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THE KNOT 5_2 AND CYCLICALLY PRESENTED GROUPS

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ABSTRACT. The cyclically presented groups which arise as fundamental groups of cyclic branched coverings of the knot 5_2 are studied. The fundamental polyhedra for these groups are described. Moreover the cyclic covering manifolds are obtained in terms of Dehn surgery and as two-fold branched coverings of the 3-sphere.

1. Introduction

The cyclically presented groups comprise a rich source of groups, which are interesting from a topological point of views. The connection between cyclically presented groups and cyclic branched coverings of knots and links was studied, in particular, in [3], [6], [10], [11], [12], [16], [19], and [20].

Let $\mathbb{F}_n = \langle x_1, \ldots, x_n | \quad \rangle$ be the free group of rank n and $\eta : \mathbb{F}_n \longrightarrow \mathbb{F}_n$ be the automorphism of order n such that $\eta(x_i) = x_{i+1}$, $i = 1, \ldots, n$, where the indices are taken mod n.

We recall [15, § 9] that for a reduced word $w \in \mathbb{F}_n$, the *cyclically presented group* $G_n(w)$ is given by

$$G_n(w) \ = \ \langle x_1, \, \ldots, \, x_n \mid w, \, \eta(w), \, \ldots, \, \eta^{n-1}(w) \rangle. \tag{1}$$

A group G is said to have a *cyclic presentation* if $G \cong G_n(w)$ for some n and w.

Clearly, the automorphism η of $\mathbb{F}_n$ induces an automorphism of $G_n(w)$. This cyclic automorphism has order dividing n and so we can consider

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the group $\widehat{G}_n(w)$ which is the split extension of $G_n(w)$ by the cyclic group of order n. It was remarked in [6] that the group $\widehat{G}_n(w)$ always has a 2-generator, 2-relator presentation of the form

$$(2) \qquad \widehat{G}_n(w) = \langle t, x \mid t^n = 1, \quad v(t,x) = 1 \rangle,$$

where $v = v(t,x)$ lies in the normal closure of t^n and x. Conversely, any group with such a presentation is the split extension of a $G_n(w)$ for some n.

Let $\mathcal{K}$ be a knot in the 3-sphere $\mathbb{S}^3$. We will say that a three-dimensional manifold M is a *n-fold cyclic branched covering of the knot* $\mathcal{K}$ if M is the n-fold cyclic branched covering of $\mathbb{S}^3$ branched over the knot $\mathcal{K}$ [2, Ch. 4], [26, §10C]. In other words, M is the covering of the orbifold $\mathcal{K}(n)$ with underlying space $\mathbb{S}^3$ and singular set the knot $\mathcal{K}$ with index n. In this case the fundamental group of the manifold M has the cyclic automorphism and the split extension is the group of the orbifold $\mathcal{K}(n)$. So, it is interesting to find the cyclic presentation for the fundamental group of the manifold, corresponding to this cyclic covering. Moreover, the problem of the distinguishing of the underlying space and singular set of the quotient orbifold arises.

For $\mathcal{K} = 3_1$, the trefoil knot, it was shown in [3] that the fundamental group of the n-fold cyclic branched covering of the knot 3_1 is isomorphic to the Sieradski group $S(n)$ with the presentation

$$(3) \qquad S(n) = \langle x_1, \ldots, x_n \mid x_i x_{i+2} = x_{i+1}, \quad i = 1, \ldots, n \rangle,$$

where the indices are taken mod n. In this case $w = x_1 x_3 x_2^{-1}$.

For $\mathcal{K} = 4_1$, the figure-eight knot, it was shown in [10] that the fundamental group of the n-fold cyclic branched covering of the knot 4_1 is isomorphic to the Fibonacci group $F(2,2n)$ with the cyclic presentation

$$F(2,2n) = \langle a_1, \ldots, a_{2n} \mid a_i a_{i+1} = a_{i+2}, \quad i = 1, \ldots, 2n \rangle,$$

where the indices are taken mod $2n$. In this case the covering manifolds are said to be *Fibonacci manifolds*. Considering elements $x_j = a_{2j}$, by $a_{2j+1} = a_{2j}^{-1} a_{2j+2}$, we get the presentation

$$(4) \quad F(2,2n) = \langle x_1, \ldots, x_n \mid x_j^{-1} x_{j+1}^2 x_{j+2}^{-1} x_{j+1} = 1, \quad j = 1, \ldots, n \rangle,$$

where the indices are taken mod n. In this case $w = x_1^{-1} x_2^2 x_3^{-1} x_2$.

In both above cases the cyclic presentations are closely connected with automorphisms of the free group $\mathbb{F}_2$, because knots 3_1 and 4_1 are fibre [2, Ch. 5C] and the commutator subgroups $[\pi(3_1), \pi(3_1)]$ and $[\pi(4_1), \pi(4_1)]$

are free groups of rank 2. In particular, it turns us to the construction of the Fibonacci manifolds by Dehn filling on once-punctured torus bundles [12]. We recall [2] that all knots fall into two different classes according to the structure of their commutator subgroups. The first of them comprises the knots whose commutator subgroups are finitely generated, and hence free, the second one those whose commutator subgroups cannot be finitely generated. The knot 5_2 is the simplest example of the second type.

In the present paper we consider finitely generated groups G_n, $n \geq 2$, with the following cyclic presentation

(5)

$$G_n = \langle x_1, \ldots, x_n \mid x_i \, x_{i+2} \, x_{i+1}^{-1} \, x_{i+2} \, x_{i+1}^{-1} \, x_i \, x_{i+1}^{-1} = 1, \quad i = 1, \ldots, n \rangle,$$

where the indices are taken mod n. In this case $w = x_1 x_3 x_2^{-1} x_3 x_2^{-1} x_1 x_2^{-1}$. We will demonstrate that these cyclically-presented groups are closely connected with the 2-bridge knot 5_2, that is the closure of the rational $(7/3)$–tangle (see [2] or [26] for knot notations).

In section 2 we will describe the fundamental polyhedron for the group G_n and demonstrate that this group is the fundamental group of a three-dimensional manifold M_n. In section 3 we will consider the split extension $\widehat{G}_n$ of G_n by the cyclic automorphism corresponding to the presentation (1) and will show that $\widehat{G}_n$ is the group of the orbifold $5_2(n)$ and the manifold M_n is the cyclic branched covering of the knot 5_2. In section 4 we will study topological properties of manifolds M_n. In particularly, these manifolds will be obtained by Dehn surgery and as two-fold branched coverings of $\mathbb{S}^3$.

2. The polyhedron $\mathcal{P}_n$ and the group G_n

In this section we construct the fundamental polyhedron (or a squashable complex according to [28], [3]) for the group G_n, and demonstrate, using the Siefert–Threlfall criterion, that G_n arises as fundamental group of a 3-manifold.

THEOREM 1. *For $n \geq 2$ the group G_n is a fundamental group of a three-dimensional manifold.*

Proof. Let us consider a polyhedron $\mathcal{P}_n$, $n \geq 2$, whose boundary, which can be regarded as the 2-sphere $\mathbb{S}^2$, consists of n triangles $T_i' =$

NA_iA_{i+1} in the north hemisphere, n quadrilaterals $Q'_i = SB_iC_iB_{i+1}$ in the south hemisphere, n triangles $T_i = A_{i-1}B_{i-1}C_{i-1}$ and n quadrilaterals $Q_i = A_iA_{i+1}B_{i+1}C_i$ in the equatorial zone, where $i = 1, \dots, n$, and all indices are taken by mod n. In this case $\mathcal{P}_n$ has $4n$ faces, $7n$ edges and $3n+2$ vertices. The polyhedron $\mathcal{P}_4$ is pictured in Figure 1. Let us consider the 1-skeleton of $\mathcal{P}_n$ with orientation and labelling of its edges in the following manner.

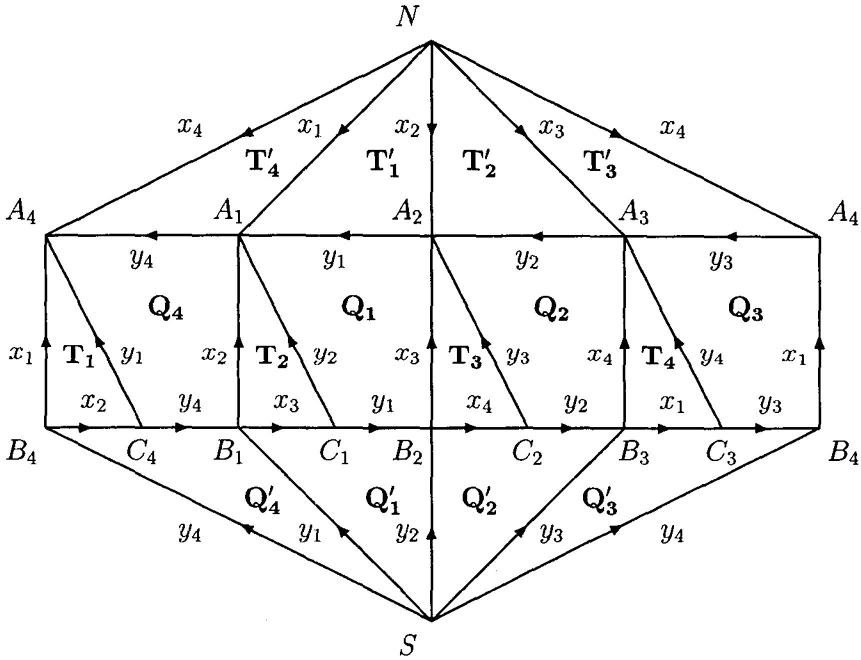

FIGURE 1. The polyhedron $\mathcal{P}_4$.

1. The oriented edges fall into $2n$ classes: x_i, $i = 1, \dots, n$, where each class x_i consists three edges, and y_i, $i = 1, \dots, n$, where each class y_i consists four edges. In this case oriented edges from the same class carry the same label.
2. For each $i = 1, \dots, n$ the boundary cycle of the triangles T_i and T'_i is $x_i^{-1}x_{i+1}y_i$ with the indices taken mod n.
3. For each $i = 1, \dots, n$ the boundary cycle of the quadrilaterals Q_i and Q'_i is $y_{i+1}^{-1}y_ix_{i+2}y_i$ with the indices taken mod n.

Now we consider pairwise identifying of faces of the polyhedron $\mathcal{P}_n$. For each $i = 1, \ldots, n$ let t_i identifies triangles T_i and T_i', and q_i identifies quadrilaterals Q_i and Q_i' such that the corresponding oriented edges of polygons carrying the same label are identified.

Therefore we get the following cycles of equivalent edges. For the label x_i :

$$B_{i-1}A_{i-1} \xrightarrow{t_i} NA_i \xrightarrow{t_{i-1}^{-1}} B_{i-2}C_{i-2} \xrightarrow{q_{i-2}^{-1}} B_{i-1}A_{i-1},$$

whence

$$t_i \, t_{i-1}^{-1} \, q_{i-2}^{-1} \; = \; 1.$$

Analogously for the label y_i :

$$A_{i+1}A_i \xrightarrow{q_i} C_iB_{i+1} \xrightarrow{q_i} SB_i \xrightarrow{q_{i-1}^{-1}} C_{i-1}A_{i-1} \xrightarrow{t_i} A_{i+1}A_i,$$

whence

$$q_i \, q_i \, q_{i-1}^{-1} \, t_i \; = \; 1.$$

The resulting complex M_n has 1 vertex, $2n$ edges, $2n$ two-cells and 1 three-cell.

There is a following criterion, due to H. Seifert and W. Threlfall [27, p. 216], for M_n to be a manifold: *A complex, which is formed by identifying the faces of a polyhedron will be a manifold if and only if its Euler characteristic equals zero.*

Applying this criterion to our case, we get that M_n is a 3-manifold and its fundamental group has the following presentation:

$$\begin{aligned} \pi_1(M_n) \; = \; \langle t_1, \ldots, t_n, q_1, \ldots, q_n \mid \; & t_i \, t_{i-1}^{-1} \, q_{i-2}^{-1} \; = \; 1, \\ & q_i \, q_i \, q_{i-1}^{-1} \, t_i \; = \; 1, \; i = 1, \ldots, n \rangle, \end{aligned} \tag{6}$$

where the indices are taken mod n.

As one can see, the group $\pi_1(M_n)$ with the presentation (6) is isomorphic to the group G_n with the presentation (5). Indeed, from relations of the first type in (6) we get $q_i = t_{i+2}t_{i+1}^{-1}$, and substituting into relations of the second type we will get

$$\pi_1(M_n) \; = \; \langle t_1, \ldots, t_n \mid t_{i+2} \, t_{i+1}^{-1} \, t_{i+2} \, t_{i+1}^{-1} \, t_i \, t_{i+1}^{-1} \, t_i \; = \; 1, \; i = 1, \ldots, n \rangle.$$

Therefore, $G_n \cong \pi_1(M_n)$ is the fundamental group of a 3-manifold. □

We remark that the polyhedron $\mathcal{P}_n$ can be considered as the natural generalization of the fundamental polyhedron for the Fibonacci group $F(2, 2n)$ constructed in [10]. Indeed, we will get the polyhedron from

[10] if assume that vertices C_i and B_{i+1} coincide for all $i = 1, \ldots, n$ and is this case faces Q_i and Q'_i become triangular.

3. The split extension of the group G_n

From the cyclic presentation (5) we see that the group G_n has the cyclic automorphism $\rho : x_i \to x_{i+1}$ of order n. This automorphism corresponds to the symmetry of order n (also denoted by ρ) of the polyhedron $\mathcal{P}_n$ such that

$$\rho : T_i \longrightarrow T_{i+1}, \ T'_i \longrightarrow T'_{i+1}, \ Q_i \longrightarrow Q_{i+1}, \ Q'_i \longrightarrow Q'_{i+1},$$

where indices are taken mod n.

In respect to the presentation (6) the automorphism ρ acts as the following:

$$\rho \, : \, t_i \longrightarrow t_{i+1}, \quad q_i \longrightarrow q_{i+1}.$$

Let us consider the split extension $\widehat{G}_n$ of group G_n by the cyclic group of automorphisms generated by ρ. The following demonstrates that the group $\widehat{G}_n$ is interesting from the topological point of views.

Denote by $5_2(n)$ the orbifold with the 3-sphere as underlying space and the knot 5_2 with index n as singular set.

THEOREM 2. *For $n \geq 2$ the group $\widehat{G}_n$ is fundamental group of the orbifold $5_2(n)$.*

Proof. From the presentation (6) we get the following presentation with notations $t = t_1$ and $q = q_1$:

$$\begin{aligned} \widehat{G}_n &= \langle \rho, t, q \mid (\rho^{-1} t \rho) t^{-1} (\rho q \rho^{-1})^{-1} = 1, \\ &\qquad q^2 (\rho q \rho^{-1})^{-1} t = 1, \quad \rho^n = 1 \rangle \\ &= \langle \rho, t, q \mid (t^{-1} \rho)^{-1} \rho (t^{-1} \rho) = \rho q, \\ &\qquad t^{-1} \rho = q^2 \rho q^{-1}, \quad \rho^n = 1 \rangle. \end{aligned}$$

Let us consider $\mu = \rho q$, that is conjugate to ρ and so, $\mu^n = 1$. Then $q = \rho^{-1} \mu$, and

$$t^{-1} \rho = (\rho^{-1} \mu)^2 \rho (\rho^{-1} \mu)^{-1} = \rho^{-1} \mu \rho^{-1} \mu \rho \mu^{-1} \rho.$$

Thus we have a 2-generator presentation for the group $\widehat{G}_n$:

$$\begin{aligned}\widehat{G}_n &= \langle \rho, \mu \mid \rho(\rho^{-1}\mu\rho^{-1}\mu\rho\mu^{-1}\rho) = (\rho^{-1}\mu\rho^{-1}\mu\rho\mu^{-1}\rho)\mu, \\ &\qquad \rho^n = 1, \quad \mu^n = 1\rangle \\ &= \langle \rho, \mu \mid \rho(\mu\rho^{-1}\mu\rho\mu^{-1}\rho) = (\mu\rho^{-1}\mu\rho\mu^{-1}\rho)\mu, \\ &\qquad \rho^n = 1, \quad \mu^n = 1\rangle.\end{aligned}$$

We recall that the group

$$\langle a, b \mid b(ab^{-1}aba^{-1}b) = (ab^{-1}aba^{-1}b)a\rangle$$

is the group of the 5_2-knot, where a and b corresponding to Figure 2.

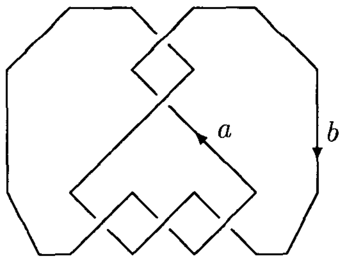

FIGURE 2. The knot 5_2.

Therefore by [9], the group $\widehat{G}_n$ is the group of the orbifold $5_2(n)$. □

THEOREM 3. *For $n \geq 2$ the manifolds M_n is the n-fold cyclic branched covering of the 3-sphere branched over the knot 5_2.*

Proof. Let us consider the above automorphism ρ of the group $G_n = \pi_1(M_n)$, and denote the corresponding homeomorphism of M_n also by ρ. Because the automorphism ρ of G_n corresponds to the symmetry of the polyhedron $\mathcal{P}_n$, we see that the $\frac{1}{n}$-piece Π_n of $\mathcal{P}_n$, pictured in Figure 3 is the fundamental polyhedron for the quotient space M_n/ρ with $\widehat{G}_n = \pi_1^{\mathrm{orb}}(M_n/\rho)$. The complex Π_n has faces NA_1B_1S, NA_2B_2S, NA_1A_2, $A_1A_2B_2C_1$, $A_1B_1C_1$, and $SB_1C_1B_2$. These faces are pairwise equivalent under the group $\widehat{G}_n$ action.

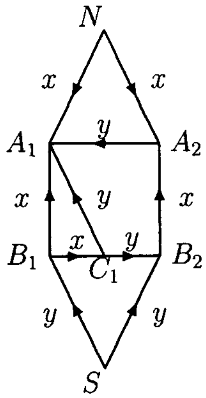

FIGURE 3.

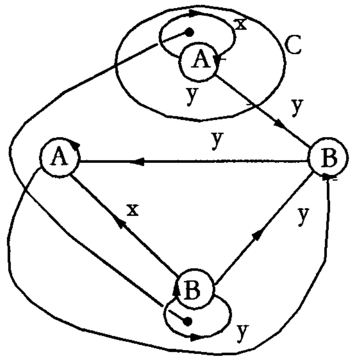

FIGURE 4.

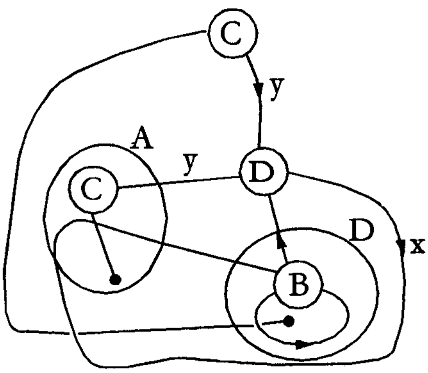

FIGURE 5.

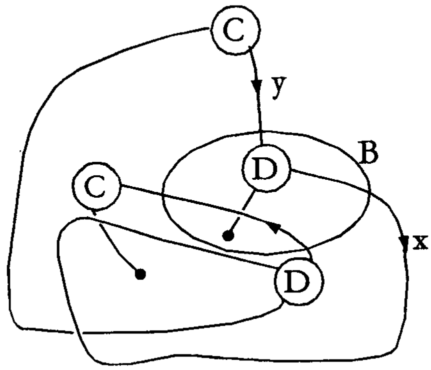

FIGURE 6.

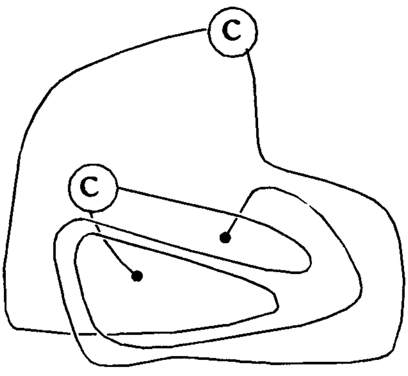

FIGURE 7.

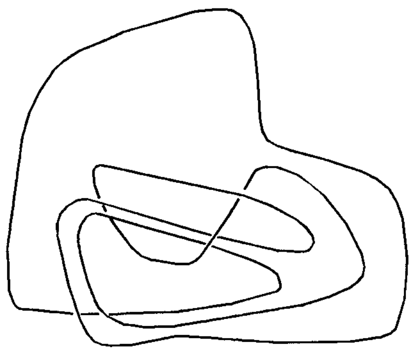

FIGURE 8.

After the identification of faces NA_1B_1S and NA_2B_2S, we can redraw the remain four faces as the regions on the 2-sphere, In this case we will get a Heegaard diagram, pictured in Figure 4. The axe of the rotation ρ is pictures as a thick curve. It lies below the diagram, inside the ball whose boundary is being identified along the two disc pairs denoted by A and B. Using the approach used in [12] for the figure-eight knot, we can modify Figure 4 to Figure 8. Figure 5 is obtained from Figure 4 by a simplification along C. Figure 6 is obtained from Figure 5 by a simplification along D. Figure 7 is obtained from Figure 6 by a cancellation of handles. It is also a Heegaard diagram for the quotient (i.e. $\mathbb{S}^3$). Cancelling the last part of handles we will get Figure 8. In all these pictures the image of the axe of ρ is a thick curve. It is easy to see by Reidemeister moves that the knot pictured in Figure 8 is the 5_2-knot.

Thus, we get that the image of the unknotted curve inside the 3-ball which connects vertices N and S, that is the image of the axe of the rotation ρ, forms the knot 5_2. Moreover, in this case the image of the 3-ball is the 3-ball again. Thus, the rotation ρ corresponds to the covering, branched over the knot 5_2. Therefore, the manifold M_n can be obtained as the n-fold regular branched covering of the 3-sphere, branched over the knot 5_2. □

We recall [13], that the orbifold $5_2(n)$, that is denoted by $(7/3)(n)$ in [13], is hyperbolic for $n \geq 3$, and it is spherical for $n = 2$.

COROLLARY 1. *The manifolds M_n is hyperbolic for $n \geq 3$, and M_2 is the lens space $L(7,3)$.*

COROLLARY 2. *The group G_n is infinite for $n \geq 3$, and $G_2 \cong \mathbb{Z}_7$.*

In virtue of the complete characterization of the arithmeticity of orbifolds $5_2(n)$ (that is the characterization of the arithmeticity of groups $\widehat{G}_n$) obtained in [13] we get the following property.

COROLLARY 3. *The group G_n is arithmetic if and only if $n = 3,4,5,6$.*

4. The topological properties of the manifolds M_n

In this section we will study the topological properties of manifolds M_n, that gives a topological approach to the studying of cyclically-presented groups G_n. This studying is analogous to the topological studying of Sieradski groups $S(n)$ and Fibonacci groups $F(2, 2n)$ given in [3], [4], [12], and [21].

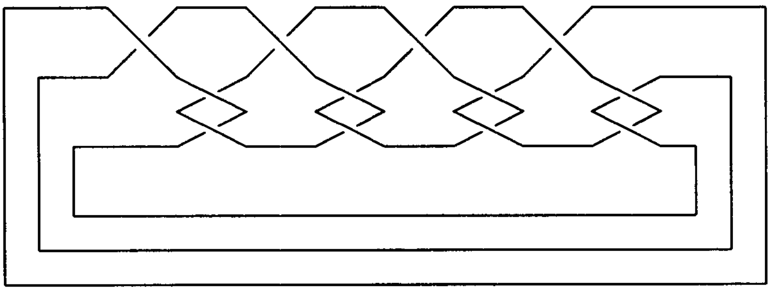

FIGURE 9. The knot $\mathcal{K}_4$.

Firstly we define a series of knots. We recall that any knot can be obtained as the closure of some braid [2]. let p and q be coprime integers, then by $\sigma_i^{p/q}$ we denote the rational p/q–tangle whose incoming arcs are i-th and $(i+1)$-th strings. For an integer $n \geq 1$ we define the n-periodic knot $\mathcal{K}_n$ as the closure of the rational 3-strings braid $(\sigma_1 \sigma_2^{1/2})^n$. We recall that the fragment $(\sigma_1 \sigma_2^{1/2})$ was used by R. Fox and E. Artin [8] (see also [1, p.48], [24, p.142]) for the construction of a wild 2-sphere. The knots can be regarded as finite n-periodic fragments of the Fox–Artin arc, so we will be say the knots $\mathcal{K}_n$ to be *Fox–Artin* knots. The diagram of the knot $\mathcal{K}_4$ is pictured in Figure 9.

Obviously, $\mathcal{K}_1$ is a trivial knot. It is easy to check directly, that the knot $\mathcal{K}_2$ is equivalent, under the Reidemeister moves, to the 2-periodic knot 5_2. The knot $\mathcal{K}_3$ is the non-alternating 3-periodic knot 9_{49} (see [2, p. 265]).

THEOREM 4. *For $n \geq 2$ the manifold M_n is the two-fold covering of the 3-sphere branched over the Fox–Artin knot $\mathcal{K}_n$.*

Proof. By theorem 3 the manifold M_n is the n-cyclic branched covering of the 3-sphere $\mathbb{S}^3$, branched over the 2-periodic knot 5_2. To describe M_n as the 2-cyclic branched covering of $\mathbb{S}^3$, branched over a n-periodic knot, we will use the following construction which is analogously to [21] and [3] where the Fibonacci groups and the Sieradski groups were topologically studied.

From the presentation of the knot 5_2 in the form $\mathcal{K}_2$ we see that the orbifold $5_2(n)$ has symmetry of order 2 such that the axe of the symmetry and the singular set of the orbifold are disjoint. Therefore, the quotient space of $5_2(n)$ under this symmetry action is the orbifold whose underlying space is $\mathbb{S}^3$, and whose singular set is the 2-component link pictured in Figure 10 with branch indices 2 and n.

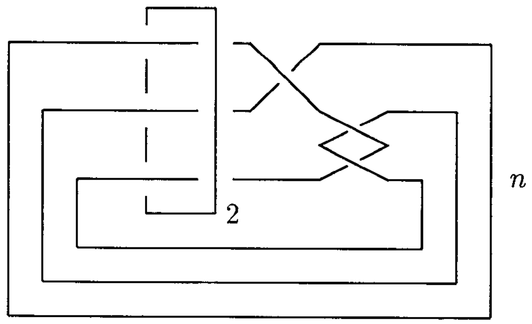

FIGURE 10. The link 7_1^2.

It is easy so see, that the singular set of the quotient orbifold is the two-component link 7_1^2, that is the 2-bridge link and can be obtained as the closure of the rational (14/3)–tangle. So, we denote the quotient orbifold by $7_1^2(2,n)$. Thus, we have the following covering diagram

$$M_n \xrightarrow{n} 5_2(n) \xrightarrow{2} 7_1^2(2,n) \tag{7}$$

and a sequence of normal subgroups

$$G_n = \pi_1(M_n) \lhd \widehat{G}_n = \pi_1(5_2(n)) \lhd \Omega(2,n) = \pi_1(7_1^2(2,n)),$$

where $|\Omega(2,n) : \widehat{G}_n| = 2$ and $|\widehat{G}_n : G_n| = n$.

According to the general properties of the 2-bridge knots and links [2], [13], by [9], we get the following presentation of the orbifold group $\Omega(2,n)$ of the orbifold $7_1^2(2,n)$:

$$\Omega(2,n) = \langle \alpha,\ \beta \mid \alpha\, w = w\, \alpha, \quad \alpha^n = \beta^2 = 1 \rangle,$$

where

$$w = \beta\,\alpha\,\beta\,\alpha\,\beta^{-1}\,\alpha^{-1}\,\beta^{-1}\,\alpha^{-1}\,\beta^{-1}\,\alpha\,\beta\,\alpha\,\beta,$$

and generators α and β are loops around components of the singular set of the orbifold $7_1^2(2,n)$ denoted in Figure 10 by n and 2, respectively.

Let us consider the group

$$\mathbb{Z}_n \oplus \mathbb{Z}_2 = \langle a \mid a^n = 1\rangle \oplus \langle b \mid b^2 = 1\rangle$$

and the epimorphism

$$\theta : \Omega(2,n) \longrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_2$$

defined by setting $\theta(\alpha) = a$ and $\theta(\beta) = b$.

By the construction of the 2-fold covering $5_2(n) \xrightarrow{2} 7_1^2(2,n)$ the loop $\beta \in \Omega(2,n)$ lifts to a trivial loop in $\widehat{G}_n$, and the loop $\alpha \in \omega(2,n)$ lifts to a loop in $\widehat{G}_n$ which generates a cyclic subgroup of order n. Thus, it follows that

$$\pi_1(5_2(n)) = \theta^{-1}\left(\langle a \mid a^n = 1\rangle\right) = \theta^{-1}\left(\mathbb{Z}_n\right).$$

For the $2n$-fold covering $M_n \xrightarrow{2n} 7_1^2(2,n)$ both loops α and β from $\Omega(2,n)$ lift to trivial loops in $G_n = \pi_1(M_n)$, hence $G_n = \operatorname{Ker}\theta$.

Let Γ_n be the subgroup of $\Omega(2,n)$ given by

$$\Gamma_n = \theta^{-1}(\langle b \mid b^2 = 1\rangle) = \theta^{-1}\left(\mathbb{Z}_2\right).$$

Then we get a sequence of normal subgroups

$$G_n \triangleleft \Gamma_n \triangleleft \Omega(2,n),$$

where $|\Omega(2,n) : \Gamma_n| = n$ and $|\Gamma_n : G_n| = 2$. We recall, that the orbifold $7_1^2(2,n)$ is spherical for $n = 2$, and hyperbolic for $n \geq 3$. Hence the group Γ_n acts by isometries on the universal covering X_n, that is the

3-sphere $\mathbb{S}^3$ for $n = 2$, and the hyperbolic space $\mathbb{H}^3$ for $n \geq 3$. Thus we get the orbifold $X_n \,/\, \Gamma_n$ and the following covering diagram

$$M_n \xrightarrow{2} X_n \,/\, \Gamma_n \xrightarrow{n} 7_1^2(2, n). \tag{8}$$

In this case the second covering is cyclic and it is branched over the component with index n of the singular set of $7_1^2(2, n)$ in Figure 10. But this component is the knot $\mathcal{K}_1$ and is trivial. So, underlying space of $X_n \,/\, \Gamma_n$ is the 3-sphere. By the construction of the n-fold covering $X_n \,/\, \Gamma_n \xrightarrow{n} 7_1^2(2, n)$ the loop $\alpha \in \Omega(2, n)$ lifts to a trivial loop in Γ_n, and the loop $\beta \in \Omega(2, n)$ lifts to a loop in Γ_n which generates a cyclic group of order 2. Because 7_1^2 is a 2-bridge link, its components are equivalent and we can exchange branch indices of components in Figure 10. Therefore, the singular set of $X_n \,/\, \Gamma_n$ is a n-periodic knot which can be obtained as the closure of the 3-string braid $(\sigma_1 \, \sigma_2^{1/2})^n$, that is the knot $\mathcal{K}_n$. Because the index of singularity is equal to 2, we denote $\mathcal{K}_n(2) = X_n \,/\, \Gamma_n$.

Comparing (7) and (8), we get that the following covering diagram is commutative:

$$\begin{array}{ccccc} & & M_n & & \\ & {}^{n}\swarrow & & \searrow^{2} & \\ 5_2(n) & & & & \mathcal{K}_n(2) \\ & {}_{2}\searrow & & \swarrow_{n} & \\ & & 7_1^2(2, n) & & \end{array}$$

In particular, we have that M_n is the 2-fold branched covering of the 3-sphere $\mathbb{S}^3$ branched over the knot $\mathcal{K}_n$, and theorem is proved. □

We remark, that the particular case $n = 3$ of the above covering diagram was proven in [23] by the direct consideration of the isometry group action on the manifold M_3.

Now we will give another topological description of the manifolds M_n. We recall the following fundamental Lickorish's theorem [17], [26, §9I]: *Every closed, orientable, connected 3-manifold may be obtained by surgery on a link in* $\mathbb{S}^3$.

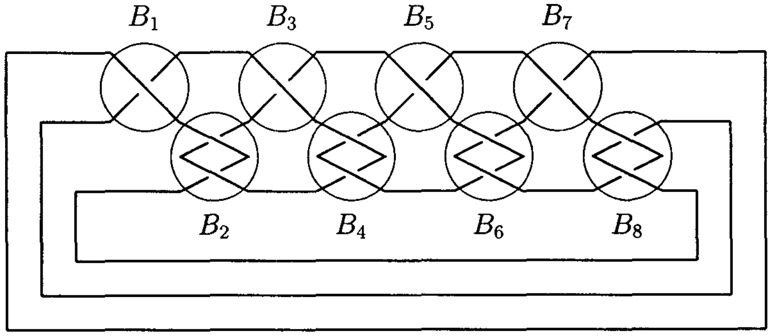

FIGURE 11.

To find such representation for manifolds M_n, we will use the approach of J. Montesinos [25]. We recall that in virtue of theorem 4, the manifold M_n, $n \geq 2$, can be obtained as the two-fold covering of the 3-sphere $\mathbb{S}^3$ branched over the n-periodic knot $\mathcal{K}_n$, that is the closure of the 3-strings braid $(\sigma_1 \sigma_2^{1/2})^n$.

We consider the neighborhoods $B_1, \ldots, B_8$ which contain cross-points of the diagram of the knot $\mathcal{K}_4$, pictured in Figure 11. Each B_i is a 3-ball such that $\partial B_i \cap \mathcal{K}_4$ consists of four points which are pairwise connected by two arcs formed by $B_i \cap \mathcal{K}_4$. Therefore, B_i can be considered as a Conway's sphere [5] and, more exactly, B_i is the 1-tangle if i is odd, and B_i is the 1/2-tangle if i is even.

Let B_i', $i = 1, \ldots, 8$, be trivial tangles such that $\partial B_i' = \partial B_i$ and the four points on the boundary $\partial B_i'$ are pairwise connected by two arcs a_i and b_i inside B_i'. Then the set

$$\left(\mathcal{K}_4 \bigcap Ext \left(\bigcup_{i=1}^{8} B_i \right) \right) \bigcup \left(\bigcup_{i=1}^{8} (a_i \cup b_i) \right)$$

is a closed unknotted curve $\mathcal{C}$ in $\mathbb{S}^3$ which diagram is pictured in Figure 12.

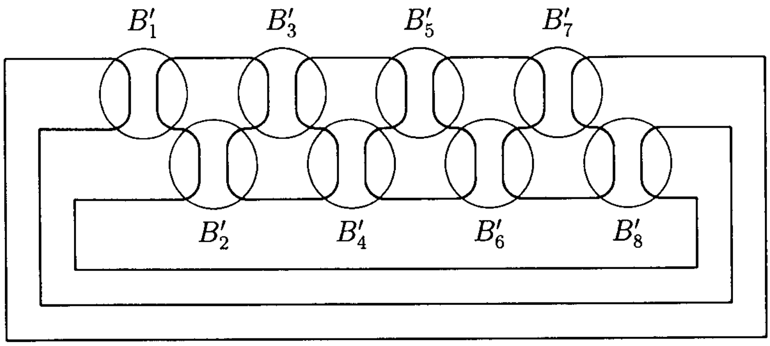

FIGURE 12.

Following [25], we can redraw the curve $\mathcal{C}$ as a horizontal line. Let us consider the two-fold coverings of $\mathbb{S}^3$ branched over the knot $\mathcal{K}_4$ and the two-fold covering of $\mathbb{S}^3$ branched over the curve $\mathcal{C}$. In this case for each $i = 1, \dots, 8$ the two-fold coverings of B_i branched over $B_i \cap \mathcal{K}_4$ and the two-fold covering of B_i' branched over $B_i \cap \mathcal{C}$ are solid torus.

Let us denote by $\tilde{L}_i$ a torus corresponding to B_i'. Using the approach of [25], we see that toruses $\tilde{L}_1, \dots, \tilde{L}_8$ are torus neighborhoods of components $L_1, \dots, L_8$ of the link pictured in Figure 13, and the two-fold covering of $\mathbb{S}^3$ branched over the knot $\mathcal{K}_4$ can be obtained by surgeries with parameters 1 and 1/2 on components $L_1, \dots, L_8$ of the link.

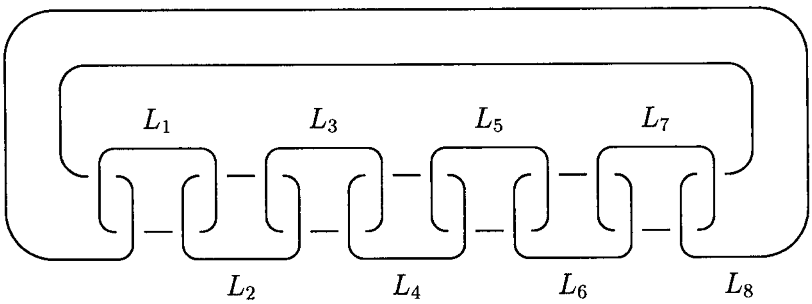

FIGURE 13.

Therefore the manifold M_4 can be obtained by Dehn surgery on the chain of circles $L_1 \cup \cdots \cup L_8$, by doing 1–surgery on circles with odd numbers, and by doing 1/2–surgery on circles with even numbers. Applying the same arguments for an arbitrary $n \geq 2$, we will get

THEOREM 5. *A manifold M_n, $n \geq 2$, can be obtained by Dehn surgery on the chain of circles $L_1 \cup \cdots \cup L_{2n}$, by doing 1–surgery on circles with odd numbers, and by doing 1/2–surgery on circles with even numbers.*

We recall that the presentation of the fundamental group of a compact manifold obtained by Dehn surgery on the link $L_1 \cup \cdots \cup L_{2n}$ was studied in [29]. In particular, for the manifolds M_n we get

COROLLARY 4. *The group G_n has the following presentation:*

(9)

$$G_n = \langle a_1, \ldots, a_{2n} \mid a_{2j+1}a_{2j+2} = a_{2j+3} \quad a_{2j}^2 a_{2j+1}^{-1} = a_{2j+2}^2 \quad j = 1, \ldots, n \rangle,$$

where indices are taken by mod n.

Let us consider the framed $2n$-component link $L_1 \cup \cdots \cup L_{2n}$ with coefficients as in the above theorem. Let us apply the twists about n components with odd number (with labels 1), which are unknotted. Then according to the Kirby-Rolfsen calculus on framed links [26, Ch. 9], we will get the alternating link with $2n$ cross-points, that is a chain of n unknotted components and all surgery coefficients are equal to $(-3/2)$ (see Figure 14 for the case $n = 4$).

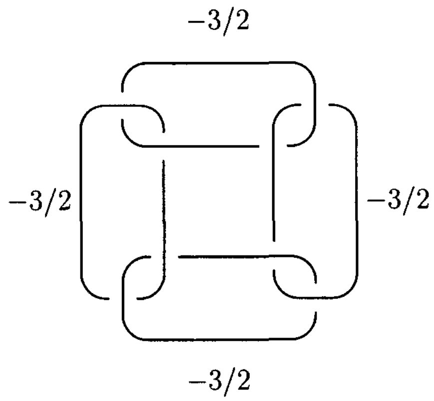

FIGURE 14.

Let us denote by $\mathcal{L}_n$ the alternating link consisting of n linked unknotted circles similar to Figure 14. Then from above considerations we get

THEOREM 6. *For $n \geq 2$ the manifold M_n can be obtained by $(-3/2)$-surgeries on components of the link $\mathcal{L}_n$.*

It is interesting to remark, that the Fibonacci manifold uniformized by Fibonacci groups $F(2, 2n)$ can be obtained by (-3)–surgeries on components of the link $\mathcal{L}_n$ [4].

Analogously, applying double-twists about n components with even numbers (which have labels 1/2) of the link $L_1 \cup \cdots \cup L_{2n}$, we will get the alternating link with $4n$ cross-points, which has n unknotted components and all surgery coefficients are equal to -3 (see Figure 15 for the case $n = 4$).

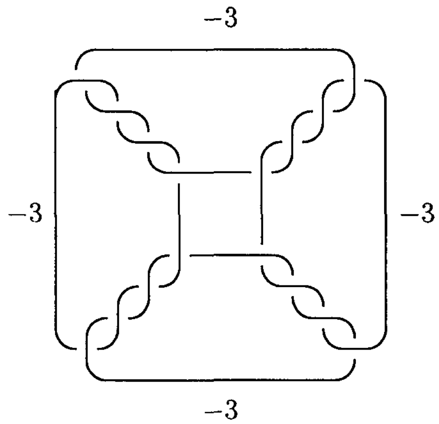

FIGURE 15.

Let us denote by $\mathcal{L}^*_n$ the alternating link consisting of n linked unknotted circles similar to Figure 15. Then from above considerations we get

THEOREM 7. *For $n \geq 2$ the manifold M_n can be obtained by -3–surgeries on components of the link $\mathcal{L}^*_n$.*

We recall that the smallest volume closed hyperbolic 3-manifold $\mathcal{M}_1$ was constructed independently by A. Fomenko and S. Matveev [7], and by J. Weeks [30]. This manifolds was described in [14] as the result of

$(-3/2)$–surgery on components of the link $\mathcal{L}_3$. Thus, from theorem 6 we get

COROLLARY 5. *The manifold M_3 is the Fomenko–Matveev–Weeks manifold $\mathcal{M}_1$.*

From theorem 3 and theorem 4 we get that the Fomenko–Matveev–Weeks manifold $\mathcal{M}_1$ can be obtained as the 3-fold cyclic covering of the 3-sphere branched over the knot 5_2, and as the 2-fold cyclic covering of the 3-sphere branched over the knot $9_{49} = \mathcal{K}_3$, that was remarked in [22]. Moreover, the isometry group action on the manifold $\mathcal{M}_1$ was studied in [23]. The validity of theorem 7 for the manifold $M_3 = \mathcal{M}_1$ was remarked in [18, p.80]

Another surgery description of the manifold $M_3 = \mathcal{M}_1$ was given in [14], where this manifold was obtained by surgeries on the components of the Whitehead link with parameters (-5) and $(-5/2)$.

References

[1] R. H. Bing, *The geometry and topology of 3-manifolds,* Amer. Math. Soc. Coll. Publ. No. **40** (1983), Providence.

[2] G. Burde and H. Zieschang, *Knots,* de Gruyter Studies in Mathematics, vol. 5, Berlin–New York, 1985.

[3] A. Cavicchioli, F. Hegenbarth and A. C. Kim, *A geometric study of Sieradski groups,* to appear in Bull. Australian Math. Soc.

[4] A. Cavicchioli and F. Spaggiari, *The classification of 3-manifolds with spines related to Fibonacci groups,* in "Algebraic Topology - Homotopy and Group Cohomology", Lect. Notes in Math., vol. 1509, Springer–Verlag, 1992, pp. 50–78.

[5] J. H. Conway, *An enumeration of knots and links,* in: Computational problems in abstract algebra, Edited by J. Leech, Pergamon Press, 1969, pp. 329–358.

[6] M. J. Dunwoody, *Cyclic presentations and 3-manifolds,* in: "Groups–Korea'94", Proceedings of the International Conference, held in Pusan, Korea, August 18–25, 1994, Edited by A. C. Kim and D. L. Johnson, de Gruyter, Berlin New York, 1995, pp. 47–55.

[7] A. T. Fomenko and S. V. Matveev, *Constant energy surfaces of Hamiltonian systems, enumeration of three-dimensional manifolds in increasing order of complexity, and computation of volumes of closed hyperbolic manifolds,* Russian Math. Surveys **43** (1988), 1, 3–24.

[8] R. H. Fox and E. Artin, *Some wild cell and spheres in three-dimensional space,* Ann. of Nath. **49** (1948), 979–990.

[9] A. Haefliger and N. D. Quach, *Une presentation de groupe fundamental d'une orbifold,* Asterisque **116** (1984), 98–107.

[10] H. Helling, A. C. Kim, and J. Mennicke, *A geometric study of Fibonacci groups,* SFB–343 Bielefeld, Diskrete Strukturen in der Mathematik, Preprint (1990).

[11] ———, *Some honeycombs in hyperbolic 3-space,* Comm. in Algebra **23** (1995), 5169–5206.

[12] H. M. Hilden, M. T. Lozano, and J. M. Montesinos, *The arithmeticity of the figure-eight knot orbifolds,* in: Topology '90, Edited by B. Apanasov, W. Neumann, A. Reid, and L. Siebenmann, de Gruyter, Berlin, 1992, pp. 169–183.

[13] ———, *On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant,* Journal Knot Theory and its Ramifications 4 (1995), 81–114.

[14] C. Hodgson and J. Weeks, *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds,* Experimental Mathematics **3** (1994), 101–114.

[15] D. L. Johnson, *Topics in the theory of group presentations,* London Math. Soc. Lecture Notes Series·**42**, Cambridge University Press, 1980.

[16] A. C. Kim and A. Yu. Vesnin, *A topological study of the Fractional Fibonacci groups,* preprint, Universität Bielefeld, Bielefeld, 1996.

[17] W. Lickorish, *A representation of orientable combinatorial 3-manifolds,* Ann. of Math. **76** (1962), 531–538.

[18] S. Lins, *Gems, computers and attractors for 3-manifolds,* World Scientific, Singapore, 1995.

[19] C. Maclachlan, *Generalizations of Fibonacci numbers, groups and manifolds,* in: Combinatorial and Geometric Group Theory, Edinburgh 1993, Edited by A. J. Duncan, N. D. Gilbert and J. Howie, London Math. Soc. Lecture Notes Series **204**, Cambridge University Press, 1995, pp. 233–238.

[20] A. Mednykh and A. Vesnin, *On the Fibonacci groups, the Turk's head links and hyperbolic 3-manifolds,* in: "Groups–Korea '94", Proceedings of the International Conference, held in Pusan, Korea, August 18–25, 1994, Edited by A. C. Kim and D. L. Johnson, de Gruyter, Berlin New York, 1995, pp. 231–239.

[21] A. D. Mednykh and A. Yu. Vesnin, *Fibonacci manifolds as two-fold coverings over the three-dimensional sphere and the Meyerhoff-Neumann conjecture,* Sibirsk. Mat. Zh. **37** (1996), no. 3, 534–542 (Russian), translated in Siberian Math. J. **37** (1996), no. 3. 461–467.

[22] ———, *On Heegaard Genus of Three-Dimensional Hyperbolic Manifolds of Small Volume,* Sibirsk. Mat. Zh. **37** (1996), no. 5, 1013–1018 (Russian), translated in Siberian Math. J. **37** (1996), no. 5, 893–897.

[23] ———, *Visualization of the Isometry Group Action on the Fomenko-Matveev-Weeks Manifold,* to appear in Journal of Lie Theory **8** (1998), pp. 16.

[24] E. E. Moise, *Geometric topology in dimensional 2 and 3,* Graduate Tests in Math., vol. 47, Springer–Verlag, 1977.

[25] J. M. Montesinos, *Surgery on links and double branched covers of* $\mathbb{S}^3$, in: Knots, Groups, and 3-Manifolds, Edited by L. P. Neuwirth, Princeton University Press, Princeton, 1975, pp. 227–259.

[26] D. Rolfsen, *Knots and links,* Publish or Perish Inc., Berkely Ca., 1976.

[27] H. Seifert and W. Threlfall, *A textbook of topology,* Academic Press, Inc., 1980.

[28] A. J. Sieradski, *Combinatorial squashings, 3-manifolds, and the third homotopy of groups,* Invent. Math. **84** (1986), 121–139.

[29] M. Takahashi, *On the presentation of the fundamental groups of 3-manifolds,* Tsukuba J. Math, **13** (1989), 175–189.

[30] J. Weeks, *Hyperbolic structures on three-manifolds,* Princeton Univ. Ph. D. Thesis, 1985.

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