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STABILITY THEOREMS FOR THE OPERATOR-VALUED FEYNMAN INTEGRAL: THE $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ THEORY

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ABSTRACT. In this paper, we prove stability theorems for the operator-valued Feynman integral of certain functionals involving some Borel measures on $(0,t)$ as a bounded linear operator from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$.

0. Introduction

In 1984, Johnson proved a bounded convergence theorem for the operator-valued function space integral [6]. As far as we know, this is the first stability theorem for the integral as a bounded linear operator on $L_2(\mathbb{R}^n)$ where n is any positive integer. In [9], Johnson and Skoug introduced stability theorems for the integral as an $\mathcal{L}(L_p(\mathbb{R}^N), L_{p'}(\mathbb{R}^N))$ theory, $1 < p \leq 2$, where N is a positive integer such that $N < \frac{2p}{2-p}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Chang studied stability theorems for the integral as a bounded linear operator from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$ [2]. In those papers mentioned above, they treated certain functionals which involve only the Lebesgue measure on the interval $(0,t)$.

In [7], Johnson and Lapidus established stability theorems for the integral as an $\mathcal{L}(L_2(\mathbb{R}^N), L_2(\mathbb{R}^N))$ theory for certain functionals involving any Borel measures on $(0,t)$. Chang and Ryu proved theorems insuring stability with respect to potentials and wave functions for the integral as a bounded linear operator on $L_p(\mathbb{R}^N)$ for certain functionals involving some Borel measures on $(0,t)$ [4].

Functionals we consider in this paper are defined in terms of potentials, wave functions and measures. We study the stability of the

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operator-valued Feynman integral as an $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ theory for the functionals involving some Borel measures on $(0,t)$ with respect to potentials, wave functions and measures.

1. Preliminaries and notations

Let $\mathbb{R}, \mathbb{C}, \mathbb{C}^+$ and $\tilde{\mathbb{C}}^+$ denote the set of all real numbers, all complex numbers, all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively. $C_0(\mathbb{R})$ will denote the space of $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ which vanish at ∞ with the supremum norm. $L_1(\mathbb{R})$ is the space of Borel measurable, $\mathbb{C}$-valued functions ψ on $\mathbb{R}$ such that $|\psi|$ is integrable with respect to the Lebesgue measure m on $\mathbb{R}$ with the norm $\|\psi\|_1 = \int |\psi|\,dm$. $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ will denote the space of bounded linear operators from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$. Let $\tilde{M}(0,t)$ denote the space of complex Borel measures η on the interval $(0,t)$ which satisfy the following conditions;

(1) If μ is the continuous part of η, the Radon-Nikodym derivative $\frac{d|\mu|}{dm}$ exists and is essentially bounded, where m is the Lebesgue measure on $(0,t)$.

(2) $\eta = \sum_{j=1}^{k} w_j \delta_{\tau_j} + \mu$, where δ_{τ_j} is the Dirac measure at $\tau_j \in (0,t)$, $0 < \tau_1 < \cdots < \tau_k < t$ and $w_j \in \mathbb{C}$ for $j = 1, 2, \cdots, k$.

Let $r \in (2,\infty]$ and $\eta \in \tilde{M}(0,t)$. Let $L_{1r:\eta}([0,t] \times \mathbb{R}) \equiv L_{1r:\eta}$ be the space of all Borel measurable $\mathbb{C}$-valued functions θ on $[0,t] \times \mathbb{R}$ such that

$$\| \theta \|_{1r:\eta} \equiv \left\{ \int_{(0,t)} \| \theta(s,\cdot) \|_1^r \, d|\eta|(s) \right\}^{\frac{1}{r}} \tag{1.1}$$

is finite. If θ is in $L_{1r:\eta}$ and $\eta = \mu + \nu$ is the Lebesgue decomposition, it is not difficult to show that $\theta \in L_{1r:\mu} \cap L_{1r:\nu}$. Let $\eta \in \tilde{M}(0,t)$. A Borel measurable $\mathbb{C}$-valued function θ on $[0,t] \times \mathbb{R}$ is said to belong to $L_{\infty 1:\eta}$ if

$$\|\theta\|_{\infty 1:\eta} = \int_{(0,t)} \|\theta(s,\cdot)\|_\infty \, d|\eta|(s) \tag{1.2}$$

is finite. For $\lambda \in \tilde{\mathbb{C}}^+, \psi \in L_1(\mathbb{R})$ and a positive real number s, let

$$(1.3) \qquad (C_{\lambda/s}\psi)(\xi) = (\frac{\lambda}{2\pi s})^{1/2} \int_{\mathbb{R}} \psi(u) \exp(-\frac{\lambda(u-\xi)^2}{2s})\, dm(u).$$

Then $C_{\lambda/s}$ is in $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ and $\|C_{\lambda/s}\| \le (|\lambda|/2\pi s)^{\frac{1}{2}}$ [8]. And as a function of λ, $C_{\lambda/s}$ is analytic in $\mathbb{C}^+$ and is weakly continuous in $\tilde{\mathbb{C}}^+$ [8]. Let θ be in $L_1(\mathbb{R})$ and let M_θ be the operator of multiplication from $C_0(\mathbb{R})$ to $L_1(\mathbb{R})$ given by $M_\theta\psi = \psi\theta$. Then M_θ is in $\mathcal{L}(C_0(\mathbb{R}), L_1(\mathbb{R}))$ and $\|M_\theta\| \le \|\theta\|_1$ [3]. It will be convenient to let $\theta(s)$ denote $M_{\theta(s,\cdot)}$ for θ in $L_{1r:\eta}$.

Let $C[0,t]$ be the space of continuous functions on $[0,t]$ and the Wiener space, $C_0[0,t]$, will consist of those x in $C[0,t]$ such that $x(0) = 0$. Integration over $C_0[0,t]$ will always be with respect to the Wiener measure m_w.

Let F be a functional from $C[0,t]$ to $\mathbb{C}$. Given $\lambda > 0, \psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

$$(1.4) \qquad (I_\lambda(F)\psi)(\xi) = \int_{C_0[0,t]} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)\, dm_w(x).$$

If $I_\lambda(F)\psi$ is in $C_0(\mathbb{R})$ as a function of ξ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose that there exists $\lambda_0 (0 < \lambda_0 < \infty)$ such that $I_\lambda(F)$ exists for all λ in $(0, \lambda_0)$ and further suppose that there exists an $\mathcal{L}$-valued function which is analytic in $\mathbb{C}^+_{\lambda_0} \equiv \{\lambda \in \mathbb{C} \,|\, Re\lambda > 0, |\lambda| < \lambda_0\}$ and agree with $I_\lambda(F)$ on $(0, \lambda_0)$. Then this $\mathcal{L}$-valued function is denoted by $I^{an}_\lambda(F)$ and is called the operator-valued analytic Wiener integral of F associated with λ. Finally, let q be in $\mathbb{R}$ with $0 < |q| < \lambda_0$. Suppose there exists an operator $J^{an}_q(F)$ in $\mathcal{L}$ such that for every ψ in $L_1(\mathbb{R})$, $J^{an}_q(F)\psi$ is the weak limit of $I^{an}_\lambda(F)\psi$ as $\lambda \to -iq$ through $\mathbb{C}^+_{\lambda_0}$. Then $J^{an}_q(F)$ is called the operator-valued Feynman integral of F associated with q.

As we continue, we will need to write

$$[w_1\theta(\tau_1, x(\tau_1)) + \cdots w_m\theta(\tau_m, x(\tau_m)) + \theta(s, x(s))]^n$$

as a product of monomials. However, we will need more refined breakdown of the sum. It will be convenient to introduce a prime notation on sum like $\sum'_{q_0+q_1+\cdots+q_{m-k}=n}$: this sum is to be over integers $q_0, q_1, \cdots, q_{m-k}$, where $q_0 \geq 0, q_1 \geq 1, \cdots, q_{m-k} \geq 1$ and $q_0 + \cdots + q_{m-k} = n$. Using this notation, we have the following equality [3]

(1.5)

$$\left[\sum_{j=1}^{m} w_j\theta(\tau_j, x(\tau_j)) + \theta(s, x(s))\right]^n$$
$$= \sum_{k=0}^{m} \sum_{1\leq z_1<\cdots<z_{m-k}\leq m} \sum'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n!}{q_0!\, q_1! \cdots q_{m-k}!}$$
$$[w_{z_1}\theta(\tau_{z_1}, x(\tau_{z_1}))]^{q_1} \cdots [w_{z_{m-k}}\theta(\tau_{z_{m-k}}, x(\tau_{z_{m-k}}))]^{q_{m-k}} [\theta(s, x(s))]^{q_0}.$$

Let $\eta \in \tilde{M}(0,t)$ and $\theta \in L_{1r:\eta}$. Set

(1.6)

$$F_n(x) := \left(\int_{(0,t)} \theta(s, x(s))\, d\eta(s)\right)^n, \ x \in C[0,t], \quad n = 0, 1, 2, \cdots.$$

Here, if $n = 0$, from the definition, we have $I_\lambda(F_0) = C_{\lambda/t}$.

We use the following two theorems from [1,3] in the sequel.

THEOREM 1.1. *Let* $\eta = \sum_{j=1}^{m} w_j\delta_{\tau_j} + \mu$ *where* δ_{τ_j} *is the Dirac measure at* $\tau_j \in (0,t), 0 < \tau_1 < \cdots < \tau_m < t$ *and* $w_j \in \mathbb{C}$ *for* $j = 1, 2, \cdots, m$. *Suppose that* $\theta(\tau_j, \cdot), j = 1, 2, \cdots, m$, *are essentially bounded. Then the operators* $I_\lambda^{an}(F_n)$ *and* $J_q^{an}(F_n)$ *exist for all* $\lambda \in \mathbb{C}^+$ *and all real*

$q \neq 0$, *respectively. Further for* $\lambda \in \mathbb{C}^+$, $\psi \in L_1(\mathbb{R})$ *and* $\xi \in \mathbb{R}$,

(1.7)
$$\begin{aligned}&(I_\lambda^{an}(F_n)\psi)(\xi)\\ &= \sum_{k=0}^{m} \sum_{1 \leq z_1 < \cdots < z_{m-k} \leq m} \sum_{q_0+q_1+\cdots+q_{m-k}=n}{}' \frac{n! w_{z_1}^{q_1} \cdots w_{z_{m-k}}^{q_{m-k}}}{q_1! \cdots q_{m-k}!} \\ &\quad \Bigg[\sum_{j_1+\cdots j_{m-k+1}=q_0} \int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} ((L_0 \circ L_1 \circ \cdots \\ &\qquad \cdots \circ L_{m-k})\psi)(\xi)\, d \mathop{\times}_{i=1}^{q_0} \mu(s_i)\Bigg],\end{aligned}$$

where

(1.8)
$$\begin{aligned}&\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}\\ &= \{(s_1,\cdots,s_{q_0}) \in (0,t)^{q_0} \mid 0 < s_1 < \cdots < s_{j_1} < \tau_{z_1} \\ &\qquad < s_{j_1+1} < \cdots < s_{j_1+\cdots+j_{m-k}} < \tau_{z_{m-k}} \\ &\qquad < s_{j_1+\cdots+j_{m-k}+1} < \cdots < s_{q_0} < t\}\end{aligned}$$

and for $(s_1,\cdots,s_{q_0}) \in \Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}$ *and* $\alpha \in \{0,1,\cdots,m-k\}$

(1.9)
$$\begin{aligned}L_\alpha = \theta(\tau_{z_\alpha})^{q_\alpha} &\circ C_{\lambda/(s_{j_1+\cdots+j_\alpha+1}-\tau_{z_\alpha})} \circ \theta(s_{j_1+\cdots+j_\alpha+1}) \circ \cdots \\ &\circ \theta(s_{j_1+\cdots+j_{\alpha+1}}) \circ C_{\lambda/(\tau_{z_{\alpha+1}}-s_{j_1}+\cdots+j_{\alpha+1})}.\end{aligned}$$

(It is convenient to let $\theta(\tau)^q$ *denote the operator of multiplication by* $[\theta(\tau,\cdot)]^q$*, that is,* $\theta(\tau)^q = M_{[\theta(\tau,\cdot)]^q}$*. We use the conventions* $\tau_0 = 0, \tau_{m+1} = t$ *and* $\theta(\tau_0)^{q_0} = 1$*, where 1 is the inclusion map.)*

For all real $q \neq 0$, $(J_q^{an}(F_n)\psi)(\xi)$ *is given by the right hand side of* (1.7) *with* $\lambda = -iq$*. Finally we have for* $\lambda \in \mathbb{C}^+$,

(1.10) $$\|I_\lambda^{an}(F_n)\| \leq B_n(|\lambda|),$$

where

(1.11)

$$B_n(|\lambda|) := (n!)^{\frac{1}{r'}}\left[\left(\frac{|\lambda|}{2\pi}\right)^{\frac{m+1}{2}} \vee \left(\frac{|\lambda|}{2\pi}\right)^{\frac{1}{2}}\right]\left[\min_{1\le j\le m}(\tau_j-\tau_{j-1})\right]^{-\frac{m+1}{2}} \left[\Gamma(l(1-\frac{r'}{2}))\right]^{-\frac{m+1}{r'}}\left[\Gamma(1-\frac{r'}{2})\right]^{\frac{m+1}{r'}}\left[\sum_{j=1}^{m}|w_j|(\|\theta(\tau_j,\cdot)\|_\infty \vee \|\theta(\tau_j,\cdot)\|_1) + \left(\sum_{j=1}^{m+1}(\tau_j-\tau_{j-1})^{1-\frac{r'}{2}}\right)^{\frac{1}{r'}}\left(\frac{|\lambda|}{2\pi}\right)^{\frac{1}{2}}\|\theta\|_{1r;\mu}\left\|\frac{d|\mu|}{dm}\right\|_\infty^{\frac{1}{r'}}\Gamma(1-\frac{r'}{2})^{\frac{1}{r'}}\right]^n,$$

where l is a positive integer such that $\Gamma(l(1-\frac{r'}{2}))$ is the minimum value of $\{\Gamma(i(1-\frac{r'}{2}))|i\in\mathbb{N}\}$, Γ is the gamma function, $\frac{1}{r}+\frac{1}{r'}=1$ and the notation $a\vee b$ means the maximum value of a and b. The inequality (1.10) also holds for $J_q^{an}(F_n)$ with $|\lambda|$ replaced by $|q|$.

Let $\lambda_0>0$ be given and $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an analytic function in $\mathbb{C}_{\lambda_0}^+$ such that $\sum_{n=0}^{\infty}|a_n|B_n(|\lambda|)<\infty$ for all λ in $\mathbb{C}_{\lambda_0}^+$.

Let

$$F(y)=f\left(\int_{(0,t)}\theta(s,y(s))d\eta(s)\right)\quad \text{for } y \text{ in } C[0,t]. \tag{1.12}$$

THEOREM 1.2. *Let $\eta=\sum_{j=1}^{m}w_j\delta_{\tau_j}+\mu$ where δ_{τ_j} is the Dirac measure at $\tau_j\in(0,t), 0<\tau_1<\cdots<\tau_m<t$ and $w_j\in\mathbb{C}$ for $j=1,2,\cdots,m$. Suppose that $\theta(\tau_j,\cdot), j=1,2,\cdots,m$, are essentially bounded. Then for $\lambda\in(0,\lambda_0)$ and $\xi\in\mathbb{R}$, $\sum_{n=0}^{\infty}a_nF_n(\lambda^{-\frac{1}{2}}x+\xi)$ converges absolutely for a.e. $x\in C_0[0,t]$. Also the operators $I_\lambda^{an}(F)$ and $J_q^{an}(F)$ exist for all*

$\lambda \in \mathbb{C}_{\lambda_0}^+$ *and for all real* q *with* $0 < |q| < \lambda_0$, *respectively. Further for* $\lambda \in \mathbb{C}_{\lambda_0}^+$

$$I_\lambda^{an}(F) = \sum_{n=0}^{\infty} a_n I_\lambda^{an}(F_n) \tag{1.13}$$

and

$$J_q^{an}(F) = \sum_{n=0}^{\infty} a_n \, J_q^{an}(F_n), \tag{1.14}$$

where F_n *is the functional defined in* (1.6). *Moreover, for* λ *in* $\mathbb{C}_{\lambda_0}^+$, *the series in* (1.13) *and* (1.14) *satisfy*

$$\|I_\lambda^{an}(F)\| \leq \sum_{n=0}^{\infty} |a_n| B_n(|\lambda|) \tag{1.15}$$

and

$$\|J_q^{an}(F)\| \leq \sum_{n=0}^{\infty} |a_n| B_n(|q|) \tag{1.16}$$

and both of them converge in the operator norm.

2. Stability theorems

Firstly, we establish the stability for the operator-valued Feynman integral of functionals involving some Borel measures on $(0,t)$ with respect to potentials.

THEOREM 2.1. *Let* η *be in* $\tilde{M}(0,t)$ *with* $\eta = \mu + \sum_{p=1}^{k} w_p \delta_{\tau_p}$. *Let* $H \in L_{1r;\eta}$ and $H(\tau_p, \cdot)$ *be essentially bounded for each* $p = 1, 2, \cdots, k$. *Let* $\theta^{(N)}, N = 1, 2, \cdots$, *be Borel measurable functions on* $[0,t] \times \mathbb{R}$ *such* *that for $\eta \times m$-a.e.*

(2.1.a) $$\theta^{(N)} \longrightarrow \theta \quad \text{as } N \to \infty$$

and

(2.1.b) $$|\theta^{(N)}| \leq |H| \quad \text{for } N = 1, 2, \cdots .$$

Then θ and $\theta^{(N)}$ belong to $L_{1r:\eta}$. Let $F_n^{(N)}$ be defined in (1.6) with θ replaced by $\theta^{(N)}$. Then for all real $q > 0$, $J_q^{an}(F_n)$ and $J_q^{an}(F_n^{(N)})$ exist for each $N \in \mathbb{N}$ and as $N \to \infty$,

$$J_q^{an}(F_n^{(N)}) \to J_q^{an}(F_n) \quad \text{in the operator norm.}$$

Proof. By (2.1), $\|\theta^{(N)}\|_{1r:\eta} \leq \|H\|_{1r:\eta}$ for $N = 1, 2, \cdots$ and so $\theta^{(N)}$ and θ are in $L_{1r:\eta}$. And $\theta(\tau_p, \cdot), p = 1, 2, \cdots, k$, are essentially bounded. Hence by Theorem 1.1, for each $N \in \mathbb{N}$, $J_q^{an}(F_n)$ and $J_q^{an}(F_n^{(N)}), N = 1, 2, \cdots$, exist for all real $q > 0$. For each $\psi \in L_1(\mathbb{R})$ and $q > 0$,

$$\begin{aligned}
&\|J_q^{an}(F_n^{(N)})\psi - J_q^{an}(F_n)\psi\|_\infty \\
&\leq \|\psi\|_1 \sum_{k=0}^{m} \sum_{1 \leq z_1 < \cdots < z_{m-k} \leq m} \sideset{}{'}\sum_{q_0 + q_1 + \cdots + q_{m-k} = n} \frac{n! |w_{z_1}|^{q_1} \cdots |w_{z_{m-k}}|^{q_{m-k}}}{q_1! \cdots q_{m-k}!} \\
&\left(\frac{|q|}{2\pi}\right)^{\frac{q_0 + m - k + 1}{2}} \Bigg[\sum_{j_1 + \cdots + j_{m-k+1} = q_0} \\
&\int_{\Delta_{q_0; j_1, \cdots, j_{m-k+1}}^{z_1, \cdots, z_{m-k}}} [s_1 \cdots (\tau_{z_1} - s_{j_1})(s_{j_1+1} - \tau_{z_1}) \cdots (t - s_{q_0})]^{-\frac{1}{2}} \\
&\int_{\mathbb{R}^{q_0 + m - k}} \Bigg| \prod_{i=1}^{q_0} \theta^{(N)}(s_i, v_i) \prod_{j=1}^{m-k} (\theta^{(N)}(\tau_{z_j}, v_j'))^{q_j} \\
&\qquad - \prod_{i=1}^{q_0} \theta(s_i, v_i) \prod_{j=1}^{m-k} (\theta(\tau_{z_j}, v_j'))^{q_j} \Bigg| d \mathop{\times}_{i=1}^{q_0} v_i \, d \mathop{\times}_{j=1}^{m-k} v_j' \, d \mathop{\times}_{i=1}^{q_0} |\mu|(s_i) \Bigg].
\end{aligned}$$

Since

$$\|J_q^{an}(F_n^{(N)})\psi - J_q^{an}(F_n)\psi\|_\infty$$
$$\le \|\psi\|_1(\|J_q^{an}(F_n^{(N)})\| + \|J_q^{an}(F_n)\|) < \infty,$$

by applying the Fubini theorem we obtain the above inequality.

(2.2)

$$\|J_q^{an}(F_n^{(N)}) - J_q^{an}(F_n)\|$$
$$\le \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum_{q_0+q_1+\cdots+q_{m-k}=n}' \frac{n!|w_{z_1}|^{q_1}\cdots|w_{z_{m-k}}|^{q_{m-k}}}{q_1!\cdots q_{m-k}!}$$
$$\left(\frac{|q|}{2\pi}\right)^{\frac{q_0+m-k+1}{2}}\left[\sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{m-k+1}}^{z_1,\cdots,z_{m-k}}} L(N)\, d\overset{q_0}{\underset{i=1}{\times}} |\mu|(s_i)\right],$$

where

(2.3)

$$L(N) = L(N;q_0;s_1,\cdots,s_{q_0})$$
$$= [s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{1}{2}}$$
$$\int_{\mathbb{R}^{q_0+m-k}}\left|\prod_{i=1}^{q_0}\theta^{(N)}(s_i,v_i)\prod_{j=1}^{m-k}(\theta^{(N)}(\tau_{z_j},v_j'))^{q_j}\right.$$
$$\left.-\prod_{i=1}^{q_0}\theta(s_i,v_i)\prod_{j=1}^{m-k}(\theta(\tau_{z_j},v_j'))^{q_j}\right| d\overset{q_0}{\underset{i=1}{\times}} v_i\, d\overset{m-k}{\underset{j=1}{\times}} v_j'.$$

We know that by (2.1.a), as $N\to\infty$,

$$\text{(2.4)}\qquad \prod_{i=1}^{q_0}\theta^{(N)}(s_i,v_i)\prod_{j=1}^{m-k}(\theta^{(N)}(\tau_{z_j},v_j'))^{q_j}$$
$$\to \prod_{i=1}^{q_0}\theta(s_i,v_i)\prod_{j=1}^{m-k}(\theta(\tau_{z_j},v_j'))^{q_j}\quad \text{a.e.}$$

Since for every $N \in \mathbb{N}, |\theta^{(N)}(s,u)| \leq |H(s,u)|$ for $\eta \times m$-a.e.(s,u),

(2.5)

$$\left|\prod_{i=1}^{q_0} \theta^{(N)}(s_i,v_i) \prod_{j=1}^{m-k} (\theta^{(N)}(\tau_{z_j},v'_j))^{q_j} - \prod_{i=1}^{q_0} \theta(s_i,v_i) \prod_{j=1}^{m-k} (\theta(\tau_{z_j},v'_j))^{q_j}\right| \leq 2\prod_{i=1}^{q_0} H(s_i,v_i) \prod_{j=1}^{m-k} H(\tau_{z_j},v'_j)^{q_j}.$$

Then, $\prod_{i=1}^{q_0} H(s_i,v_i)\prod_{j=1}^{m-k} H(\tau_{z_j},v'_j)^{q_j}$ is $\underset{i=1}{\overset{q_0}{\times}} v_i \underset{j=1}{\overset{m-k}{\times}} v'_j$-integrable. In view of (2.4) and (2.5), the dominated convergence theorem gives $L(N) \to 0$ as $N \to \infty$.

Now, we claim that

$$\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} L(N) d \underset{i=1}{\overset{q_0}{\times}} |\mu|(s_i) \to 0 \qquad \text{as } N \to \infty. \tag{2.6}$$

For a.e.$(s_1,\cdots,s_{q_0}) \in \Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}$, by (2.5) and the Fubini theorem

(2.7)

$$\begin{aligned}
&|L(N)| \\
&\leq 2[s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{1}{2}} \\
&\int_{\mathbb{R}^{q_0+m-k}} \prod_{i=1}^{q_0} H(s_i,v_i) \prod_{j=1}^{m-k} (H(\tau_{z_j},v'_j))^{q_j}\, d \underset{i=1}{\overset{q_0}{\times}} v_i\, d \underset{j=1}{\overset{m-k}{\times}} v'_j \\
&\leq 2[s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{1}{2}} \\
&\prod_{j=1}^{m-k} (\|H(\tau_{z_j},\cdot)\|_\infty^{q_j-1}\cdot\|H(\tau_{z_j},\cdot)\|_1) \prod_{i=1}^{q_0} \|H(s_i,\cdot)\|_1.
\end{aligned}$$

Then, by *Hölder's inequality*, the right hand side of the inequality in (2.7) is $\times_{i=1}^{q_0}\mu$-integrable. Hence, by the dominated convergence theorem, (2.6) is established. Therefore, $J_q^{an}(F_n^{(N)}) \to J_q^{an}(F_n)$ as $N \to \infty$ in the operator norm. Thus, we have proved the theorem. □

REMARK 2.2. For each $n \in \mathbb{N}$, define $B_n^*(|q|)$ by

(2.8)

$$
\begin{aligned}
&B_n^*(|q|) \\
&:\equiv (n!)^{\frac{1}{r'}} \left[\left(\frac{|q|}{2\pi}\right)^{\frac{m+1}{2}} \vee \left(\frac{|q|}{2\pi}\right)^{\frac{1}{2}} \right] \left[\min_{1\le j\le m} (\tau_j - \tau_{j-1}) \right]^{-\frac{m+1}{2}} \\
&\left[\Gamma(l(1-\frac{r'}{2}))\right]^{-\frac{m+1}{r'}} \left[\Gamma(1-\frac{r'}{2})\right]^{\frac{m+1}{r'}} \left[\sum_{j=1}^{m} |w_j|(\|H(\tau_j,\cdot)\|_\infty \vee \|H(\tau_j,\cdot)\|_1) \right. \\
&\left. + \left(\sum_{j=1}^{m+1} (\tau_j - \tau_{j-1})^{1-\frac{r'}{2}}\right)^{\frac{1}{r'}} \left(\frac{|q|}{2\pi}\right)^{\frac{1}{2}} \|H\|_{1r;\mu} \left\|\frac{d|\mu|}{dm}\right\|_\infty^{\frac{1}{r'}} \left[\Gamma(1-\frac{r'}{2})\right]^{\frac{1}{r'}} \right]^n .
\end{aligned}
$$

Then $\|J_q^{an}(F_n^{(N)})\| \le B_n^*(|q|)$ and $\|J_q^{an}(F_n)\| \le B_n^*(|q|)$.

Let $\lambda_0 > 0$ be given and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in $\mathbb{C}_{\lambda_0}^+$ such that $\sum_{n=0}^{\infty} |a_n| B_n(|q|) < \infty$ for all real q with $0 < |q| < \lambda_0$. Let

$$F(y) = f\left(\int_{(0,t)} \theta(s, y(s))\, d\eta(s)\right) \quad \text{for } y \text{ in } C[0,t] \tag{2.9}$$

and

$$F^{(N)}(y) = f\left(\int_{(0,t)} \theta^{(N)}(s, y(s))\, d\eta(s)\right) \quad \text{for } y \text{ in } C[0,t]. \tag{2.10}$$

THEOREM 2.3. *Let the hypotheses of* Theorem 2.1 *be satisfied. Then for each real* $q > 0, J_q^{an}(F)$ *and* $J_q^{an}(F^{(N)})$, $N = 1, 2, \cdots$, *exist where* F *and* $F^{(N)}$ $N = 1, 2, \cdots$, *are given by* (2.9) *and* (2.10), *respectively. Moreover, as* $N \to \infty$

$$J_q^{an}(F^{(N)}) \to J_q^{an}(F) \quad \text{in the operator norm.} \tag{2.11}$$

Proof. By Theorem 1.2, for each real $q > 0, J_q^{an}(F)$ and $J_q^{an}(F^{(N)})$, $N = 1, 2, \cdots$, exist. And further they can be represented as

$$J_q^{an}(F) = \sum_{n=0}^{\infty} a_n J_q^{an}(F_n)$$

and

$$J_q^{an}(F^{(N)}) = \sum_{n=0}^{\infty} a_n J_q^{an}(F_n^{(N)}) \quad \text{for all } N \in \mathbb{N}.$$

Since $\|J_q^{an}(F) - J_q^{an}(F^{(N)})\| \leq 2 \sum_{n=0}^{\infty} |a_n| B_n^*(q)$ by Remark 2.2, we have

$$\begin{aligned} (2.12) \qquad & \lim_{N\to\infty} J_q^{an}(F^{(N)}) \\ &= \lim_{N\to\infty} \sum_{n=0}^{\infty} a_n J_q^{an}(F_n^{(N)}) \\ &= \sum_{n=0}^{\infty} \lim_{N\to\infty} a_n J_q^{an}(F_n^{(N)}) \\ &= \sum_{n=0}^{\infty} a_n J_q^{an}(F_n) \\ &= J_q^{an}(F) \quad \text{in the operator norm.} \end{aligned}$$

Thus, the proof of Theorem 2.3 is complete. □

Now, we consider the stability for the operator-valued Feynman integral of functionals with respect to wave functions.

THEOREM 2.4. *Let $\{\psi^{(N)}\}$ be a sequence in $L_1(\mathbb{R})$ and $\| \psi^{(N)} - \psi \|_1 \to 0$ as $N \to \infty$. Then for $N \in \mathbb{N}, J_q^{an}(F)\psi$ and $J_q^{an}(F^{(N)})\psi^{(N)}$ exist in $C_0(\mathbb{R})$ for real q with $0 < |q| < \lambda_0$. Moreover,*

$$(2.13) \qquad \|J_q^{an}(F^{(N)})\psi^{(N)} - J_q^{an}(F)\psi\|_\infty \to 0 \quad \text{as } N \to \infty.$$

Proof. By Theorem 1.2, $J_q^{an}(F)\psi$ and $J_q^{an}(F^{(N)})\psi^{(N)}$ exist in $C_0(\mathbb{R})$ for $q > 0$. By Theorem 2.3 and $\|J_q^{an}(F^{(N)})\psi^{(N)} - J_q^{an}(F)\psi\|_\infty \leq \|J_q^{an}(F^{(N)})\|\|\psi^{(N)} - \psi\|_1 + \|J_q^{an}(F^{(N)}) - J_q^{an}(F)\|\|\psi\|_1$, we prove the theorem. □

THEOREM 2.5. *Suppose that $\{q_N\}$ is a sequence of real numbers which converges to a nonzero real number q with $0 < |q| < \lambda_0$. Then as $N \to \infty$,*

$$J_{q_N}^{an}(F) \to J_q^{an}(F) \quad \text{in the operator norm} . \tag{2.14}$$

Proof. Let q be in $\mathbb{R}$ with $0 < |q| < \lambda_0$ and $\psi \in L_1(\mathbb{R})$. Then, from (1.6) and (1.12) we have

(2.15)

$$\begin{aligned}
&(J_{q_N}^{an}(F)\psi)(\xi) - (J_q^{an}(F)\psi)(\xi) \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum\nolimits'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n! w_{z_1}^{q_1}\cdots w_{z_{m-k}}^{q_{m-k}}}{q_1!\cdots q_{m-k}!} \\
&\sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{m-k+1}}^{z_1,\cdots,z_{m-k}}} [s_1\cdots(t-s_{q_0})]^{-\frac{1}{2}} \\
&\Bigg\{\Bigg[(\frac{-iq_N}{2\pi})^{\frac{q_0+m-k+1}{2}} - (\frac{-iq}{2\pi})^{\frac{q_0+m-k+1}{2}}\Bigg] \\
&\int_{\mathbb{R}^{q_0+m-k+1}} \prod_{i=1}^{q_0} \theta(s_i,v_i) \prod_{j=1}^{m-k} (\theta(\tau_{z_j},v_j'))^{q_j}\, \psi(v_{m-k+1}) \\
&\exp\Bigg(\frac{iq_N}{2}(\frac{(v_1-\xi)^2}{s_1}+\cdots+\frac{(v'_{m-k+1}-v_{q_0})^2}{(t-s_{q_0})})\Bigg) d\overset{q_0}{\underset{i=1}{\times}} v_i\, d\overset{m-k+1}{\underset{j=1}{\times}} v_j' \\
&+ (\frac{-iq}{2\pi})^{\frac{q_0+m-k+1}{2}} \int_{\mathbb{R}^{q_0+m-k+1}} \prod_{i=1}^{q_0} \theta(s_i,v_i) \prod_{j=1}^{m-k} (\theta(\tau_{z_j},v_j'))^{q_j}\, \psi(v_{m-k+1}) \\
&\Bigg[\exp\Bigg(\frac{iq_N}{2}(\frac{(v_1-\xi)^2}{s_1}+\cdots+\frac{(v'_{m-k+1}-v_{q_0})^2}{(t-s_{q_0})})\Bigg) \\
&- \exp\Bigg(\frac{iq}{2}(\frac{(v_1-\xi)^2}{s_1}+\cdots+\frac{(v'_{m-k+1}-v_{q_0})^2}{(t-s_{q_0})^2})\Bigg)\Bigg] \\
&d\overset{q_0}{\underset{i=1}{\times}} v_i\, d\overset{m-k+1}{\underset{j=1}{\times}} v_j' \Bigg\} d\overset{q_0}{\underset{i=1}{\times}} \mu(s_i).
\end{aligned}$$

Let $\delta = \frac{1}{2}\min\{|q|, \lambda_0 - |q|\}$. By the hypotheses, there exists a positive integer M_0 such that if $N \geq M_0$, we have

$$(2.16)\qquad \begin{aligned} &\left|(\frac{-iq_N}{2\pi})^{\frac{q_0+m-k+1}{2}} - (\frac{-iq}{2\pi})^{\frac{q_0+m-k+1}{2}}\right| \\ &\leq (\frac{1}{2\pi})^{\frac{q_0+m-k+1}{2}}\left[|q_N|^{\frac{q_0+m-k+1}{2}} + |q|^{\frac{q_0+m-k+1}{2}}\right] \\ &\leq 2(\frac{|q|+\delta}{2\pi})^{\frac{q_0+m-k+1}{2}}. \end{aligned}$$

For each $n \in \mathbb{N}$,

(2.17)

$$\begin{aligned} &\|J_{q_N}^{an}(F_n) - J_q^{an}(F_n)\| \\ &\leq \sum_{k=0}^{m} \sum_{1\leq z_1<\cdots<z_{m-k}\leq m} \sum_{q_0+q_1+\cdots+q_{m-k}=n}' \sum_{j_1+\cdots+j_{m-k+1}=q_0} \\ &\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} [s_1\cdots(t-s_{q_0})]^{-\frac{1}{2}} \\ &\Bigg\{\left|(\frac{-iq_N}{2\pi})^{\frac{q_0+m-k+1}{2}} - (\frac{-iq}{2\pi})^{\frac{q_0+m-k+1}{2}}\right| \\ &\int_{\mathbb{R}^{q_0+m-k}} \prod_{i=1}^{q_0}|\theta(s_i,v_i)| \prod_{j=1}^{m-k}|\theta(\tau_{z_j},v_j')|^{q_j}\, d\overset{q_0}{\underset{i=1}{\times}} v_i\, d\overset{m-k}{\underset{j=1}{\times}} v_j' \\ &+\left|(\frac{-iq}{2\pi})\right|^{\frac{q_0+m-k+1}{2}} \int_{\mathbb{R}^{q_0+m-k}} \prod_{i=1}^{q_0}|\theta(s_i,v_i)| \prod_{j=1}^{m-k}|\theta(\tau_{z_j},v_j')|^{q_j} \\ &\left|\exp\left(\frac{iq_N}{2}(\frac{(v_1-\xi)^2}{s_1} - \cdots - \frac{(v'_{m-k+1}-v_{q_0})^2}{(t-s_{q_0})})\right)\right. \\ &\left.-\exp\left(\frac{iq}{2}(\frac{(v_1-\xi)^2}{s_1} - \cdots - \frac{(v'_{m-k+1}-v_{q_0})^2}{(t-s_{q_0})})\right)\right| \\ &d\overset{q_0}{\underset{i=1}{\times}} v_i\, d\overset{m-k}{\underset{j=1}{\times}} v_j'\Bigg\}\, d\overset{q_0}{\underset{i=1}{\times}} |\mu|(s_i). \end{aligned}$$

Then, for N such that $N > M_0$,

(2.18)

$$\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} [s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{1}{2}}$$
$$\Bigg\{\Big|(\frac{-iq_N}{2\pi})^{\frac{q_0+m-k+1}{2}}-(\frac{-iq}{2\pi})^{\frac{q_0+m-k+1}{2}}\Big|$$
$$\int_{\mathbb{R}^{q_0+m-k}}\prod_{i=1}^{q_0}|\theta(s_i,v_i)|\prod_{j=1}^{m-k}|\theta(\tau_{z_j},v'_j)|^{q_j}\,d\mathop{\times}_{i=1}^{q_0}v_i\,d\mathop{\times}_{j=1}^{m-k}v'_j$$
$$+\Big(\frac{|q|}{2\pi}\Big)^{\frac{q_0+m-k+1}{2}}\int_{\mathbb{R}^{q_0+m-k}}\prod_{i=1}^{q_0}|\theta(s_i,v_i)|\prod_{j=1}^{m-k}|\theta(\tau_{z_j},v'_j)|^{q_j}$$
$$\Big|\exp\Big(\frac{iq_N}{2}(\frac{(v_1-\xi)^2}{s_1}-\cdots-\frac{(v'_{m-k+1}-v_{q_0})^2}{(t-s_{q_0})})\Big)$$
$$-\exp\Big(\frac{iq}{2}(\frac{(v_1-\xi)^2}{s_1}-\cdots-\frac{(v'_{m-k+1}-v_{q_0})^2}{(t-s_{q_0})})\Big)\Big|$$
$$d\mathop{\times}_{i=1}^{q_0}v_i\,d\mathop{\times}_{j=1}^{m-k}v'_j\Bigg\}\,d\mathop{\times}_{i=1}^{q_0}|\mu|(s_i)$$
$$\le\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} [s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{1}{2}}$$
$$\Big[2(\frac{|q|+\delta}{2\pi})^{\frac{q_0+m-k+1}{2}}+(\frac{|q|}{2\pi})^{\frac{q_0+m-k+1}{2}}\Big]$$
$$\prod_{i=1}^{q_0}\|\theta(s_i,\cdot)\|_1\prod_{j=1}^{m-k}\|\theta(\tau_{z_j},\cdot)\|_\infty^{q_j-1}\,\|\theta(\tau_{z_j},\cdot)\|_1\Big\|\frac{d|\mu|}{dm}\Big\|_\infty^{q_0}\,d\mathop{\times}_{i=1}^{q_0}s_i$$
$$=\Big[2(\frac{|q|+\delta}{2\pi})^{\frac{q_0+m-k+1}{2}}+(\frac{|q|}{2\pi})^{\frac{q_0+m-k+1}{2}}\Big]\Big\|\frac{d|\mu|}{dm}\Big\|_\infty^{q_0}$$
$$\prod_{j=1}^{m-k}\|\theta(\tau_{z_j},\cdot)\|_\infty^{q_j-1}\,\|\theta(\tau_{z_j},\cdot)\|_1\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}}\prod_{i=1}^{q_0}\|\theta(s_i,\cdot)\|_1$$
$$[s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{1}{2}}d\mathop{\times}_{i=1}^{q_0}s_i.$$

Since $[s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{1}{2}}\|\theta(s_i,\cdot)\|_1$ is $\times_{i=1}^{q_0}s_i$-

integrable and $q_N \to q$, by the *Hölder's inequality* and the dominated convergence theorem, $J_{q_N}^{an}(F_n^{(N)})$ converges to $J_q^{an}(F_n)$ in the operator norm as $N \to \infty$.

Therefore, since $\|J_q^{an}(F) - J_q^{an}(F^{(N)})\| \leq 2\sum_{n=0}^{\infty} a_n B_n^*(q) < \infty$

$$\begin{aligned}(2.19)\qquad \lim_{N\to\infty} J_{q_N}^{an}(F) &= \lim_{N\to\infty} \sum_{n=0}^{\infty} a_n J_{q_N}^{an}(F_n) \\ &= \sum_{n=0}^{\infty} a_n \lim_{N\to\infty} J_{q_N}^{an}(F_n) = \sum_{n=0}^{\infty} a_n J_q^{an}(F_n) \\ &= J_q^{an}(F) \qquad \text{in the operator norm.}\end{aligned}$$

□

COROLLARY 2.6. *Suppose that the hypotheses of* Theorem 2.1 *and* Theorem 2.5 *are satisfied. Then as* $N \to \infty$

$$(2.20)\qquad J_{q_N}^{an}(F^{(N)}) \to J_q^{an}(F) \text{ in the operator norm .}$$

Proof. We may assume that $|q_N| < \lambda_0$ for sufficiently large N,

$$\|J_{q_N}^{an}(F^{(N)}) - J_q^{an}(F)\| \leq \|J_{q_N}^{an}(F^{(N)}) - J_{q_N}^{an}(F)\| + \|J_{q_N}^{an}(F) - J_q^{an}(F)\|.$$

Since $\|J_{q_N}^{an}(F^{(N)}) - J_{q_N}^{an}(F)\| \to 0$ as $N \to \infty$ for each q_N and $\|J_{q_N}^{an}(F) - J_q^{an}(F)\| \to 0$ as $N \to \infty$, thus $J_{q_N}^{an}(F^{(N)}) \to J_q^{an}(F)$ as $N \to \infty$ in the operator norm. □

COROLLARY 2.7. *Suppose that the hypotheses of* Theorem 2.1, Theorem 2.3 *and* Theorem 2.4 *hold. Then as* $N \to \infty$,

$$(2.21)\qquad \|J_{q_N}^{an}(F^{(N)})\psi_N - J_q^{an}(F)\psi\|_\infty \to 0.$$

Finally, we treat the stability theorem for the operator-valued Feynman integral with respect to measures.

THEOREM 2.8. *Let θ be a continuous function bounded by c and let η and $\eta_N, N = 1, 2, \cdots$ be in $\tilde{M}(0,t)$. Assume that*

$$\eta_N \to \eta \quad \textit{weakly}. \tag{2.22}$$

Let F be defined as in (2.9) and F_N be defined as in (2.9) with η replaced by η_N. Then

$$I_\lambda^{an}(F_N) \to I_\lambda^{an}(F) \quad \textit{in the operator norm,} \tag{2.23}$$

uniformly in λ on all compact subset of $\mathbb{C}_{\lambda_0}^+$.

Proof. For $\psi \in L_1(\mathbb{R})$, $\xi \in \mathbb{R}$ and $\lambda > 0$,

(2.24)
$$(I_\lambda(F_N)\psi)(\xi) = \int_{C_0[0,t]} f\left(\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}}y(s)+\xi)\, d\eta_N(s)\right) \psi(\lambda^{-\frac{1}{2}}y(t)+\xi)\, dm_w(y)$$

and we have similar result for F by replacing η_N by η in (2.24). Given $y \in C[0,t]$, the function $\theta(s, \lambda^{-\frac{1}{2}}y(s) + \xi)$ is bounded by c and it is continuous as a function of s. Hence, by (2.22),

$$\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}}y(s) + \xi) d\eta_N(s) \to \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}}y(s) + \xi) d\eta(s). \tag{2.25}$$

Since f is continuous,

(2.26)
$$f\left(\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}}y(s) + \xi) d\eta_N(s)\right) \psi(\lambda^{-\frac{1}{2}}y(t) + \xi) \to f\left(\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}}y(s) + \xi) d\eta_N(s)\right) \psi(\lambda^{-\frac{1}{2}}y(t) + \xi).$$

By the uniform boundedness principle and (2.22),

$$\left|\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}}y(s) + \xi)\, d\eta_N(s)\right|$$

is bounded by M with $M = c \cdot \sup_N \|\eta_N\|$. Thus

(2.27)
$$\left| f\left(\int_{(o,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi)\, d\eta_m(s) \right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi) \right| \leq M_1 |\psi(\lambda^{-\frac{1}{2}} y(t) + \xi)|,$$

where $M_1 = \sup_{|z| \leq M} |f(z)| < \infty$. Recall that $|\psi(\lambda^{-\frac{1}{2}} y(t) + \xi)|$ is Wiener integrable. In view of (2.26) and (2.27), the dominated convergence theorem yields

(2.28) $$(I_\lambda(F_N)\psi)(\xi) \longrightarrow (I_\lambda(F)\psi)(\xi) \quad \text{as } N \to \infty \text{ for a.e.} \xi \in \mathbb{R}.$$

Thus,

(2.29) $$I_\lambda(F_N)\psi \to I_\lambda(F)\psi \quad \text{in} \quad C_0(\mathbb{R}).$$

Now, $I_\lambda^{an}(F_N)$ is analytic for $\lambda \in \mathbb{C}_{\lambda_0}^+$. By Wiener integration formula [10],

(2.30) $$\|I_\lambda^{an}(F_N)\psi\|_\infty \leq M_1 \left(\frac{\lambda_0}{2\pi t} \right)^{\frac{1}{2}} \|\psi\|_1 \quad N = 1, 2, \cdots .$$

Hence, by (2.29) and (2.30), Vitali Theorem [5] gives the result for $\lambda \in \mathbb{C}_{\lambda 0}^+$. □

The conclusion of Theorem 2.8 can be reinforced provided that we assume that measures converge in the strong sense.

THEOREM 2.9. *Assume that $\eta_N \to \eta$ in norm. Then, under the hypotheses of* Theorem 2.8

(2.31) $$I_\lambda^{an}(F_N) \to I_\lambda^{an}(F)$$

uniformly in λ on all compact subset of $\mathbb{C}_{\lambda_0}^+$.

Proof. Given $y \in C[0,t]$ and $\xi \in \mathbb{R}$

(2.32)
$$\left|\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) d\eta_N(s) - \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) d\eta(s)\right| \le \|\eta_N - \eta\| \|\theta\|_\infty.$$

For $\psi \in L_1(\mathbb{R})$, a.e. $\xi \in \mathbb{R}$ and $\lambda > 0$

(2.33)
$$\begin{aligned} &\left| f\left(\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) d\eta_N(s)\right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi) \right. \\ &\left. - f\left(\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) d\eta(s)\right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi) \right| \\ &\le T_N |\psi(\lambda^{-\frac{1}{2}} y(t) + \xi)|, \end{aligned}$$

where $T_N = \sup\{|f(z_1) - f(z_2)| \mid |z_1 - z_2| \le \|\eta_N - \eta\| \|\theta\|_\infty\}$. Thus, for $\lambda > 0$

(2.34)
$$\begin{aligned} &\|I_\lambda(F_N)\psi - I_\lambda(F)\psi\|_\infty \\ &\le \int_{C_0[0,t]} T_N |\psi(\lambda^{-\frac{1}{2}} y(t) + \xi)| dm_w(y) \\ &\le T_N \left(\frac{\lambda_0}{2\pi t}\right)^{\frac{1}{2}} \|\psi\|_1. \end{aligned}$$

Since ψ is arbitrary and $T_N \to 0$, $I_\lambda(F_N) \to I_\lambda(F)$ in the operator norm topology. Since $I_\lambda^{an}(F_N)$ is analytic in $\mathbb{C}_{\lambda 0}^+$ and $\|I_\lambda^{an}(F_N)\| \le \|\eta_N\| \|\theta\|_\infty$, the Vitali theorem [5] yields (2.31). Thus, the proof of theorem is complete. □

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