}$$

It may be also decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}. \tag{2.4}$$

Since $det(^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor $^*h_{\lambda\mu}$ by

$$^*h_{\lambda\mu}\,{}^*h^{\lambda\nu} \stackrel{\text{def}}{=} \delta^\nu_\mu. \tag{2.5}$$

In n-$^*g$-UFT we use both $^*h^{\lambda\nu}$ and $^*h_{\lambda\mu}$ as tensors for raising and/or lowering indices of all tensors defined in X_n in the usual manner. We then have

$$^*k_{\lambda\mu} = {}^*k^{\rho\sigma}\,{}^*h_{\lambda\rho}\,{}^*h_{\mu\sigma}, \quad {}^*g_{\lambda\mu} = {}^*g^{\rho\sigma}\,{}^*h_{\lambda\rho}\,{}^*h_{\mu\sigma} \tag{2.6a}$$

so that

$$^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu}. \tag{2.6b}$$

Principle B. The differential geometric structure on X_n is imposed by the tensor $^*g^{\lambda\nu}$ by means of a connection $\Gamma^\nu_{\lambda\mu}$ defined by a system of equations[3]

$$D_\omega\,{}^*g^{\lambda\mu} = -2S_{\omega\alpha}{}^\mu\,{}^*g^{\lambda\alpha}. \tag{2.7a}$$

[3]It has been proved that system (2.7a) is equivalent to

$$D_\omega\, g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha\, g_{\lambda\alpha} \tag{2.7b}$$

which is also equivalent to the original Einstein's equations

$$\partial_\omega\, g_{\lambda\mu} - \Gamma^\alpha_{\lambda\omega}\, g_{\alpha\mu} - \Gamma^\alpha_{\omega\mu}\, g_{\lambda\alpha} = 0. \tag{2.7c}$$

The equivalence of $(2.7a, b, c)$ was also shown by Hlavatý ([13]).

Here D_ω denotes the symbol of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$. Under certain conditions the system (2.7a) admits a unique solutions $\Gamma^\nu_{\lambda\mu}$.

Principle C. In order to obtain $^*g^{\lambda\nu}$ involved in the solution for $\Gamma^\nu_{\lambda\mu}$ certain conditions are imposed. These conditions may be condensed to

$$(2.8)\qquad S_\lambda \stackrel{\text{def}}{=} S_{\lambda\alpha}{}^\alpha = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu}Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0$$

where Y_λ is an arbitrary vector, and $R_{\omega\mu\lambda}{}^\nu$ together with $R_{\mu\lambda}$ and $V_{\omega\mu}$ are the curvature tensors of X_n defined by

$$(2.9)\qquad R_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\alpha{}^\nu{}_{[\mu}\Gamma_{|\lambda|}{}^\alpha{}_{\omega]})$$

$$(2.10)\qquad R_{\mu\lambda} \stackrel{\text{def}}{=} R_{\alpha\mu\lambda}{}^\alpha, \quad V_{\omega\mu} \stackrel{\text{def}}{=} R_{\omega\mu\alpha}{}^\alpha.$$

In the following remark, we summarize the main differences between n-g-UFT and n-$^*g$-UFT.

REMARK 2.1. In $\begin{cases} n-g-UFT \\ n-{}^*g-UFT \end{cases}$, the algebraic structure on X_n is imposed by the tensor $\begin{cases} g_{\lambda\mu} \\ {}^*g^{\lambda\nu} \end{cases}$, and $\begin{cases} \text{the tensor } h_{\lambda\mu} \text{ and} \\ \quad \text{its inverse tensor } h^{\lambda\nu} \\ \text{the tensor } {}^*h^{\lambda\nu} \text{ and} \\ \quad \text{its inverse tensor } {}^*h_{\lambda\mu} \end{cases}$ are used for raising and/or lowering the indices of tensors in X_n . On the other hand, the differential geometric structure on X_n is imposed by $\begin{cases} g_{\lambda\mu} \text{ in } n\text{-}g\text{-UFT} \\ {}^*g^{\lambda\nu} \text{ in } n\text{-}{}^*g\text{-UFT} \end{cases}$ through the Einstein's connection $\Gamma^\nu_{\lambda\mu}$ satisfying $\begin{cases} (2.7b) \\ (2.7a) \end{cases}$. Therefore, if the system $\begin{cases} (2.7b) \\ (2.7a) \end{cases}$ admits a unique solution, the connection $\Gamma^\nu_{\lambda\mu}$ will be expressed in terms of $\begin{cases} g_{\lambda\mu} \text{ in } n\text{-}g\text{-UFT} \\ {}^*g^{\lambda\nu} \text{ in } n\text{-}{}^*g\text{-UFT} \end{cases}$ in virtue of $\begin{cases} (2.7b) \\ (2.7a) \end{cases}$.

(b) Some notations and results

The following quantities are frequently used in our further considerations:

(2.11a) $$ {}^*\mathfrak{g} = det({}^*g_{\lambda\mu}), \quad {}^*\mathfrak{h} = det({}^*h_{\lambda\mu}), \quad {}^*\mathfrak{k} = det({}^*k_{\lambda\mu}) $$

(2.11b) $$ {}^*g = \frac{{}^*\mathfrak{g}}{{}^*\mathfrak{h}}, \quad {}^*k = \frac{{}^*\mathfrak{k}}{{}^*\mathfrak{h}} $$

(2.11c) $$ \sigma = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases} $$

(2.11d) $$ K_p = {}^*k_{[\alpha_1}{}^{\alpha_1}\, {}^*k_{\alpha_2}{}^{\alpha_2} \cdots {}^*k_{\alpha_p]}{}^{\alpha_p}, \qquad (p = 0, 1, 2, \cdots) $$

(2.11e) $$ {}^{(0)*}k_\lambda{}^\nu = \delta_\lambda^\nu, \quad {}^{(p)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\alpha \; {}^{(p-1)*}k_\alpha{}^\nu, \quad (p = 1, 2, \cdots). $$

In a general X_n it was proved that

(2.12a) $$ {}^{(p)*}k_{\lambda\mu} = (-1)^p \, {}^{(p)*}k_{\mu\lambda}, \qquad (p = 0, 1, 2, \cdots) $$

(2.12b) $$ K_0 = 1, \quad K_n = {}^*k \quad \text{if n is even,} \quad \text{and} $$
$$ K_p = 0 \quad \text{if p is odd} $$

(2.12c) $$ {}^*g = \sum_{s=0}^{n-\sigma} K_s $$

(2.12d) $$ \sum_{s=0}^{n-\sigma} K_s \, {}^{(n-s)*}k_\lambda{}^\mu = 0. $$

Here and in what follows, *the index s is assumed to take the values* $0, 2, 4, \cdots$ *in the specified range.*

We also use the following useful abbreviations for an arbitrary tensor $T^{\cdots}_{\cdots}$, for $p = 1, 2, 3, \cdots$:

$$\text{(2.13a)} \qquad {}^{(p)}T^{\nu\cdots}_{\cdots} \stackrel{\text{def}}{=} {}^{(p-1)*}k^{\nu}{}_{\alpha}\ T^{\alpha\cdots}_{\cdots}\ .$$

In virtue of (2.11e), an easy inspection gives

$$\text{(2.13b)} \qquad {}^{(p)}T^{\nu\cdots}_{\cdots} = {}^*k^{\nu}{}_{\alpha}\ {}^{(p-1)}T^{\alpha\cdots}_{\cdots}$$

$$\text{(2.13c)} \qquad {}^{(p)}T_{\lambda\cdots}^{\ \cdots} = {}^{(p-1)*}k_{\lambda}{}^{\alpha}\ T_{\alpha\cdots}^{\ \cdots} = {}^*k_{\lambda}{}^{\alpha}\ {}^{(p-1)}T_{\alpha\cdots}^{\ \cdots}\ .$$

We note that definition (2.11e) is a special case of (2.13). In particular, for an arbitrary vector Y_λ we have

$$\text{(2.14a)} \qquad {}^{(p)}Y^{\nu} = {}^{(p-1)*}k^{\nu}{}_{\alpha}\,Y^{\alpha} = {}^*k^{\nu}{}_{\alpha}\,{}^{(p-1)}Y^{\alpha}$$

$$\text{(2.14b)} \qquad {}^{(p)}Y_{\lambda} = {}^{(p-1)*}k_{\lambda}{}^{\alpha}\,Y_{\alpha} = {}^*k_{\lambda}{}^{\alpha}\,{}^{(p-1)}Y_{\alpha}\ .$$

(c) The SE-connection and $^*g$-SE-manifold in n-$^*g$-UFT

In this subsection, we display an useful representation of the SE-connection in n-$^*g$-UFT. All results presented in this subsection are due to [11].

DEFINITION 2.2. A connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be *semi-symmetric* if its torsion tensor $S_{\lambda\mu}{}^{\nu}$ is of the form

$$\text{(2.15)} \qquad S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda}\,X_{\mu]}$$

for an arbitrary vector $X_\lambda \neq 0$, which is not an gradient vector. A connection which is both semi-symmetric and Einstein[4] is called a *SE-connection.* An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by the tensor $^*g^{\lambda\nu}$ by means of a SE-connection, is called an *n-dimensional* **g-SE-manifold.* We denote this manifold by $^*g$-SEX$_n$ in our further considerations.

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[4]A connection is said to be *Einstein* if it satisfies the system of Einstein's equations (2.7).

THEOREM 2.3. *Under the condition* (2.15), *the system of equations* (2.7) *is equivalent to*

$$\Gamma^{\nu}_{\lambda\mu} = {}^*\{{}^{\ \nu}_{\lambda\mu}\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu} \tag{2.16}$$

and

$$\nabla_{\omega}\, {}^*k_{\lambda\mu} = 2\, {}^*h_{\omega[\lambda}\, X_{\mu]} + 2\, {}^*k_{\omega[\mu}\, {}^{(2)}X_{\lambda]} \tag{2.17}$$

where ∇_{ω} *is the symbolic vector of the covariant derivative with respect to the Christoffel symbols* ${}^*\{{}^{\ \nu}_{\lambda\mu}\}$ *defined by* ${}^*h_{\lambda\mu}$, *and*

$$S_{\lambda\mu}{}^{\nu} = 2\, \delta^{\nu}_{[\lambda}\, X_{\mu]}, \quad U^{\nu}{}_{\lambda\mu} = -{}^*h_{\lambda\mu}\, {}^{(2)}X^{\nu}. \tag{2.18}$$

In order to state next theorem, we need the following symmetric tensor:

$$A_{\lambda\mu} \stackrel{\text{def}}{=} (1-n)\, {}^*h_{\lambda\mu} + {}^{(2)*}k_{\lambda\mu}. \tag{2.19}$$

Since the tensor $A_{\lambda\mu}$ was shown to be of rank n, there exists a unique inverse tensor $B^{\lambda\nu}$, satisfying

$$A_{\lambda\mu}\, B^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{2.20}$$

THEOREM 2.4. *A necessary and sufficient condition for the system* (2.7) *to admit exactly one SE-connection of the form* (2.16) *is that the tensor field* ${}^*g^{\lambda\nu}$ *satisfies*

$$\nabla_{\omega}\, {}^*k_{\lambda\mu} = 2({}^*h_{\omega[\lambda}\, B_{\mu]}{}^{\alpha} + {}^*k_{\omega[\mu}\, {}^{(2)}B_{\lambda]}{}^{\alpha})\nabla_{\beta}\, {}^*k_{\alpha}{}^{\beta}. \tag{2.21}$$

If this condition is satisfied, then

$$X_{\lambda} = B_{\lambda}{}^{\alpha}\, C_{\alpha} \tag{2.22}$$

where

$$C_{\lambda} = \nabla_{\beta}\, {}^*k_{\lambda}{}^{\beta}. \tag{2.23}$$

AGREEMENT 2.5. *We assume that our further considerations in the present paper are restricted to the following two conditions:*

(i) *The quantity*

$$\phi = \sqrt{n-1} \tag{2.24}$$

is not a basic scalar in n-$^*g$-UFT.

(ii) *The condition (2.21) is always satisfied by the tensor field $^*g^{\lambda\nu}$.*

The situation that these two conditions are imposed on our $^*g$-SEX$_n$ are described in this paper by the words "present conditions".

We now state the following two theorems under the present conditions, which give surveyable tensorial representations of the SE-vector X_λ and the unique SE-connection $\Gamma^\nu_{\lambda\mu}$ in terms of the tensor field $^*g^{\lambda\nu}$.

THEOREM 2.6. *Under the present conditions, the SE-vector X_λ of $^*g$-SEX$_n$ is given by*

$$X_\lambda = \theta\psi \sum_{s=2}^{n-\sigma} H_s \, {}^{(n-s+\sigma+1)}C_\lambda - \sigma\theta C_\lambda \tag{2.25}$$

where the vector C_λ is defined by (2.23) and

$$H_0 \overset{\text{def}}{=} 0, \quad H_s \overset{\text{def}}{=} (n-1)H_{s-2} + K_{s-2} \; {}^{5} \qquad (s = 2, 4, \cdots, n+2-\sigma) \tag{2.26}$$

$$\theta = \frac{1}{1+(n-2)\sigma}, \quad \psi = -\frac{1}{H_{n+2-\sigma}}. \tag{2.27}$$

THEOREM 2.7. *Under the present conditions, the unique SE-connection $\Gamma^\nu_{\lambda\mu}$ of $^*g$-SEX$_n$ may be given by*

$$\begin{aligned}\Gamma^\nu_{\lambda\mu} = {}^*\{{}^{\,\nu}_{\lambda\mu}\} + \theta\psi \sum_{s=2}^{n-\sigma} H_s \, (2\, {}^{(n-s+\sigma+1)}C_{[\mu}\, \delta^\nu_{\lambda]} \\ - {}^*h_{\lambda\mu}\, {}^{(n-s+\sigma+2)}C^\nu) + \sigma\theta({}^*h_{\lambda\mu}\, {}^{(2)}C^\nu - 2\delta^\nu_{[\lambda}\, C_{\mu]}).\end{aligned} \tag{2.28}$$

[5]A direct calculation shows that

$$H_{n+2-\sigma} = K_0\phi^{n-\sigma} + K_2\phi^{n-2-\sigma} + \cdots + K_{n-\sigma-2}\phi^2 + K_{n-\sigma} \neq 0.$$

3. Two recurrence relations, and the vectors X_λ, S_λ, and U_λ

This section is concerned with two useful recurrence relations and identities satisfied by the vector X_λ, given by (2.15), and the vectors

$$S_\lambda \stackrel{\text{def}}{=} S_{\lambda\alpha}{}^{\alpha}, \quad U_\lambda \stackrel{\text{def}}{=} U^{\alpha}{}_{\lambda\alpha}. \tag{3.1}$$

THEOREM 3.1. *In $^*g$-SEX_n under the present conditions, the following recurrence relations hold:*

$$\sum_{s=0}^{n-\sigma} K_s\, {}^{(n-s+1)}X_\lambda = 0, \tag{3.2}$$

$${}^{(p)}X_\lambda = (n-1)\, {}^{(p-2)}X_\lambda + {}^{(p-2)}C_\lambda, \qquad (p = 3, 4, 5, \cdots) \tag{3.3}$$

where the vector C_λ is given by (2.23).

Proof. The relation (3.2) is a direct consequence of (2.12d) and (2.14b). In order to prove (3.3), multiply $A_\omega{}^{\lambda}$ to both sides of (2.22) and make use of (2.20) to obtain

$$X_\alpha\, A_\omega{}^{\alpha} = C_\omega. \tag{3.4a}$$

Substitution of (2.19) for $A_\omega{}^{\alpha}$ into (3.4a) gives

$${}^{(3)}X_\omega = (n-1)\, X_\omega + C_\omega. \tag{3.4b}$$

The relation (3.3) may be easily obtained from (3.4b) making use of (2.14b). □

We need the following theorem in order to prove Theorem (3.3).

THEOREM 3.2. *We have*

$$ln\, \mathfrak{g} = 2\, ln^*\mathfrak{h} - ln^*\mathfrak{g} \tag{3.5}$$

$$g_{\mu\nu}(\partial_\lambda {}^*g^{\mu\nu}) = -2\,\partial_\lambda\, ln^*\mathfrak{h} + \partial_\lambda\, ln^*\mathfrak{g}. \tag{3.6}$$

Proof. In virtue of (2.3) and (2.6a), we have

$$\mathfrak{g}\, det(^*g^{\lambda\nu}) = 1, \qquad ^*\mathfrak{g} = {}^*\mathfrak{h}^2\, det(^*g^{\lambda\nu}). \tag{3.7}$$

The relation (3.5) follows immediately from (3.7). The relation (3.6) is a consequence of (3.5) and

$$g_{\mu\nu}(\partial_\lambda\, ^*g^{\mu\nu}) = -\, ^*g^{\mu\nu}(\partial_\lambda\, g_{\mu\nu}) = -\frac{1}{\mathfrak{g}}(\partial_\lambda\, \mathfrak{g}) = -\partial_\lambda\, (ln\, \mathfrak{g})$$

which may be obtained from (2.3). □

THEOREM 3.3. *In $^*g$-SEX$_n$, the vectors S_λ and U_λ are given by*

$$S_\lambda = (1-n)X_\lambda \tag{3.8}$$

$$U_\lambda = -\, ^{(2)}X_\lambda = -\frac{1}{2}\, \partial_\lambda\, ln^*g. \tag{3.9}$$

Proof. Putting $\nu = \mu$ in (2.15) and (2.18), we have (3.8) and the first relation of (3.9), respectively. In order to prove the second relation of (3.9), consider the following Einstein's equations, which are equivalent to (2.7a):

$$\partial_\lambda\, ^*g^{\mu\nu} + \Gamma^{\mu}_{\alpha\lambda}\, ^*g^{\alpha\nu} + \Gamma^{\nu}_{\lambda\alpha}\, ^*g^{\mu\alpha} = 0. \tag{3.10}$$

Multiplying $g_{\mu\nu}$ to both sides of (3.10) and making use of (2.3) and (3.6), we have

$$g_{\mu\nu}(\partial_\lambda\, ^*g^{\mu\nu}) + \Gamma^{\alpha}_{\alpha\lambda} + \Gamma^{\alpha}_{\lambda\alpha} = 0$$

or equivalently

$$\Gamma^{\alpha}_{\lambda\alpha} = \partial_\lambda\, ln^*\mathfrak{h} - \frac{1}{2}\partial_\lambda\, ln^*\mathfrak{g} + S_\lambda. \tag{3.11}$$

On the other hand, in virtue of the classical result

$$^*\{^{\ \alpha}_{\lambda\alpha}\} = \tfrac{1}{2}\, ln(\partial_\lambda\, ^*\mathfrak{h}) \tag{3.12}$$

the relation (2.16) gives

$$\Gamma^{\alpha}_{\lambda\alpha} = \frac{1}{2}\, ln(\partial_\lambda\, ^*\mathfrak{h}) + S_\lambda + U_\lambda. \tag{3.13}$$

The second relation of (3.9) immediately follows from (3.11) and (3.13). □

THEOREM 3.4. *In $^*g$-SEX_n, the following relations hold for $p, q = 1, 2, \cdots$:*

$$^{(p+1)}S_\lambda = (1-n)\,^{(p+1)}X_\lambda = (n-1)\,^{(p)}U_\lambda \tag{3.14}$$

$$^{(p)}U_\alpha\,^{(q)}X^\alpha = (-1)^{p+1}\,^{(p+q-1)*}k_{\beta\gamma}\,X^\beta X^\gamma. \tag{3.15}$$

In particular,

$$^{(p)}U_\alpha\,^{(q)}X^\alpha = 0, \qquad \text{if } p+q-1 \text{ is odd.} \tag{3.16}$$

Proof. The relations (3.14) are direct consequences of (3.8), (3.9), and (2.14). Making use of (3.14), the relation (3.15) follows as in the following way:

$$^{(p)}U_\alpha\,^{(q)}X^\alpha = -^{(p+1)}U_\alpha\,^{(q)}X^\alpha = (-1)^{p+1}\,^{(p+q-1)*}k_{\beta\gamma}\,X^\beta X^\gamma.$$

The statement (3.16) may be proved from (3.15), since $^{(p+q-1)*}k_{\beta\gamma}$ is skew-symmetric if $p+q-1$ is odd. □

THEOREM 3.5. *In $^*g$-SEX_n, the following relations hold:*

$$D_\lambda X_\mu = \nabla_\lambda X_\mu \tag{3.17}$$

$$D_{[\lambda}X_{\mu]} = \nabla_{[\lambda}X_{\mu]} = \partial_{[\lambda}X_{\mu]} \tag{3.18}$$

$$\nabla_{[\lambda}U_{\mu]} = 0, \quad D_{[\lambda}U_{\mu]} = 2U_{[\lambda}X_{\mu]} = -2\,^{(2)}X_{[\lambda}\,X_{\mu]}. \tag{3.19}$$

Proof. In virtue of (2.14) and Theorem 2.3, the relation (3.17) follows as in the following way:

$$\begin{aligned} D_\lambda X_\mu &= \nabla_\lambda X_\mu - X_\alpha\, S_{\mu\lambda}{}^\alpha - X_\alpha\, U^\alpha{}_{\mu\lambda} \\ &= \nabla_\lambda X_\mu - 2X_{[\mu}\,X_{\lambda]} + {}^*h_{\mu\lambda}({}^*k_{\alpha\beta}\,X^\alpha X^\beta) = \nabla_\lambda X_\mu. \end{aligned}$$

The relations (3.18) are direct consequences of (3.17). Since $\partial_{[\lambda}U_{\mu]} = 0$ in virtue of the second relation of (3.9), we have the first relation of (3.19). Similarly, the second relations of (3.19) may be proved in virtue of (3.9). □

4. The SE-curvature tensor of $^*g$-SEX$_n$

The n-dimensional *SE-curvature tensor* $R_{\omega\mu\lambda}{}^{\nu}$ of $^*g$-SEX$_n$ is the curvature tensor defined by the SE-connection $\Gamma^{\nu}_{\lambda\mu}$ under the present conditions. A lengthy, but precise and surveyable tensorial representation of $R_{\omega\mu\lambda}{}^{\nu}$ in terms of $^*g^{\lambda\nu}$ and their first two derivatives may be obtained by simply substituting (2.16) for $\Gamma^{\nu}_{\lambda\mu}$ into (2.9).

In this section, we present more concise and useful tensorial representation of $R_{\omega\mu\lambda}{}^{\nu}$ in terms of $^*g^{\lambda\nu}$ and the SE-vector X_λ, and prove two identities involving it.

THEOREM 4.1. *Under the present conditions, the SE-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $^*g$-SEX$_n$ may be given by*

$$R_{\omega\mu\lambda}{}^{\nu} = {}^*H_{\omega\mu\lambda}{}^{\nu} + M_{\omega\mu\lambda}{}^{\nu} + N_{\omega\mu\lambda}{}^{\nu} \tag{4.1}$$

where

$$^*H_{\omega\mu\lambda}{}^{\nu} = 2(\partial_{[\mu}\, {}^*\{{}^{\,\nu}_{\omega]\lambda}\} + {}^*\{{}^{\;\nu}_{\alpha[\mu}\}\, {}^*\{{}^{\,\alpha}_{\omega]\lambda}\}) \tag{4.2a}$$

$$M_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{\lambda}\, \partial_{[\mu} X_{\omega]} + \delta^{\nu}_{[\mu}\, \nabla_{\omega]} X_\lambda - {}^*h_{\lambda[\omega}\, \nabla_{\mu]}\, {}^{(2)}X^{\nu}) \tag{4.2b}$$

$$N_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{[\mu}\, X_{\omega]} X_\lambda + {}^*h_{\lambda[\omega}\, {}^{(2)}X_{\mu]}\, {}^{(2)}X^{\nu}). \tag{4.2c}$$

Proof. Substitute (2.16) into (2.9) and make use of (4.2a) to obtain

$$\begin{aligned} R_{\omega\mu\lambda}{}^{\nu} = {} & 2(\partial_{[\mu}\, {}^*\{{}^{\,\nu}_{\omega]\lambda}\} + X_{\omega]}\, \delta^{\nu}_{\lambda} - \delta^{\nu}_{\omega]}\, X_\lambda + U^{\nu}{}_{\omega]\lambda}) + \\ & + 2({}^*\{{}^{\;\nu}_{\alpha[\mu}\} + \delta^{\nu}_{\alpha}\, X_{[\mu} - X_\alpha\, \delta^{\nu}_{[\mu} + U^{\nu}{}_{\alpha[\mu}) \times \\ & \times ({}^*\{{}^{\,\alpha}_{\omega]\lambda}\} + X_{\omega]}\, \delta^{\alpha}_{\lambda} - \delta^{\alpha}_{\omega]}\, X_\lambda + U^{\alpha}{}_{\omega]\lambda}) \\ = {} & {}^*H_{\omega\mu\lambda}{}^{\nu} + 2\delta^{\nu}_{\lambda}\partial_{[\mu} X_{\omega]} + 2\, (\delta^{\nu}_{[\mu}\, \partial_{\omega]} X_\lambda - \delta^{\nu}_{[\mu}\, {}^*\{{}^{\,\alpha}_{\omega]\lambda}\} X_\alpha) + \\ & + 2(\partial_{[\mu}\, U^{\nu}{}_{\omega]\lambda} + {}^*\{{}^{\;\alpha}_{\lambda[\omega}\}\, U^{\nu}{}_{\mu]\alpha} + {}^*\{{}^{\;\nu}_{\alpha[\mu}\}\, U^{\alpha}{}_{\omega]\lambda}) + \\ & + 2(\delta^{\nu}_{[\mu}\, X_{\omega]} X_\lambda - X_\alpha\, \delta^{\nu}_{[\mu}\, U^{\alpha}{}_{\omega]\lambda} + U^{\nu}{}_{\alpha[\mu}\, U^{\alpha}{}_{\omega]\lambda}). \end{aligned} \tag{4.3}$$

In virtue of the second relation of (2.18), the sum of the second, third and fourth terms on the right-hand side of (4.3) is $M_{\omega\mu\lambda}{}^{\nu}$. On the other hand, using (2.18), the first relation of (3.9), and (3.16), we have

$$U^{\nu}{}_{\lambda\mu} = -{}^{*}h_{\lambda\mu}\,{}^{(2)}X^{\nu} = {}^{*}h_{\lambda\mu}\,U^{\nu} \tag{4.4}$$

$$-X_{\alpha}\,\delta^{\nu}_{[\mu}\,U^{\alpha}{}_{\omega]\lambda} = -\delta^{\nu}_{[\mu}\,{}^{*}h_{\omega]\lambda}\,(X_{\alpha}\,U^{\alpha}) = 0 \tag{4.5a}$$

$$\begin{aligned} U^{\nu}{}_{\alpha[\mu}\,U^{\alpha}{}_{\omega]\lambda} &= ({}^{*}h_{\alpha[\mu}\,{}^{(2)}X^{\nu})({}^{*}h_{\omega]\lambda}\,{}^{(2)}X^{\alpha}) \\ &= {}^{*}h_{\lambda[\omega}\,{}^{(2)}X_{\mu]}\,{}^{(2)}X^{\nu}. \end{aligned} \tag{4.5b}$$

Substituting (4.5a, b) into the fifth term of (4.3), we find that it is equal to $N_{\omega\mu\lambda}{}^{\nu}$. Consequently, our proof of the theorem is completed. □

THEOREM 4.2. *Under the present condtions, the SE-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $^{*}g$-SEX$_n$ is a tensor involved in the following identity:*

$$R_{[\omega\mu\lambda]}{}^{\nu} = 4\delta^{\nu}_{[\lambda}\partial_{\mu}X_{\omega]} \tag{4.6}$$

Proof. The relation (4.1) gives

$$R_{[\omega\mu\lambda]}{}^{\nu} = {}^{*}H_{[\omega\mu\lambda]}{}^{\nu} + M_{[\omega\mu\lambda]}{}^{\nu} + N_{[\omega\mu\lambda]}{}^{\nu}. \tag{4.7}$$

On the other hand, in virtue of (4.2) we have

$${}^{*}H_{[\omega\mu\lambda]}{}^{\nu} = M_{[\omega\mu\lambda]}{}^{\nu} = 0, \quad N_{[\omega\mu\lambda]}{}^{\nu} = 4\delta^{\nu}_{[\mu}\partial_{\omega}X_{\lambda]}. \tag{4.8}$$

Our identity (4.6) is a consequence of (4.7) and (4.8). □

THEOREM 4.3. (Generalized Bianchi's identity in $^{*}g$-SEX$_n$) *Under the present conditions, the SE-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $^{*}g$-SEX$_n$ satisfies the following identity:*

$$D_{[\xi}R_{\omega\mu]\lambda}{}^{\nu} = -4X_{[\xi}\,{}^{*}H_{\omega\mu]\lambda}{}^{\nu} + Z_{[\xi\omega\mu]\lambda}{}^{\nu} \tag{4.9}$$

where

(4.10)

$$\begin{aligned} \frac{1}{8}Z_{\xi\omega\mu\lambda}{}^{\nu} = \{&\delta^{\nu}_{\lambda}X_{\xi}\,\partial_{\omega}X_{\mu} + X_{\xi}\,\delta^{\nu}_{\omega}\,\nabla_{\mu}X_{\lambda} \\ &- X_{\xi}\,\nabla_{\omega}({}^{*}h_{\mu\lambda}\,{}^{(2)}X^{\nu})\} - {}^{*}h_{\lambda\xi}\,X_{\omega}\,{}^{(2)}X_{\mu}\,{}^{(2)}X^{\nu}. \end{aligned}$$

Proof. On a manifold X_n to which an Einstein's connection is connected, Hlavatý proved the following identity([13], p.129):

$$D_{[\xi} R_{\omega\mu]\lambda}{}^{\nu} = -2S_{[\xi\omega}{}^{\beta} R_{\mu]\beta\lambda}{}^{\nu}. \tag{4.11}$$

In virtue of (2.15) and (4.1), the identity (4.11) may be written as

$$\begin{aligned} D_{[\xi} R_{\omega\mu]\lambda}{}^{\nu} &= -2S_{[\xi\omega}{}^{\beta}\, {}^*H_{\mu]\beta\lambda}{}^{\nu} - 2S_{[\xi\omega}{}^{\beta} M_{\mu]\beta\lambda}{}^{\nu} - 2S_{[\xi\omega}{}^{\beta} N_{\mu]\beta\lambda}{}^{\nu} \\ &= -4X_{[\omega}\, {}^*H_{\mu\xi]\lambda}{}^{\nu} - 4X_{[\xi} M_{\omega\mu]\lambda}{}^{\nu} - 4X_{[\xi} N_{\omega\mu]\lambda}{}^{\nu}. \end{aligned} \tag{4.12}$$

In virtue of (4.2b) the second term on the right-hand side of (4.12) may be expressed in the form

$$\begin{aligned} &- 4X_{[\xi} M_{\omega\mu]\lambda}{}^{\nu} \\ &\quad = -8(\delta^{\nu}_{\lambda} X_{[\xi}\partial_{\mu} X_{\omega]} + X_{[\xi}\, \delta^{\nu}_{\mu} \nabla_{\omega]} X_{\lambda} + X_{[\xi} \nabla_{\mu} U^{\nu}{}_{\omega]\lambda}). \end{aligned} \tag{4.13a}$$

The relation (4.2c) enables one to write the third term on the right-hand side of (4.12) as follows:

$$-4X_{[\xi} N_{\omega\mu]\lambda}{}^{\nu} = 8\, {}^*h_{\lambda[\omega} X_{\xi}\, {}^{(2)}X_{\mu]}\, {}^{(2)}X^{\nu}. \tag{4.13b}$$

We now substitute (4.13a,b) into (4.12) and make use of (4.10) to complete the proof of the theorem. □

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